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Research Article

Convexity and Proximinality in Banach Space

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By the continuity of preduality map, we give some necessary and sufficient conditions of the strongly convex and very convex spaces, respectively. Using nearly strong convexity of X, we give some equivalent conditions that every element in X is strongly unique of order p, bounded strongly unique of order p, and locally strongly unique of order p.

1. Notations and Definitions

Let X be a Banach space and let X^* be its dual space. Let us denote by $B(X)$ and $S(X)$ the closed unit ball and the unit sphere of X, and by $B(x,r)$ the closed ball centered at x of radius $r > 0$. Let $x \in S(X)$, $D(x) = \{f \in S(X^*) : f(x) = 1\}$. Let us denote by $NA(X)$ the set of all norm-attaining functionals in X^* and $S_0(X^*) = S(X^*) \cap NA(X)$.

For a subset $C \subset X$, the metric projection $P_C : X \to 2^C$ is defined by $P_C(x) = \{y \in C : \|x - y\| = d(x,C)\}$, where $d(x,C) = \inf\{\|x - y\| : y \in C\}$, $x \in X$. If $P_C(x) \neq \emptyset$ for each $x \in X$, then C is said to be proximinal. If $P_C(x)$ is at most a singleton for each $x \in X$, then C is said to be semi-Chebyshev. If C is a simultaneously proximinal and semi-Chebyshev set, then C is said to be a Chebyshev set.

Definition 1.1 (see [1]). One says that X is strongly convex (resp., very convex/nearly strongly convex/nearly very convex) if for any $x \in S(X)$ and $\{x_n\} \subset B(X)$ convergence $x^*(x_n) \to 1$ as $n \to \infty$ for some $x^* \in D(x)$ implies that $x_n \to x$ as $n \to \infty$ (resp., $x_n \xrightarrow{w} x$ as $n \to \infty$/$\{x_n\}$ is relatively compact/$\{x_n\}$ is weakly relatively compact).

Definition 1.2 (see [2]). One says that X is midpoint locally uniformly rotund (MLUR) (resp., weakly midpoint locally uniformly rotund (WMLUR)) if any $x_0, x_n, y_n \in S(X)$, $(n = 1, 2, \ldots)$, $x_n + y_n \to 2x_0$ as $n \to \infty$ implies that $x_n - y_n \to \theta$ as $n \to \infty$ (resp., $x_n - y_n \xrightarrow{w} \theta$ as $n \to \infty$).

Wu and Li defined strong convexity in [3], and Wang and Zhang in [4] defined very convexity, nearly strong convexity, and nearly very convexity which are two generalizations of locally uniformly rotund (LUR) and weakly locally uniformly rotund (WLUR) spaces. By [3, 5], we have the following relations:

$$\begin{tikzcd}
\text{LUR} \arrow[r] \arrow[dd] & \text{Strong convexity} \arrow[rr] \arrow[dd] \arrow[dr] & & \text{MLUR} \arrow[d] \arrow[dd] \\
& & \text{Nearly strong convexity} \arrow[d] & \text{R} \\
& & \text{Nearly very convexity} & \\
\text{WLUR} \arrow[r] & \text{Very convexity} \arrow[rr] \arrow[ur] & & \text{WMLUR} \arrow[u]
\end{tikzcd}$$

Sullivan defined very rotund space in [6]. A Banach space X is said to be very rotund if no $x^* \in S(X^*)$ is simultaneously a norming element for some $x \in S(X)$ and $x^{**} \in S(X^{**})$, where $x \neq x^{**}$. Z. H. Zhang and C. J. Zhang proved that very rotund space coincides with very convex space in [5]. In [3–5, 7], many results of these four classes of convexities were proved. In particular, Zhang and Shi proved that they have important applications in approximation theory in [1]. In [8], Bandyapadhyay et al. also proposed two generalizations of locally uniformly rotund space, which are called almost locally uniformly rotund space and weakly almost locally uniformly rotund space. A Banach space X is said to be ALUR (resp., WALUR) if for any $x \in S(X)$, $\{x_n\} \subset B(X)$, and $\{x_n\} \subset B(X^*)$, the condition $\lim_m \lim_n x_m^*((x_n+x)/2) = 1$ implies $x_n \to x$ (resp., $x_n \xrightarrow{w} x$). Many properties of these two classes of convexities were studied in [8–10] too. Recently, we proved that almost locally uniformly rotund space is equivalent to strongly convex space and that weakly almost locally uniformly rotund space is equivalent to very convex space [7]. Thus, we unified the results of the studies about the strongly convex space (resp., very convex space) and the almost locally uniform rotundity (resp., weakly almost locally uniform rotundity). This shows that these convexities have important effects on and applications in geometry of Banach space and approximation theory.

A sequence $\{z_n\} \subset C$ is said to be minimizing for $x \in X \setminus C$ if $\|x - z_n\| \to d(x, C)$ as $n \to \infty$.

Definition 1.3 (see [10]). Let C be a closed subset (resp., a weakly closed subset) of X and $x_0 \in X \setminus C$.

(1) One says that C is approximatively (resp., weakly approximatively) compact for x_0 if every minimizing sequence $\{z_n\} \subset C$ for x_0 has a convergent (resp., weakly convergent) subsequence.

(2) One says that C is strongly Chebyshev (resp., weakly strongly Chebyshev) for x_0 if every minimizing sequence $\{z_n\} \subset C$ for x_0 is convergent (resp., weakly convergent).

If C is approximatively compact (resp., weakly approximatively compact/strongly Chebyshev/weakly strongly Chebyshev) for every $x \in X \setminus C$, we say that C is approximatively compact (resp., weakly approximatively compact/strongly Chebyshev/weakly strongly Chebyshev) in X.

Definition 1.4 (see [11]). Let $G \in X$, $x \in X \setminus G$, and $1 \le p < \infty$.

(1) $g_0 \in P_G(x)$ is said to be strongly unique at x if there exists a constant $r_p > 0$ such that

$$\|x - g\| \ge \|x - g_0\| + r_p \|g - g_0\|, \tag{1.1}$$

for any $g \in G$.

(2) $g_0 \in G$ is said to be strongly unique of order p at x if there exists an $r_p = r_p(x) > 0$ such that

$$\|x - g\|^p \ge \|x - g_0\|^p + r_p \|g - g_0\|^p, \tag{1.2}$$

for any $g \in G$.

(3) $g_0 \in G$ is said to be bounded strongly unique of order p at x if given any $N > 0$, there exists an $r_{p,N} = r_{p,N}(x)$ such that

$$\|x - g\| \ge \|x - g_0\| + r_{p,N} \|g - g_0\|^p, \tag{1.3}$$

for any $g \in G$, $\|g - g_0\| \le N$.

(4) $g_0 \in G$ is said to be locally strongly unique of order p at x if there exist $N > 0$ and $r_{p,N} = r_{p,N}(x)$ such that

$$\|x - g\| \ge \|x - g_0\| + r_{p,N} \|g - g_0\|^p, \tag{1.4}$$

for any $g \in G$, $\|g - g_0\| \le N$.

In order to study the uniqueness of best approximation in nonlinear approximation theory, Wulbert [12] defined the strong uniqueness of best approximation. Smarzewski [13] and Schmidt [14] defined the strongly unique of order p and the bounded strongly unique of order p, respectively. By [11], we know that the strongly unique of order p and the bounded strongly unique of order p all are generalizations of the strongly unique. The strongly unique of order p and the bounded strongly unique of order p imply the locally strongly unique of order p, but converse implied relation is not generally true. When $p = 1$, the strongly unique of order p and the bounded strongly unique of order p all are strongly unique. The locally strongly unique of order 1 is not equivalent to the strongly unique.

Definition 1.5 (see [10]). For $x^* \in S(X^*)$ and $x \in S(X)$, let one define the following maps:

$$D(x) = \{x^* \in S(X^*) : x^*(x) = 1\}, \qquad D^{-1}(x^*) = \{x \in S(X) : x^*(x) = 1\}. \tag{1.5}$$

D is called the duality map and D^{-1} is called the preduality map. Naturally, D^{-1} is defined only on $S_0(X^*)$.

Definition 1.6 (see [10]). The preduality map $D^{-1} : S(X^*) \to S(X)$ is said to be upper semicontinuous $(n - \tau)$ on $S_0(X^*)$ if for any $x^* \in S_0(X^*)$ and any τ open set W with $D^{-1}(x^*) \subset W$, there exists $\varepsilon > 0$ such that $D^{-1}(y^*) \subset W$ whenever $\|x^* - y^*\| < \varepsilon$, where τ stands for norm or weak topology.

2. Convexity and Continuity of the Preduality Map

Using the Bronsted-Rockafeller Theorem (see [15, Theorem 3.18, page 51]), we can prove the following lemma.

Lemma 2.1. *Suppose that $\varepsilon > 0$, $x_0 \in S(X)$, $x_0^* \in S(X^*)$, and $x_0^*(x_0) > 1 - \varepsilon$, then there are $x_\varepsilon \in S(X)$ and $x_\varepsilon^* \in D(x_\varepsilon)$ such that*

$$\|x_\varepsilon - x_0\| < 2\sqrt{\varepsilon}, \qquad \left\|x_\varepsilon^* - x_0^*\right\| < 2\sqrt{\varepsilon}. \tag{2.1}$$

Theorem 2.2. *Let X be a Banach space. X is strongly convex if and only if the preduality map $D^{-1} : S_0(X^*) \to S(X)$ is singlevalued and continuous.*

Proof. Necessity. Since strong convexity implies strict convexity, D^{-1} is singlevalued. Suppose that $\{x_n^*\} \subset S_0(X^*)$, $x^* \in S_0(X^*)$ with $x_n^* \to x^*$, then there exist $\{x_n\} \subset S(X)$ and $x \in S(X)$ such that $x_n^*(x_n) = 1 = x^*(x)$. We have that

$$x^*\left(D^{-1}(x_n^*)\right) = (x^* - x_n^* + x_n^*)\left(D^{-1}(x_n^*)\right) = (x^* - x_n^*)\left(D^{-1}(x_n^*)\right) + x_n^*\left(D^{-1}(x_n^*)\right). \tag{2.2}$$

Since

$$\left|(x^* - x_n^*)D^{-1}(x_n^*)\right| \le \|x_n^* - x^*\| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty, \tag{2.3}$$

and $x_n^*(D^{-1}(x_n^*)) = x_n^*(x_n) = 1$, we have that

$$x^*\left(D^{-1}(x_n^*)\right) = x^*(x_n) \longrightarrow 1, \quad \text{as } n \longrightarrow \infty. \tag{2.4}$$

Since X is strongly convex, we deduce that $x_n \to x$ as $n \to \infty$, that is, $D^{-1}(x_n^*) \to D^{-1}(x^*)$ as $n \to \infty$. This means that D^{-1} is continuous.

Sufficiency. Let $\{x_n\} \subset S(X), x \in S(X)$ with $x^*(x_n) \to 1$ as $n \to \infty$ for some $x^* \in D(x)$. Since D^{-1} is singlevalued, by Lemma 2.1, there exist $\{y_n\} \subset S(X)$ and $\{y_n^*\} \subset S_0(X^*)$ such that $D^{-1}(y_n^*) = y_n$ and

$$\|y_n - x_n\| \longrightarrow 0, \quad \left\|y_n^* - x^*\right\| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{2.5}$$

Since D^{-1} is continuous, we have that $D^{-1}(y_n^*) \to D^{-1}(x^*)$ as $n \to \infty$, that is, $y_n \to x$ as $n \to \infty$. So $x_n \to x$ as $n \to \infty$, which means that X is strongly convex. □

Using Lemma 2.1, in a similar way to prove Theorem 2.2, we can prove the following result.

Theorem 2.3. *Let X be a Banach space. X is very convex if and only if the preduality map $D^{-1} : S_0(X^*) \to S(X)$ is singlevalued and weakly continuous.*

Lemma 2.4. *Let C be a convex set of a strongly convex Banach space X. The following are equivalent:*

(1) *C is proximinal;*

(2) *C is weakly approximatively compact;*

(3) *C is approximatively compact;*

(4) *C is strongly Chebyshev.*

Proof. We only need to prove (1)⇒(4).

Let $x \in X \setminus C$, $\{z_n\} \subset C$ such that

$$\lim_{n\to\infty} \|x - z_n\| = d(x, C). \tag{2.6}$$

In order to finish the proof, we will show that there exists some $y_0 \in C$ such that $\|y_0 - z_n\| \to 0$ as $n \to \infty$.

Step 1. If $x = 0$, then $r = d(0, C) > 0$ and

$$B(0, r) \cap C = P_C(0) \neq \emptyset, \qquad \operatorname{int} B(0, r) \cap C = \emptyset, \tag{2.7}$$

where $B(0, r) = \{y : \|y\| \leq r\}$. By the separation theorem [2] and definition of norm, there exists an $f \in S(X^*)$ such that

$$\sup\{f(y) : y \in C\} \leq \inf\{f(y) : y \in B(0, r)\} = -\|f\| r = -\|y_0\|, \tag{2.8}$$

for any $y_0 \in P_C(0)$. Hence, we have that

$$-\|y_0\| \leq f(y_0) \leq \sup\{f(y) : y \in C\} \leq -\|y_0\|. \tag{2.9}$$

This shows that $f \in D(-y_0/\|y_0\|)$. From $z_n \in C$, we get $f(z_n) \leq f(y_0)$. Combining it with the condition $\lim_{n\to\infty}\|0 - z_n\| = d(0, C)$, it follows that

$$\|0 - y_0\| = f(0 - y_0) \leq f(0 - z_n) \leq \|0 - z_n\| \longrightarrow d(0, C) = \|0 - y_0\|, \tag{2.10}$$

as $n \to \infty$. Hence, $f(-z_n/\|z_n\|) \to 1$ as $n \to \infty$. Since X is strongly convex, $-z_n/\|z_n\| \to -y_0/\|y_0\|$ as $n \to \infty$, that is, $\|z_n - y_0\| \to 0$ as $n \to \infty$.

Step 2. If $x \neq 0$, we set $C' = x - C$. It is clear that C' is proximinal and $\{x - z_n\} \subset C'$ is a minimizing sequence for 0. By Step 1, there exists $y_0' \in C'$, such that $\|y_0' - (x - z_n)\| \to 0$ as $n \to \infty$. This shows that $\|(x - y_0') - z_n\| \to 0$ as $n \to \infty$ and $x - y_0' = y_0 \in C$.

□

Similarly to the proof of Lemma 2.4, we can prove the following result.

Lemma 2.5. *Let C be a convex set of a nearly strongly convex Banach space X. The following are equivalent:*

(1) *C is proximinal;*

(2) *C is weakly approximatively compact;*

(3) *C is approximatively compact.*

Lemma 2.6. *Let $x \in S(X)$, $f \in D(x)$, then hyperplane $H = \{y \in X : f(y) = 1\}$ is a proximinal convex subset in X.*

Proof. Let

$$\ker f = \{y \in X : f(y) = 0\}. \tag{2.11}$$

We will prove that $\ker f$ is a proximinal convex subset in X. For any $z \in X$, if $d(z, \ker f) = 0$, since $\ker f$ is a closed subspace, we know that $z \in \ker f$. Hence, $z \in P_{\ker f}(z)$. If $d(z, \ker f) > 0$, since $X = \{\alpha z\} + \ker f$, there exist $\lambda \in \mathbb{R}$ and $y_0 \in \ker f$ such that $d(z, \ker f)x = \lambda z + y_0$. For any $y \in X$, $d(y, \ker f) = |f(y)|$, and $f(x) = 1$, we have that $d(x, \ker f) = 1$ and $f(d(z, \ker f)x) = d(d(z, \ker f)x, \ker f)$. Therefore, we have that

$$d(z, \ker f) = d(\lambda z + y_0, \ker f) = |\lambda| d(z, \ker f). \tag{2.12}$$

This means that $|\lambda| = 1$. Hence, we have

$$d(z, \ker f) = \|d(z, \ker f)x\| = \|\lambda z + y_0\| = \|z + \lambda y_0\|. \tag{2.13}$$

It follows that $-\lambda y_0 \in P_{\ker f}(z)$, which means that $\ker f$ is proximinal set.

Furthermore, we will prove that hyperplane $H = \{y \in X : f(y) = 1\}$ is a proximinal convex subset in X. For $x_0 \in H$, $H = x_0 + \ker f$. For any $z \in X$,

$$d(z, H) = \inf_{y \in \ker f} \|z - (x_0 + y)\| = d(z - x_0, \ker f). \tag{2.14}$$

Since $\ker f$ is a proximinal subset, there exists a $y_0 \in \ker f$ such that

$$d(z - x_0, \ker f) = \|z - (x_0 + y_0)\|. \tag{2.15}$$

Therefore, $x_0 + y_0 \in P_H(z)$, which means that H is proximinal set in X. □

Theorem 2.7. *Let X be a Banach space. X is a nearly strongly convex if and only if the preduality map $D^{-1} : S_0(X^*) \to S(X)$ is upper semicontinuous $(n - n)$ on $S_0(X^*)$ with norm compact images, where n stands for norm topology.*

Proof. Necessity. Arbitrarily take $x^* \in S_0(X^*)$. Then by Lemma 2.6, we know that $\ker x^* = Y$ is a proximinal convex subset in X. Since X is nearly strongly convex, by Lemma 2.5, we know that Y is approximatively compact. Suppose that D^{-1} is not upper semicontinuous $(n - n)$ at x^*, then for some open set W in X with $D^{-1}(x^*) \subset W$, there exists $\{x_n^*\} \subset S(X^*)$ such that $x_n^* \to x^*$ as $n \to \infty$ and $D^{-1}(x_n^*) \not\subseteq W$ for all n. Let $z_n \in D^{-1}(x_n^*) \setminus W$. Fix $x \in D^{-1}(x^*)$. Let $x_n = x^*(z_n)x - z_n$, then $\{x_n\} \subset Y$ is a minimizing sequence for x. Since Y is approximatively compact, $\{x_n\}$ has a convergent subsequence. So $\{z_n\}$ has convergent subsequence converging to z. Thus, $z \in D^{-1}(x^*) \subset W$, but $z_n \in X \setminus W$ is closed, which is a contradiction. By the assumption, we easily know that the image of D^{-1} is compact.

Sufficiency. Let $\{x_n\} \subset S(X), x \in S(X)$ with $x^*(x_n) \to 1$ as $n \to \infty$ for some $x^* \in D(x)$. By Lemma 2.1, there exist $\{y_n\} \subset S(X)$ and $\{y_n^*\} \subset S(X^*)$ such that $y_n \in D^{-1}(y_n^*)$ and

$$\|y_n - x_n\| \longrightarrow 0, \quad \|y_n^* - x^*\| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{2.16}$$

Since $D^{-1}(x^*)$ is compact, $D^{-1}(x^*)$ is proximinal. Let $z_n \in D^{-1}(x^*)$ such that $\|y_n - z_n\| = d(y_n, D^{-1}(x^*))$. In virtue of D^{-1} being upper semicontinuous $(n-n)$ on $S_0(X^*)$ and $\|y_n^* - x^*\| \to 0$, for any $\varepsilon > 0$, there is n_0 such that for all $n \geq n_0$ $d(y_n, D^{-1}(x^*)) < \varepsilon$, that is, $\|y_n - z_n\| < \varepsilon$, which means that $y_n - z_n \to \theta$ as $n \to \infty$. Combining this with the compactness of $D^{-1}(x^*)$, $\{z_n\}$ has convergent subsequence, and hence $\{y_n\}$ has convergent subsequence. By $y_n - x_n \to \theta$ as $n \to \infty$, $\{x_n\}$ has convergent subsequence, which means that X is nearly strongly convex. □

Theorem 2.8. *Let X be a Banach space. X is nearly very convex if and only if the preduality map $D^{-1} : S_0(X^*) \to S(X)$ is upper semicontinuous $(n - w)$ on $S_0(X^*)$ with weakly compact images.*

Proof. In a similar way to the proof of Theorem 2.7, the necessity can be proved.

Sufficiency. Let $\{x_n\} \subset S(X)$ with $x^*(x_n) \to 1$ as $n \to \infty$ for some $x^* \in D(x)$. By Lemma 2.1, there are $\{y_n\} \subset S(X)$ and $\{y_n^*\} \subset S(X^*)$ such that $y_n \in D^{-1}(y_n^*)$ and

$$\|y_n - x_n\| \longrightarrow 0, \quad \|y_n^* - x^*\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{2.17}$$

Suppose that $\{x_n\}$ does not have weakly convergent subsequence. By (2.17), we know that $\{y_n\}$ does not have weakly convergent subsequence. Without loss of generality, we can assume that

$$|f(y_n - y_m)| \geq \varepsilon_0 \quad (\forall n,\ m \in N,\ n \neq m), \tag{2.18}$$

for some $f \in X^* \setminus \{\theta\}$ and $\varepsilon_0 > 0$. Since $D^{-1}(x^*)$ is weakly compact, there exists $\{z_i\}_{i=1}^k \subset D^{-1}(x^*)$, such that $D^{-1}(x^*) \subset \cup_{i=1}^k \{y \in X : |f(y - z_i)| < \varepsilon_0/2\}$. Combining upper semicontinuity $(n-w)$ of D^{-1} with $\|y_n^* - x^*\| \to 0$, $\{y_n\}_{n \geq n_0} \subset \cup_{i=1}^k \{y \in X : |f(y - z_i)| < \varepsilon_0/2\}$ for some $n_0 \in N$. So, there are subsequence $\{y_{n_k}\}_{k \geq 1} \subset \{y_n\}_{n \geq n_0}$ and i_0 $(1 \leq i_0 \leq k)$ such that $\{y_{n_k}\}_{k \geq 1} \subset \{y \in X : |f(y - z_{i_0})| < \varepsilon_0/2\}$. Hence,

$$|f(y_{n_k} - y_{n_l})| \leq |f(y_{n_k} - z_{i_0})| + |f(z_{i_0} - y_{n_l})| < \frac{\varepsilon_0}{2} + \frac{\varepsilon_0}{2} = \varepsilon_0, \tag{2.19}$$

a contradiction with (2.18). □

3. Convexity and Proximinality

Theorem 3.1. *Let C be a weakly approximatively compact subset of nearly strongly convex Banach space X, $x \in X \setminus C$, and $g_0 \in C$. If g_0 is the unique element of best approximation of x, then the following are equivalent:*

(1) *g_0 is strongly unique of order p at x;*

(2) *g_0 is bounded strongly unique of order p at x;*

(3) *g_0 is locally strongly unique of order p at x.*

Proof. (1)⇒(2). Let $r_p > 0$ be such that

$$\|x-g\|^p \geq \|x-g_0\|^p + r_p\|g-g_0\|^p, \tag{3.1}$$

for any $g \in C$. For any $N > 0$ and each $g \in C, \|g-g_0\| \leq N$, since

$$\|x-g\| \leq \|x-g_0\| + \|g-g_0\| \leq \|x-g_0\| + N, \tag{3.2}$$

by Lagrange's mean value theorem,

$$\frac{\|x-g\|^p - \|x-g_0\|^p}{\|x-g\| - \|x-g_0\|} \leq p(\|x-g_0\| + N)^{p-1}. \tag{3.3}$$

Set

$$r_{p,N} = \frac{r_p}{p}(\|x-g_0\| + N)^{1-p}. \tag{3.4}$$

Then

$$\|x-g\| - \|x-g_0\| \geq r_{p,N}\|g-g_0\|^p. \tag{3.5}$$

Because $r_{p,N}$ is independent of g, g_0 is bounded strongly unique of order p at x.

(2)⇒(1). From

$$\lim_{\|g\|\to\infty} \frac{\|x-g\|^p - \|x-g_0\|^p}{\|g-g_0\|^p} \geq \lim_{\|g\|\to\infty}\left[\left(1-\frac{\|x-g_0\|}{\|g-g_0\|}\right)^p - \left(\frac{\|x-g_0\|}{\|g-g_0\|}\right)^p\right] = 1, \tag{3.6}$$

we can take $N > 0$ such that

$$\|x-g\|^p \geq \|x-g_0\|^p + \frac{1}{2}\|g-g_0\|^p, \quad \forall g \in C,\ \|g-g_0\| \geq N. \tag{3.7}$$

Again, by Lagrange's mean value theorem, we have

$$\frac{\|x-g\|^p - \|x-g_0\|^p}{\|x-g\| - \|x-g_0\|} \geq p\|x-g_0\|^{p-1}. \tag{3.8}$$

Therefore,

$$\begin{aligned}\|x-g\|^p &\geq \|x-g_0\|^p + p\|x-g_0\|^{p-1}\cdot[\|x-g\| - \|x-g_0\|] \\ &\geq \|x-g_0\|^p + p\|x-g_0\|^{p-1}\cdot r_{p,N}\|g-g_0\|^p,\end{aligned} \tag{3.9}$$

for any $g \in C$ and $\|g-g_0\| \leq N$. Set $r_p = \min\{1/2, p\|x-g_0\|^{p-1}r_{p,N}\}$, then

$$\|x-g\|^p \geq \|x-g_0\|^p + r_p\|g-g_0\|^p, \tag{3.10}$$

for any $g \in C$. That shows g_0 is strongly unique of order p at x.

Now, we only need to prove (3)⇒(2). Let $N_0 > 0, r_{p,N_0} > 0$ such that

$$\|x - g\| \geq \|x - g_0\| + r_{p,N_0}\|g - g_0\|^p, \tag{3.11}$$

for all $g \in C$ and $\|g - g_0\| \leq N_0$.

If the condition (2) is not true, there exist $N > N_0$, $g_n \in C$ with $N_0 < \|g_n - g\| \leq N$ such that

$$\|x - g_n\| < \|x - g_0\| + \frac{1}{n}\|g_n - g_0\|^p, \tag{3.12}$$

then, because $\{g_n\}$ is bounded,

$$\|x - g_0\| \leq \lim_n \|x - g_n\| \leq \lim_n \left[\|x - g_0\| + \frac{1}{n}\|g_n - g_0\|^p\right] = \|x - g_0\|, \tag{3.13}$$

that is, $\lim_n \|x - g_n\| = d(x, C)$. This shows that $\{g_n\}$ is a minimizing sequence for x. Since C is weakly approximatively compact, there is a weakly convergent subsequence of $\{g_n\}$. Without loss of generality, we can assume that

$$g_n \xrightarrow{w} \overline{g_0} \in C. \tag{3.14}$$

Based on weak lower semi-continuity of norm, we get that

$$\|x - \overline{g_0}\| \leq \lim_n \|x - g_n\| = \|x - g_0\|. \tag{3.15}$$

Hence, $\overline{g_0} \in P_C(x)$, consequently, $\overline{g_0} = g_0$. Take $f \in D((x - \overline{g_0})/\|x - \overline{g_0}\|)$, then

$$f\left(\frac{x - g_n}{\|x - g_n\|}\right) \longrightarrow f\left(\frac{x - \overline{g_0}}{\|x - \overline{g_0}\|}\right) = 1. \tag{3.16}$$

Since X is nearly strongly convex, there exists $\{x - g_{n_k}\} \subset \{x - g_n\}$ such that $x - g_{n_k} \to x - \overline{g_0}$, that is, $g_{n_k} \to \overline{g_0}$. It follows that $\|\overline{g_0} - g_0\| \geq N_0$. This is a contradiction with the fact that $\overline{g_0} = g_0$. □

Corollary 3.2. *Let C be a weakly approximatively compact subset of strongly convex Banach space X, $x \in X \setminus C$, and $g_0 \in P_C(x)$, then the following are equivalent:*

(1) *g_0 is strongly unique of order p at x;*

(2) *g_0 is bounded strongly unique of order p at x;*

(3) *g_0 is locally strongly unique of order p at x.*

If C is a convex set, we have the following result.

Theorem 3.3. *Let C be a convex subset of a nearly strongly convex Banach space X, $x \in X \setminus C$, and $g_0 \in C$. If g_0 is the unique element of best approximation of x, then the following are equivalent:*

(1) *g_0 is strongly unique of order p at x;*

(2) *g_0 is bounded strongly unique of order p at x;*

(3) *g_0 is locally strongly unique of order p at x.*

Proof. By the proof of Theorem 3.1, we have (1)⇔(2). Now, we only need to prove (3)⇒(2).

If the condition (2) is not true, there exist $N > N_0$, $g_n \in C$ with $N_0 < \|g_n - g\| \leq N$ such that

$$\|x - g_n\| < \|x - g_0\| + \frac{1}{n}\|g_n - g_0\|^p. \tag{3.17}$$

In the same way of the proof of (3)⇒(2) in Theorem 3.1, we can also prove that $\{g_n\}$ is a minimizing sequence for x. By Lemma 2.5, C is approximatively compact. Hence, there exists a convergent subsequence of $\{g_n\}$. Without loss of generality, we can assume that

$$g_n \longrightarrow \overline{g_0} \in C. \tag{3.18}$$

Consequently, $\overline{g_0} = g_0$, but $\|\overline{g_0} - g_0\| \geq N_0$, which is a contradiction. □

Corollary 3.4. *Let C be a convex set of a strongly convex Banach space X. $x \in X \setminus C$ and $g_0 \in P_C(x)$. The following are equivalent:*

(1) *g_0 is strongly unique of order p at x;*

(2) *g_0 is bounded strongly unique of order p at x;*

(3) *g_0 is locally strongly unique of order p at x.*

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