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Research Article

Spectral Analysis of q-Sturm-Liouville Problem with the Spectral Parameter in the Boundary Condition

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This paper is concerned with q-Sturm-Liouville boundary value problem in the Hilbert space with a spectral parameter in the boundary condition. We construct a self-adjoint dilation of the maximal dissipative q-difference operator and its incoming and outcoming spectral representations, which make it possible to determine the scattering matrix of the dilation. We prove theorems on the completeness of the system of eigenvalues and eigenvectors of operator generated by boundary value problem.

1. Introduction

Spectral analysis of Sturm-Liouville and Schrödinger differential equations with a spectral parameter in the boundary conditions has been analyzed intensively (see [1–16]). Then spectral analysis of discrete equations became an interesting subject in this field. So there is a substantial literature on this subject (see [10, 17–19]).

There has recently been great interest in quantum calculus and many works have been devoted to some problems of q-difference equation. In particular, we refer the reader to consult the reference [20] for some definitions and theorems on q-derivative, q-integration, q-exponential function, q-trigonometric function, q-Taylor formula, q-Beta and Gamma functions, Euler-Maclaurin formula, anf so forth. In [21], Adıvar and Bohner investigated the eigenvalues and the spectral singularities of non-selfa-djoint q-difference equations of second order with spectral singularities. In [12], Huseynov and Bairamov examined the properties of eigenvalues and eigenvectors of a quadratic pencil of q-difference equations. In [22], Agarwal examined spectral analysis of self-adjoint equations. In [23], Shi and Wu presented several classes of explicit self-adjoint Sturm-Liouville difference operators with

either a non-Hermitian leading coefficient function, or a non-Hermitian potential function, or a nondefinite weight function, or a non-self-adjoint boundary condition. In [24], Annaby and Mansour studied a q-analogue of Sturm-Liouville eigenvalue problems and formulated a self-adjoint q-difference operator in a Hilbert space. They also discussed properties of the eigenvalues and the eigenfunctions.

In this paper, we consider q-Sturm-Liouville Problem and define an adequate Hilbert space. Our main target of the present paper is to study q-Sturm-Liouville boundary value problem in case of dissipation at the right endpoint of $(0, a)$ and with the spectral parameter at zero. The maximal dissipative q-Sturm-Liouville operator is constructed using [25, 26] and Lax-Phillips scattering theory in [27]. Then we constructed a functional model of dissipative operator by means of the incoming and outcoming spectral representations and defined its characteristic function in terms of the solutions of the corresponding q-Sturm-Liouville equation. By combining the results of Nagy-Foiaş and Lax-Phillips, characteristic function is expressed with scattering matrix and the dilation of dissipative operator is set up. Finally, we give theorems on completeness of the system of eigenvectors and associated vectors of the dissipative q-difference operator.

Let q be a positive number with $0 < q < 1$, $A \subset \mathbb{R}$, and $a \in \mathbb{C}$. A q-difference equation is an equation that contains q-derivatives of a function defined on A. Let $y(x)$ be a complex-valued function on $x \in A$. The q-difference operator D_q is defined by

$$D_q y(x) = \frac{y(qx) - y(x)}{\mu(x)}, \quad \forall x \in A, \tag{1.1}$$

where $\mu(x) = (q-1)x$. The q-derivative at zero is defined by

$$D_q y(0) = \lim_{n\to\infty} \frac{y(q^n x) - y(0)}{q^n x}, \quad x \in A, \tag{1.2}$$

if the limit exists and does not depend on x. A right inverse to Dq, the Jackson q-integration, is given by

$$\int_0^x f(t) d_q t = x(1-q)\sum_{n=0}^{\infty} q^n f(q^n x), \quad x \in A, \tag{1.3}$$

provided that the series converges, and

$$\int_a^b f(t) d_q t = \int_0^b f(t) d_q t - \int_0^a f(t) d_q t, \quad a, b \in A. \tag{1.4}$$

Let $L_q^2(0, a)$ be the space of all complex-valued functions defined on $[0, a]$ such that

$$\|f\| := \left(\int_0^a |f(x)| d_q x\right)^{1/2} < \infty. \tag{1.5}$$

The space $L_q^2(0,a)$ is a separable Hilbert space with the inner product

$$(f,g) := \int_0^a f(x)\overline{g(x)}d_qx, \quad f,g \in L_q^2(0,a). \tag{1.6}$$

We will consider the basic Sturm-Liouville equation

$$l(y) := -\frac{1}{q}D_{q^{-1}}D_qy(x) + v(x)y(x), \quad 0 \le x \le a < +\infty, \tag{1.7}$$

where $v(x)$ is defined on $[0,a]$ and continuous at zero. The q-Wronskian of $y_1(x)$, $y_2(x)$ is defined to be

$$W_q(y_1,y_2)(x) := y_1(x)D_qy_2(x) - y_2(x)D_qy_1(x), \quad x \in [0,a]. \tag{1.8}$$

Let L_0 denote the closure of the minimal operator generated by (1.7) and by D_0 its domain. Besides, we denote by D the set of all functions $y(x)$ from $L_q^2(0,a)$ such that $y(x)$ and $D_qy(x)$ are continuous in $[0,a)$ and $l(y) \in L_q^2(0,a)$; D is the domain of the maximal operator L. Furthermore, $L = L_0^*$ [2, 4, 13]. Suppose that the operator L_0 has defect index $(2,2)$.

For every $y,z \in D$ we have q-Lagrange's identity [24]

$$(Ly,z) - (y,Lz) = [y,\overline{z}](a) - [y,\overline{z}](0), \tag{1.9}$$

where $[y,\overline{z}] := y(x)\overline{D_{q^{-1}}z(x)} - D_{q^{-1}}y(x)\overline{z(x)}$.

2. Construction of the Dissipative Operator

Consider boundary value problem governed by

$$(ly) = \lambda y, \quad y \in D, \tag{2.1}$$

subject to the boundary conditions

$$y(a) - hD_{q^{-1}}y(a) = 0, \quad \operatorname{Im} h > 0, \tag{2.2}$$

$$\alpha_1 y(0) - \alpha_2 D_{q^{-1}}y(0) = \lambda\left(\alpha_1' y(0) - \alpha_2' D_{q^{-1}}y(0)\right), \tag{2.3}$$

where λ is spectral parameter and $\alpha_1, \alpha_2, \alpha_1', \alpha_2' \in \mathbb{R}$ and α is defined by

$$\alpha := \begin{vmatrix} \alpha_1' & \alpha_1 \\ \alpha_2' & \alpha_2 \end{vmatrix} = \alpha_1'\alpha_2 - \alpha_1\alpha_2' > 0. \tag{2.4}$$

For convenience we assume

$$\begin{aligned} R_0(y) &:= \alpha_1 y(0) - \alpha_2 D_{q^{-1}} y(0), \\ R_0'(y) &:= \alpha_1' y(0) - \alpha_2' D_{q^{-1}} y(0), \\ N_1^a(y) &:= y(a), \\ N_2^a(y) &:= D_{q^{-1}} y(a), \\ N_1^0(y) &:= y(0), \\ N_2^0(y) &:= D_{q^{-1}} y(0)_\infty, \\ R_a(y) &:= N_2^a(y) - h N_1^a(y). \end{aligned} \tag{2.5}$$

Lemma 2.1. *For arbitrary $y, z \in D$, let one suppose that $R_0(\overline{z}) = \overline{R_0(z)}, R_0'(\overline{z}) = \overline{R_0'(z)}$, then one has the following.*

Proof.

$$[y, z]_0 = \frac{1}{\alpha}\left[R_0(y)\overline{R_0'(z)} - R_0'(y)\overline{R_0(z)}\right], \tag{2.6}$$

$$\begin{aligned} &\frac{1}{\alpha}\left[R_0(y)R_0'\overline{(z)} - R_0'(y)\overline{R_0(z)}\right] \\ &\quad = \frac{1}{\alpha}\begin{bmatrix} (\alpha_1 y(0) - \alpha_2 D_{q^{-1}} y(0))\left(\overline{\alpha_1' z(0) - \alpha_2' D_{q^{-1}} z(0)}\right) \\ -(\alpha_1' y(0) - \alpha_2' D_{q^{-1}} y(0))\left(\overline{\alpha_1 z(0) - \alpha_2 D_{q^{-1}} z(0)}\right) \end{bmatrix} \\ &\quad = \frac{1}{\alpha}\left[(\alpha_1'\alpha_2 - \alpha_1\alpha_2')\left(\overline{y(0)} D_{q^{-1}} z(0) - D_{q^{-1}} y(0)\overline{z(0)}\right)\right] \\ &\quad = [y, z]_0. \end{aligned} \tag{2.7}$$

Let θ_1, θ_2 denote the solutions of (2.1) satisfying the conditions

$$N_1^0(\theta_2) = \alpha_2 - \alpha_2'\lambda, \qquad N_2^0(\theta_2) = \alpha_1 - \alpha_1'\lambda, \qquad N_1^a(\theta_1) = h, \qquad N_2^a(\theta_1) = 1. \tag{2.8}$$

Then from (2.3) we have

$$\begin{aligned} \Delta(\lambda) &= [\theta_1, \theta_2]_x = -[\theta_2, \theta_1]_x = -[\theta_2, \theta_1]_0 \\ &= -\frac{1}{\alpha}\left[R_0(\theta_1)\overline{R_0'(\theta_2)} - R_0'(\theta_1)\overline{R_0(\theta_2)}\right] \\ &= R_0(\theta_2) - \lambda R_0'(\theta_2), \end{aligned}$$

$$\begin{aligned}\Delta(\lambda) &= [\theta_1,\theta_2]_x = -[\theta_2,\theta_1]_x = -[\theta_2,\theta_1]_a \\ &= -\left(y(a)\overline{D_{q^{-1}}z(a)} - z(a)\overline{D_{q^{-1}}y(a)}\right) \\ &= -\left(y(a) - h\overline{D_{q^{-1}}y(a)}\right) = -\left(N_2^a(\theta_1) - hN_1^a(\theta_1)\right).\end{aligned} \tag{2.9}$$

We let

$$G(x,\xi,\lambda) = \frac{-1}{\Delta(\lambda)}\begin{Bmatrix}\theta_2(\xi,\lambda)\theta_1(x,\lambda), & x<\xi \\ \theta_1(x,\lambda)\theta_2(\xi,\lambda), & \xi<x\end{Bmatrix}. \tag{2.10}$$

It can be shown that $G(x,\xi,\lambda)$ satisfies (2.1) and boundary conditions (2.2)–(2.3). $G(x,\xi,\lambda)$ is a Green function of the boundary value problem (2.1)–(2.3). Thus, we obtain that the $G(x,\xi,\lambda)$ is a Hilbert-Schmidt kernel and the solution of the boundary value problem can be expressed by

$$y(x,\lambda) = \int_0^a G(x,\xi,\lambda)y(\xi,\lambda)d\xi = R_\lambda y. \tag{2.11}$$

Thus R_λ is a Hilbert Schmidt operator on space $L_q^2(0,a)$. The spectrum of the boundary value problem coincides with the roots of the equation $\Delta(\lambda) = 0$. Since Δ is analytic and not identical to zero, it means that the function Δ has at most a countable number of isolated zeros with finite multiplicity and possible limit points at infinity. □

Suppose that $f^{(1)} \in L_2[0,a)$, $f^{(2)} \in \mathbb{C}$, then we denote linear space $H = L_q^2(0,a) \oplus \mathbb{C}$ with two component of elements of $\widehat{f} = \binom{f^{(1)}}{f^{(2)}}$. If $\alpha > 0$ and $\widehat{f} = \binom{f^{(1)}}{f^{(2)}}$, $\widehat{g} = \binom{g^{(1)}}{g^{(2)}} \in H$, then the formula

$$\left(\widehat{f},\widehat{g}\right) = \int_0^a f^{(1)}(x)\overline{g}^{(1)}d_qx + \frac{1}{\alpha}f^{(2)}\overline{g}^{(2)} \tag{2.12}$$

defines an inner product in Hilbert space H. Let us define operator of $A_h : H \to H$ with equalities suitable for boundary value problem

$$\begin{gathered}D(A_h) = \left\{\widehat{f} = \begin{pmatrix}f^{(1)}\\ f^{(2)}\end{pmatrix} \in H : f^{(1)} \in D, R_a\left(f^{(1)}\right) = 0, f^{(2)} = R_0'\left(f^{(1)}\right)\right\}, \\ A_h\widehat{f} = \widetilde{l}\left(\widehat{f}\right) := \begin{pmatrix}l\left(f^{(1)}\right)\\ R_0\left(f^{(1)}\right)\end{pmatrix}.\end{gathered} \tag{2.13}$$

Remind that a linear operator A_h with domain $D(A_h)$ in Hilbert space H is called dissipative if $\operatorname{Im}(A_hf,f) \geq 0$ for all $f \in D(A_h)$ and maximal dissipative if it does not have a proper extension.

Definition 2.2. If the system of vectors of $y_0, y_1, y_2, \ldots, y_n$ corresponding to the eigenvalue λ_0 is

$$\begin{gathered} l(y_0) = \lambda_0 y_0, \qquad R_0(y_0) - \lambda R_0'(y_0) = 0, \qquad R_a(y_0) = 0, \\ l(y_s) - \lambda_0 y_s - y_{s-1} = 0, \qquad R_0(y_s) - \lambda R_0'(y_s) - R_0'(y_{s-1}) = 0, \\ R_a(y_s) = 0, \qquad s = 1, 2, \ldots, n, \end{gathered} \tag{2.14}$$

then the system of vectors of $y_0, y_1, y_2, \ldots, y_n$ corresponding to the eigenvalue λ_0 is called a chain of eigenvectors and associated vectors of boundary value problem (2.2)–(2.12).

Since the operator A_h is dissipative in H and from Definition 2.2, we have the following.

Lemma 2.3. *The eigenvalue of boundary value problem* (2.1)–(2.3) *coincides with the eigenvalue of dissipative A_h operator. Additionally each chain of eigenvectors and associated vectors $y_0, y_1, y_2, \ldots, y_n$ corresponding to the eigenvalue λ_0 corresponds to the chain eigenvectors and associated vectors $\widehat{y}_0, \widehat{y}_1, \widehat{y}_2, \ldots, \widehat{y}_n$ corresponding to the same eigenvalue λ_0 of dissipative A_h operator. In this case, the equality*

$$\widehat{y}_k = \begin{pmatrix} y_k \\ R_0'(y_k) \end{pmatrix}, \quad k = 0, 1, 2, \ldots, n \tag{2.15}$$

holds.

Proof. $\widehat{y}_0 \in D(A_h)$ and $A_h \widehat{y}_0 = \lambda_0 \widehat{y}_0$, then the equality $l(y_0) = \lambda_0 y_0$, $R_0(y_0) - \lambda R_0'(y_0) = 0$, $R_1(y_0) = R_2(y_0) = 0$ takes place; that is, y_0 is an eigenfunction of the problem. Conversely, if conditions (2.14) are realized, then $\left(\begin{smallmatrix} y_0 \\ R_0'(y_0) \end{smallmatrix}\right) = \widehat{y}_0 \in D(A_h)$ and $A_h \widehat{y}_0 = \lambda_0 \widehat{y}_0$, $\widehat{y}_0$ is an eigenvector of the operator A_h. If $\widehat{y}_0, \widehat{y}_1, \widehat{y}_2, \ldots, \widehat{y}_n$ are a chain of the eigenvectors and associated vectors of the operator A_h corresponding to the eigenvalue λ_0, then by implementing the conditions $\widehat{y}_k \in D(A_h)(k = 0, 1, 2, \ldots, n)$ and equality $A_h \widehat{y}_0 = \lambda_0 \widehat{y}_0$, $A_h \widehat{y}_s = \lambda_0 \widehat{y}_s + \widehat{y}_{s-1}$, $s = 1, 2, \ldots, n$, we get the equality (2.15), where $y_0, y_1, y_2, \ldots, y_n$ are the first components of the vectors $\widehat{y}_0, \widehat{y}_1, \widehat{y}_2, \ldots, \widehat{y}_n$. On the contrary, on the basis of the elements $y_0, y_1, y_2, \ldots, y_n$ corresponding to (2.1)–(2.3), one can construct the vectors $\widehat{y}_k = \left(\begin{smallmatrix} y_k \\ R_0'(y_k) \end{smallmatrix}\right)$ for which $\widehat{y}_k \in D(A_h)(k = 0, 1, 2, \ldots, n)$ and $A_h \widehat{y}_0 = \lambda_0 \widehat{y}_0$, $A_h \widehat{y}_s = \lambda_0 \widehat{y}_s + \widehat{y}_{s-1}$, $s = 1, 2, \ldots, n$. □

Theorem 2.4. *The operator A_h is maximal dissipative in the space H.*

Proof. Let $\widehat{y} \in D(A_h)$. From (2.6), we have

$$\begin{aligned} (A_h \widehat{y}, \widehat{y}) - (\widehat{y}, A_h \widehat{y}) &= [y_1, y_1]_a - [y_1, \overline{y_1}]_0 + \frac{1}{\alpha}\left[R_0(y_1) R_0'\overline{(y_1)} - R_0'(y_1)\overline{R_0(y_1)}\right] \\ &= [y_1, y_1]_a = 2 \operatorname{Im} h\left(D_{q^{-1}} y_1(a)\right)^2. \end{aligned} \tag{2.16}$$

It follows from that $\operatorname{Im}(A_h\widehat{y},\widehat{y}) = \operatorname{Im} h(D_{q^{-1}}y_1(a))^2 \geq 0$, A_h is a dissipative operator in H. Let us prove that A_h is maximal dissipative operator in the space H. It is sufficient to check that

$$(A_h - \lambda I)D(A_h) = H, \qquad \operatorname{Im}\lambda < 0. \tag{2.17}$$

To prove (2.17), let $F \in H$, $\operatorname{Im}\lambda < 0$ and put

$$\Gamma = \begin{pmatrix} \left(\widetilde{G}_x, \overline{F}\right) \\ R_0'\left[\left(\widetilde{G}_x, \overline{F}\right)\right] \end{pmatrix}, \tag{2.18}$$

where

$$\begin{gathered} \widetilde{G}_x = \begin{pmatrix} G(x,\xi,\lambda) \\ R_0'[G(x,\xi,\lambda)] \end{pmatrix} = \begin{pmatrix} G(x,\xi,\lambda) \\ \dfrac{-1}{\Delta(\lambda)}\theta_1(x,\lambda)\alpha \end{pmatrix}, \\ G(x,\xi,\lambda) = \frac{-1}{\Delta(\lambda)}\begin{Bmatrix} \theta_2(\xi,\lambda)\theta_1(x,\lambda), & x < \xi \\ \theta_1(x,\lambda)\theta_2(\xi,\lambda), & \xi < x \end{Bmatrix}. \end{gathered} \tag{2.19}$$

The function $x \to (G(x,\xi,\lambda), \overline{F_1})$ satisfies the equation $l(y) - \lambda y = F_1 (0 \leq x < \infty)$ and the boundary conditions (2.1)–(2.3). Moreover, for all $F \in H$ and for $\operatorname{Im}\lambda < 0$, we arrive at $\Gamma \in D(A_h)$. For each $F \in H$ and for $\operatorname{Im}\lambda < 0$, we have $(A_h - \lambda I)\Gamma = F$. Consequently, in the case of $\operatorname{Im}\lambda < 0$, the result is $(A_h - \lambda I)D(A_h) = H$. Hence, Theorem 2.4 is proved. □

3. Self-Adjoint Dilation of Dissipative Operator

We first construct the self-adjoint dilation of the operator A_h. Let us add the "incoming" and "outgoing" subspaces $D_- = L^2(-\infty, 0]$ and $D_+ = L^2[0,\infty)$ to $H = L_q^2(0,a) \oplus \mathbb{C}$. The orthogonal sum $\mathrm{H} = D_- \oplus H \oplus D_+$ is called *main Hilbert space of the dilation*. In the space $\mathscr{H}$ we consider the operator $\mathcal{L}_h$ on the set $D(\mathcal{L}_h)$, its elements consisting of vectors $w = \langle \varphi_-, y, \varphi_+ \rangle$, generated by the expression

$$\mathcal{L}\langle \varphi_-, \widehat{y}, \varphi_+ \rangle = \left\langle i\frac{d\varphi_-}{d\xi}, \widetilde{l}(\widehat{y}), i\frac{d\varphi_+}{d\xi} \right\rangle. \tag{3.1}$$

satisfying the conditions: $\varphi_- \in W_2^1(-\infty, 0]$, $\varphi_+ \in W_2^1[0,\infty)$, $\widehat{y} \in H$, $\widehat{y} = \left(\begin{smallmatrix} y_1(x) \\ y_2 \end{smallmatrix}\right)$, $y_1 \in D$, $y_2 = R_0(y_1)$, and

$$y(a) - hD_{q^{-1}}y(a) = \beta\varphi_-(0), \qquad y(a) - \overline{h}D_{q^{-1}}y(a) = \beta\varphi_+(0), \tag{3.2}$$

where $W_2^1(\cdot,\cdot)$ are Sobolev spaces and $\beta^2 := 2\operatorname{Im} h$, $\beta > 0$. Then we have the following.

Theorem 3.1. *The operator $\mathcal{L}_h$ is self-adjoint in $\mathscr{H}$ and it is a self-adjoint dilation of the operator A_h.*

Proof. We first prove that $\mathcal{L}_h$ is symmetric in $\mathcal{H}$. Namely $(\mathcal{L}_h f, g)_{\mathcal{H}} - (f, \mathcal{L}_h g)_{\mathcal{H}} = 0$. Let $f, g \in D(\mathcal{L}_h)$, $f = \langle \varphi_-, \widehat{y}, \varphi_+ \rangle$ and $g = \langle \psi_-, \widehat{z}, \psi_+ \rangle$. Then we have

$$\begin{aligned}
(\mathcal{L}_h f, g)_{\mathcal{H}} - (f, \mathcal{L}_h g)_{\mathcal{H}} &= (\mathcal{L}\langle \varphi_-, \widehat{y}, \varphi_+ \rangle, \langle \psi_-, \widehat{z}, \psi_+ \rangle) - (\langle \varphi_-, \widehat{y}, \varphi_+ \rangle, \mathcal{L}\langle \psi_-, \widehat{z}, \psi_+ \rangle), \\
&= [y_1, \overline{z_1}]_a - [y_1, \overline{z_1}]_0 + \frac{1}{\alpha}\left[R_0(y_1)\overline{R_0'(z_1)} - R_0'(y_1)\overline{R_0(z_1)}\right] \\
&\quad + i\psi_-(0)\overline{\varphi}_-(0) - i\varphi_+(0)\overline{\varphi}_+(0), \\
(\mathcal{L}_h f, g)_{\mathcal{H}} - (f, \mathcal{L}_h g)_{\mathcal{H}} &= [y_1, \overline{z_1}]_a + i\psi_-(0)\overline{\varphi}_-(0) - i\varphi_+(0)\overline{\varphi}_+(0).
\end{aligned} \tag{3.3}$$

On the other hand,

$$\begin{aligned}
i\psi_-(0)\overline{\varphi}_-(0) - i\varphi_+(0)\overline{\varphi}_+(0) &= \frac{i}{\beta^2}\left(y(a) - hD_{q^{-1}}y(a)\right)\overline{\left(z(a) - hD_{q^{-1}}z(a)\right)} \\
&\quad - \frac{i}{\beta^2}\left(y(a) - \overline{h}D_{q^{-1}}y(a)\right)\overline{\left(z(a) - \overline{h}\ D_{q^{-1}}z(a)\right)}, \\
&= \frac{i}{\beta^2}\left[\left(h - \overline{h}\right)y(a)\overline{D_{q^{-1}}z(a)} - D_{q^{-1}}y(a)\overline{z(a)}\right].
\end{aligned} \tag{3.4}$$

By (3.3), we have

$$i\psi_-(0)\overline{\varphi}_-(0) - i\varphi_+(0)\overline{\varphi}_+(0) = -[y_1, \overline{z_1}]_a. \tag{3.5}$$

From equalities (3.3) and (3.5), we have $(\mathcal{L}_h f, g)_{\mathcal{H}} - (f, \mathcal{L}_h g)_{\mathcal{H}} = 0$. Thus, $\mathcal{L}_h$ is a symmetric operator. To prove that $\mathcal{L}_h$ is self-adjoint, we need to show that $\mathcal{L}_h \subseteq \mathcal{L}_h^*$. We consider the bilinear form $(\mathcal{L}_h f, g)_{\mathcal{H}}$ on elements $g = \langle \psi_-, \widehat{z}, \psi_+ \rangle \in D(\mathcal{L}_h^*)$, where $f = \langle \varphi_-, \widehat{y}, \varphi_+ \rangle \in D(\mathcal{L}_h)$, $\varphi_\mp \in W_2^1(\mathbb{R}_\mp)$,$\varphi_\mp(0) = 0$. Integrating by parts, we get $\mathcal{L}_h^* g = \langle i(d\psi_-/d\xi), \widehat{z}^*, i(d\psi_+/d\xi) \rangle$, where $\psi_\mp \in W_2^1(\mathbb{R}_\mp)$, $\widehat{z}^* \in H$. Similarly, if $f = \langle 0, \widehat{y}, 0 \rangle \in D(\mathcal{L}_h)$, then integrating by parts in $(\mathcal{L}_h f, g)_{\mathcal{H}}$, we obtain

$$\mathcal{L}_h^* g = \mathcal{L}^*\langle \psi_-, \widehat{z}, \psi_+ \rangle = \left\langle i\frac{d\psi_-}{d\xi}, \widetilde{l}(\widehat{z}), i\frac{d\psi_+}{d\xi} \right\rangle, \quad z_1 \in D, z_2 = R_0'(z_1). \tag{3.6}$$

Consequently, we have $(\mathcal{L}_h f, g)_{\mathcal{H}} = (f, \mathcal{L}_h g)_{\mathcal{H}}$, for each $f \in D(\mathcal{L}_h)$ by (3.6), where the operator $\mathcal{L}$ is defined by (3.1). Therefore, the sum of the integrated terms in the bilinear form $(\mathcal{L}_h f, g)_{\mathcal{H}}$ must be equal to zero:

$$[y_1, \overline{z_1}]_a - [y_1, \overline{z_1}]_0 + \frac{1}{\alpha}\left[R_0(y_1)\overline{R_0'(z_1)} - R_0'(y_1)\overline{R_0(z_1)}\right] + i\varphi_-(0)'\overline{\varphi}_-(0) - i\varphi'_+(0)\overline{\varphi}_+(0) = 0. \tag{3.7}$$

Then by (2.6), we get

$$[y_1, \overline{z_1}]_a + i\varphi_-(0)'\overline{\varphi}_-(0) - i\varphi'_+(0)\overline{\varphi}_+(0) = 0. \tag{3.8}$$

From the boundary conditions for $\mathcal{L}_h$, we have

$$y(a) = \beta\varphi_-(0) - \frac{h_1}{i\beta}\left(\varphi_-(0) - \varphi_+(0)\right),$$
$$D_{q^{-1}}y(a) = \frac{i}{\beta}\left(\varphi_-(0) - \varphi_+(0)\right). \tag{3.9}$$

Afterwards, by (3.8) we get

$$\beta\varphi_-(0) - \frac{h_1}{i\beta}\left(\varphi_-(0) - \varphi_+(0)\right)\overline{z(a)} - \frac{i}{\beta}\left(\varphi_-(0) - \varphi_+(0)\right)\overline{D_{q^{-1}}z(a)}$$
$$= i\varphi_+(0)\overline{\varphi}_+(0) - i\varphi_-(0)\overline{\varphi}_-(0). \tag{3.10}$$

Comparing the coefficients of $\varphi_-(0)$ in (3.10), we obtain

$$\frac{i\beta^2 - h_1}{\beta}z(a) + \frac{1}{\beta}\overline{D_{q^{-1}}z(a)} = \varphi_-(0) \tag{3.11}$$

or

$$z(a) - hD_{q^{-1}}z(a) = \beta\psi_-(0). \tag{3.12}$$

Similarly, comparing the coefficients of $\varphi_+(0)$ in (3.10) we get

$$z(a) - \overline{h}D_{q^{-1}}z(a) = \beta\psi_+(0). \tag{3.13}$$

Therefore conditions (3.12) and (3.13) imply $D(\mathcal{L}_h^*) \subseteq D(\mathcal{L}_h)$, hence $\mathcal{L}_h = \mathcal{L}_h^*$.

The self-adjoint operator $\mathcal{L}_h$ generates on $\mathcal{H}$ a unitary group $U_t = \exp(i\mathcal{L}_h t)$ $(t \in \mathbb{R}_+ = (0,\infty))$. Let us denote by $P : \mathcal{H} \to H$ and $P_1 : H \to \mathcal{H}$ the mapping acting according to the formulae $P : \langle\varphi_-, \widehat{y}, \varphi_+\rangle \to \widehat{y}$ and $P_1 : \widehat{y} \to \langle 0, \widehat{y}, 0\rangle$. Let $Z_t := PU_tP_1, t \geq 0$, by using U_t. The family $\{Z_t\}(t \geq 0)$ of operators is a strongly continuous semigroup of completely nonunitary contraction on H. Let us denote by B_h the generator of this semigroup: $B_h\widehat{y} = \lim_{t\to+0}(it)^{-1}(Z_t\widehat{y} - \widehat{y})$. The domain of B_h consists of all the vectors for which the limit exists. The operator B_h is dissipative. The operator $\mathcal{L}_h$ is called the self-adjoint dilation of B_h (see [2, 9, 18]). We show that $B_h = A_h$, hence $\mathcal{L}_h$ is self-adjoint dilation of B_h. To show this, it is sufficient to verify the equality

$$P(\mathcal{L}_h - \lambda I)^{-1}P_1\widehat{y} = (A_h - \lambda I)^{-1}\widehat{y}, \quad \widehat{y} \in H, \operatorname{Im} h < 0. \tag{3.14}$$

For this purpose, we set $(\mathcal{L}_h - \lambda I)^{-1}P_1\widehat{y} = g = \langle\psi_-, \widehat{z}, \psi_+\rangle$ which implies that $(\mathcal{L}_h - \lambda I)g = P_1\widehat{y}$, and hence $\widetilde{l}(\widehat{z}) - \lambda\widehat{z} = \widehat{y}, \psi_-(\xi) = \psi_-(0)e^{-i\lambda\xi}$ and $\psi_+(\xi) = \psi_+(0)e^{-i\lambda\xi}$. Since $g \in D(\mathcal{L}_h)$, then $\psi_- \in L^2(-\infty, 0)$, and it follows that $\psi_-(0) = 0$, and consequently z satisfies the boundary condition $z(a) - hD_{q^{-1}}z(a) = 0$. Therefore, $\widehat{z} \in D(A_h)$, and since point λ with $\operatorname{Im}\lambda < 0$ cannot

be an eigenvalue of dissipative operator, it follows that $\varphi_+(0)$ is obtained from the formula $\varphi_+(0) = \beta^{-1}(z(a) - \overline{h}D_{q^{-1}}z(a))$. Thus

$$(\mathcal{L}_h - \lambda I)^{-1}P_1\widehat{y} = \left\langle 0, (A_h - \lambda I)^{-1}\widehat{y}, \beta^{-1}\left(z(a) - \overline{h}D_{q^{-1}}z(a)\right)\right\rangle \tag{3.15}$$

for $\widehat{y}$ and $\operatorname{Im}\lambda < 0$. On applying the mapping P, we obtain (3.14), and

$$\begin{aligned}(A_h - \lambda I)^{-1} &= P(\mathcal{L}_h - \lambda I)^{-1}P_1 = -iP\int_0^\infty U_t e^{-i\lambda t}dtP_1 \\ &= -i\int^\infty Z_t e^{-i\lambda t}dt = (B_h - \lambda I)^{-1}, \quad \operatorname{Im}\lambda < 0,\end{aligned} \tag{3.16}$$

so this clearly shows that $A_h = B_h$. □

The unitary group $\{U_t\}$ has an important property which makes it possible to apply it to the Lax-Phillips [27], that is, it has orthogonal incoming and outcoming subspaces $D_- = \langle L^2(-\infty, 0), 0, 0\rangle$ and $D_+ = \langle 0, 0, L^2(0,\infty)\rangle$ having the following properties:

(1) $U_tD_- \subset D_-$, $t \le 0$ and $U_tD_+ \subset D_+$, $t \ge 0$;

(2) $\cap_{t\le 0}U_tD_- = \cap_{t\ge 0}U_tD_+ = \{0\}$;

(3) $\cup_{t\ge 0}U_tD_- = \cup_{t\le 0}U_tD_+ = \mathcal{H}$;

(4) $D_- \perp D_+$.

To be able to prove property (1) for D_+ (the proof for D_- is similar), we set $\mathcal{R}_\lambda = (\mathcal{L}_h - \lambda I)^{-1}$. For all λ, with $\operatorname{Im}\lambda < 0$ and for any $f = \langle 0, 0, \varphi_+\rangle \in D_+$, we have

$$\mathcal{R}_\lambda f = \left\langle 0, 0, -ie^{-i\lambda\xi}\int_0^\xi e^{i\lambda s}\varphi_+(s)ds\right\rangle, \tag{3.17}$$

as $\mathcal{R}_\lambda f \in D_+$. Therefore, if $g \perp D_+$, then

$$0 = (\mathcal{R}_\lambda f, g)_{\mathcal{H}} = -i\int_0^\infty e^{-i\lambda t}(U_t f, g)_{\mathcal{H}}dt, \quad \operatorname{Im}\lambda < 0 \tag{3.18}$$

which implies that $(U_t f, g)_{\mathcal{H}} = 0$ for all $t \ge 0$. Hence, for $t \ge 0$, $U_tD_+ \subset D_+$, and property (1) has been proved.

In order to prove property (2), we define the mappings $P^+ : \mathcal{H} \to L^2(0,\infty)$ and $P_1^+ : L^2(0,\infty) \to D_+$ as follows: $P^+ : \langle \varphi_-, \widehat{y}, \varphi_+\rangle \to \varphi_+$ and $P_1^+ : \varphi \to \langle 0, 0, \varphi\rangle$, respectively. We take into consideration that the semigroup of isometries $U_t^+ := P^+U_tP_1^+ (t \ge 0)$ is a one-sided shift in $L^2(0,\infty)$. Indeed, the generator of the semigroup of the one-sided shift V_t in $L^2(0,\infty)$ is the differential operator $i(d/d\xi)$ with the boundary condition $\varphi(0) = 0$. On the other hand, the generator S of the semigroup of isometries $U_t^+ (t \ge 0)$ is the operator $S\varphi = P^+\mathcal{L}_hP_1^+\varphi =$

$P^+\mathcal{L}_h\langle 0,0,\varphi\rangle = P^+\langle 0,0,i(d/d\xi)\varphi\rangle = i(d/d\xi)\varphi$, where $\varphi \in W_2^1(0,\infty)$ and $\varphi(0) = 0$. Since a semigroup is uniquely determined by its generator, it follows that $U_t^+ = V_t$, and hence

$$\bigcap_{t\geq 0} U_t D_+ = \left\langle 0,0,\bigcap_{t\leq 0} V_t L^2(0,\infty)\right\rangle = \{0\}, \tag{3.19}$$

so, the proof of property (2) is completed.

Definition 3.2. The linear operator A with domain $D(A)$ acting in the Hilbert space H is called *completely non-self-adjoint (or simple)* if there is no invariant subspace $M \subseteq D(A)(M \neq \{0\})$ of the operator A on which the restriction A to M is self-adjoint.

To prove property (3) of the incoming and outcoming subspaces, let us prove following lemma.

Lemma 3.3. *The operator A_h is completely non-self-adjoint (simple).*

Proof. Let $H' \subset H$ be a nontrivial subspace in which A_h induces a self-adjoin t operator A'_h with domain $D(A'_h) = H' \cap D(A_h)$. If $\widehat{f} \in D(A'_h)$, then $\widehat{f} \in D(A'^*_h)$ and

$$\begin{aligned}\frac{d}{dt}\left\|e^{iA'_h t}\widehat{f}\right\|_H^2 &= \frac{d}{dt}\left(e^{iA'_h t}\widehat{f}, e^{iA'_h t}\widehat{f}\right)_H \\ &= i\left(A'_h e^{iA'_h t}\widehat{f}, e^{iA'_h t}\widehat{f}\right) - i\left(e^{iA'_h t}\widehat{f}, A'_h e^{iA'_h t}\widehat{f}\right)\end{aligned} \tag{3.20}$$

and taking $\widehat{g} = e^{iA'_h t}\widehat{f}$, we have

$$\begin{aligned}0 &= i\left(A'_h\widehat{g},\widehat{g}\right)_H - i\left(\widehat{g}, A'_h\widehat{g}\right)_H \\ &= i\left[g_1,\overline{g_1}\right]_a - i\left[g_1,\overline{g_1}\right]_0 + \frac{i}{\alpha}\left[R_0(g_1)\overline{R'_0(g_1)} - R'_0(y_1)\overline{R_0(g_1)}\right] \\ &= -2\operatorname{Im} h\left(D_{q^{-1}}y_1(a)\right)^2 \\ &= -\beta^2\left(D_{q^{-1}}y_1(a)\right)^2.\end{aligned} \tag{3.21}$$

Since $\widehat{f} \in D(A'_h)$, A'_h holds condition above. Moreover, eigenvectors of the operator A'_h should also hold this condition. Therefore, for the eigenvectors $\widehat{y}(\lambda)$ of the operator A_h acting in H' and the eigenvectors of the operator A'_h, we have $D_{q^{-1}}y_1(a) = 0$. From the boundary conditions, we get $y_1(a) = 0$ and $\widehat{y}(x,\lambda) = 0$. Consequently, by the theorem on expansion in the eigenvectors of the self-adjoint operator A'_h, we obtain $H' = \{0\}$. Hence the operator A_h is simple. The proof is completed. □

Let us define $H_- = \overline{\cup_{t\geq 0} U_t D_-}$, $H_+ = \overline{\cup_{t\leq 0} U_t D_+}$.

Lemma 3.4. *The equality $H_- + H_+ = \mathcal{H}$ holds.*

Proof. Considering property (1) of the subspace D_+, it is easy to show that the subspace $\mathcal{H}' = \mathcal{H} \ominus (H_- + H_+)$ is invariant relative to the group $\{U_t\}$ and has the form $\mathcal{H}' = \langle 0, H', 0\rangle$, where H' is a subspace in H. Therefore, if the subspace $\mathcal{H}'$ (and hence also H') was nontrivial, then the unitary group $\{U_t\}$ restricted to this subspace would be a unitary part of the group $\{U'_t\}$, and hence the restriction B'_h of B_h to H' would be a self-adjoint operator in H'. Since the operator B_h is simple, it follows that $H' = \{0\}$. The lemma is proved. □

Assume that $\varphi(\lambda)$ and $\psi(\lambda)$ are solutions of $l(y) = \lambda y$ satisfying the conditions

$$\begin{gathered}\varphi_1(0,\lambda) = 0, \qquad \varphi_2(0,\lambda) = 1, \qquad \psi_1(0,\lambda) = 1, \qquad \psi_2(0,\lambda) = 0. \\ \theta(x,\lambda) = \varphi(x,\lambda) + m_a(\lambda)\psi(x,\lambda) \in L^2_q(0,a), \quad \operatorname{Im}\lambda > 0.\end{gathered} \tag{3.22}$$

The Titchmarsh-Weyl function $m_a(\lambda)$ is a meromorphic function on the complex plane $\mathbb{C}$ with a countable number of poles on the real axis. Further, it is possible to show that the function $m_a(\lambda)$ possesses the following properties: $\operatorname{Im} m_a(\lambda) \geq 0$ for all $\operatorname{Im}\lambda > 0$, and $\overline{m_a(\lambda)} = m_a(\overline{\lambda})$ for all $\lambda \in \mathbb{C}$, except the real poles $m_a(\lambda)$. We set

$$S_h(\lambda) := \frac{m_a(\lambda) - h}{m_a(\lambda) - \overline{h}}, \tag{3.23}$$

$$U^-_\lambda(x,\xi,\zeta) = \langle e^{-i\lambda\xi}, (m_a(\lambda) - h)^{-1}\alpha\theta(x,\lambda), \overline{S_h}(\lambda)e^{-i\lambda\zeta}\rangle. \tag{3.24}$$

We note that the vectors $U^-_\lambda(x,\xi,\zeta)$ for real λ do not belong to the space $\mathcal{H}$. However, $U^-_\lambda(x,\xi,\zeta)$ satisfies the equation $\mathcal{L}U = \lambda U$ and the corresponding boundary conditions for the operator $\mathcal{L}_H$. By means of vector $U^-_\lambda(x,\xi,\zeta)$, we define the transformation $F_- : f \to \widetilde{f}_-(\lambda)$ by

$$(F - f)(\lambda) := \widetilde{f}_-(\lambda) := \frac{1}{\sqrt{2\pi}}(f, U^-_{\overline{\lambda}})_{\mathcal{H}}, \tag{3.25}$$

on the vectors $f = \langle \varphi_-, \widehat{y}, \varphi_+\rangle$ in which $\varphi_-(\xi)$, $\varphi_+(\zeta)$, $y(x)$ are smooth, compactly supported functions.

Lemma 3.5. *The transformation F_- isometrically maps H_- onto $L^2(\mathbb{R})$. For all vectors $f, g \in H_-$ the Parseval equality and the inversion formulae hold:*

$$(f,g)_{\mathcal{H}} = \left(\widetilde{f}_-, \widetilde{g}_-\right)_{L^2} = \int_{-\infty}^{\infty} \widetilde{f}_-(\lambda)\overline{\widetilde{g}_-(\lambda)}d\lambda, \qquad f = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \widetilde{f}_-(\lambda)U^-_{\overline{\lambda}}d\lambda, \tag{3.26}$$

where $\widetilde{f}_-(\lambda) = (F_- f)(\lambda)$ and $\widetilde{g}_-(\lambda) = (F_- g)(\lambda)$.

Proof. For $f, g \in D_-$, $f = \langle \varphi_-, 0, 0\rangle$, $g = \langle \psi_+, 0, 0\rangle$, with Paley-Wiener theorem, we have

$$\widetilde{f}_-(\lambda) = \frac{1}{\sqrt{2\pi}}(f, U^-_{\overline{\lambda}})_{\mathcal{H}} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} \varphi_-(\xi)e^{-i\lambda\xi}d\xi \in H^2_-, \tag{3.27}$$

and by using usual Parseval equality for Fourier integrals

$$(f,g)_{\mathscr{H}} = \int_{-\infty}^{\infty} \varphi_-(\xi)\overline{\varphi_-(\xi)}d\xi = \int_{-\infty}^{\infty} \widetilde{f}_-(\lambda)\overline{\widetilde{g}_-(\lambda)}d\lambda = (F_-f, F_-g)_{L^2}. \tag{3.28}$$

Here, $H_\pm^2$ denote the Hardy classes in $L^2(\mathbb{R})$ consisting of the functions analytically extendible to the upper and lower half-planes, respectively.

We now extend to the Parseval equality to the whole of H_-. We consider in H_- the dense set of H'_- of the vectors obtained as follows from the smooth, compactly supported functions in D_- : $f \in H'_-$ if $f = U_T f_0$, $f_0 = \langle \varphi_-, 0, 0\rangle$, $\varphi_- \in C_0^\infty(-\infty, 0)$, where $T = T_f$ is a nonnegative number depending on f. If $f, g \in H'_-$, then for $T > T_f$ and $T > T_g$ we have $U_{-T}f, U_{-T}g \in D_-$; moreover, the first components of these vectors belong to $C_0^\infty(-\infty, 0)$. Therefore, since the operators $U_t (t \in \mathbb{R})$ are unitary, by the equality

$$F_-U_t f = (U_t f, U_{\overline{\lambda}})_{\mathscr{H}} = e^{i\lambda t}(f, U_\lambda^-)_{\mathscr{H}} = e^{i\lambda t}F_-f, \tag{3.29}$$

we have

$$\begin{gathered}(f,g)_{\mathscr{H}} = (U_{-T}f, U_{-T}g)_{\mathscr{H}} = (F_-U_{-T}f, F_-U_{-T}g)_{L^2} \\ \left(e^{i\lambda T}F_-f, e^{i\lambda T}F_-g\right)_{L^2} = \left(\widetilde{f}, \widetilde{g}\right)_{L^2}.\end{gathered} \tag{3.30}$$

By taking the closure (3.30), we obtain the Parseval equality for the space H_-. The inversion formula is obtained from the Parseval equality if all integrals in it are considered as limits in the of integrals over finite intervals. Finally $F_-H_- = \overline{\cup_{t\geq 0}F_-U_tD_-} = \overline{\cup_{t\geq 0}e^{i\lambda t}H_-^2} = L^2(\mathbb{R})$, that is, F_- maps H_- onto the whole of $L^2(\mathbb{R})$. The lemma is proved. □

We set

$$U_\lambda^+(x,\xi,\zeta) = \left\langle S_h(\lambda)e^{-i\lambda\xi}, \left(m_a(\lambda) - \overline{h}\right)^{-1}\alpha\theta(x,\lambda), e^{-i\lambda\zeta}\right\rangle. \tag{3.31}$$

We note that the vectors $U_\lambda^+(x,\xi,\zeta)$ for real λ do not belong to the space $\mathscr{H}$. However, $U_\lambda^+(x,\xi,\zeta)$ satisfies the equation $\mathcal{L}U = \lambda U$ and the corresponding boundary conditions for the operator $\mathcal{L}_H$. With the help of vector $U_\lambda^+(x,\xi,\zeta)$, we define the transformation $F_+ : f \to \widetilde{f}_+(\lambda)$ by $(F_+f)(\lambda) := \widetilde{f}_+(\lambda) := (1/\sqrt{2\pi})(f, U_\lambda^+)_{\mathscr{H}}$ on the vectors $f = \langle \varphi_-, \widehat{y}, \varphi_+\rangle$ in which $\varphi_-(\xi)$, $\varphi_+(\zeta)$ and $y(x)$ are smooth, compactly supported functions.

Lemma 3.6. *The transformation F_+ isometrically maps H_+ onto $L^2(\mathbb{R})$. For all vectors $f, g \in H_+$ the Parseval equality and the inversion formula hold:*

$$(f,g)_{\mathscr{H}} = \left(\widetilde{f}_+, \widetilde{g}_+\right)_{L^2} = \int_{-\infty}^{\infty} \widetilde{f}_+(\lambda)\overline{\widetilde{g}_+(\lambda)}d\lambda, \quad f = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \widetilde{f}_+(\lambda)U_\lambda^+ d\lambda, \tag{3.32}$$

where $\widetilde{f}_+(\lambda) = (F_+f)(\lambda)$ and $\widetilde{g}_+(\lambda) = (F_+g)(\lambda)$.

Proof. The proof is analogous to Lemma 3.5. □

It is obvious that the matrix-valued function $S_h(\lambda)$ is meromorphic in $\mathbb{C}$ and all poles are in the lower half-plane. From (3.23), $|S_h(\lambda)| \leq 1$ for $\operatorname{Im}\lambda > 0$; and $S_h(\lambda)$ is the unitary matrix for all $\lambda \in \mathbb{R}$. Therefore, it explicitly follows from the formulae for the vectors U_λ^- and U_λ^+ that

$$U_\lambda^+ = S_h(\lambda)U_\lambda^-. \tag{3.33}$$

It follows from Lemmas 3.5 and 3.6 that $H_- = H_+$. Together with Lemma 3.4, this shows that $H_- = H_+ = \mathscr{H}$; therefore, property (3) above has been proved for the incoming and outcoming subspaces. Finally property (4) is clear.

Thus, the transformation F_- isometrically maps H_- onto $L^2(\mathbb{R})$ with the subspace D_- mapped onto H_-^2 and the operators U_t are transformed into the operators of multiplication by $e^{i\lambda t}$. This means that F_- is the incoming spectral representation for the group $\{U_t\}$. Similarly, F_+ is the outgoing spectral representation for the group $\{U_t\}$. It follows from (3.33) that the passage from the F_- representation of an element $f \in \mathscr{H}$ to its F_+ representation is accomplished as $\widetilde{f}_+(\lambda) = S_h^{-1}(\lambda)\widetilde{f}_-(\lambda)$. Consequently, according to [27] we have proved the following.

Theorem 3.7. *The function $S_h^{-1}(\lambda)$ is the scattering matrix of the group $\{U_t\}$ (of the self-adjoint operator L_H).*

Let $S(\lambda)$ be an arbitrary nonconstant inner function on the upper half-plane (the analytic function $S(\lambda)$ on the upper half-plane $\mathbb{C}_+$ is called *inner function* on $\mathbb{C}_+$ if $|S_h(\lambda)| \leq 1$ for all $\lambda \in \mathbb{C}_+$ and $|S_h(\lambda)| = 1$ for almost all $\lambda \in \mathbb{R}$). Define $K = H_+^2 \ominus SH_+^2$. Then $K \neq \{0\}$ is a subspace of the Hilbert space H_+^2. We consider the semigroup of operators $Z_t (t \geq 0)$ acting in K according to the formula $Z_t\varphi = P[e^{i\lambda t}\varphi], \varphi = \varphi(\lambda) \in K$, where P is the orthogonal projection from H_+^2 onto K. The generator of the semigroup $\{Z_t\}$ is denoted by

$$T\varphi = \lim_{t \to +0} (it)^{-1}(Z_t\varphi - \varphi), \tag{3.34}$$

in which T is a maximal dissipative operator acting in K and with the domain $D(T)$ consisting of all functions $\varphi \in K$, such that the limit exists. The operator T is called *a model dissipative operator* (we remark that this model dissipative operator, which is associated with the names of Lax-Phillips [27], is a special case of a more general model dissipative operator constructed by Nagy and Foiaş [26]). The basic assertion is that $S(\lambda)$ is the *characteristic function* of the operator T.

Let $K = \langle 0, H, 0\rangle$, so that $\mathscr{H} = D_- \oplus K \bigoplus D_+$. It follows from the explicit form of the unitary transformation F_- under the mapping F_-

$$\begin{gathered} \mathscr{H} \longrightarrow L^2(\mathbb{R}), \qquad f \longrightarrow \widetilde{f}_-(\lambda) = (F_- f)(\lambda), \qquad D_- \longrightarrow H_-^2, \qquad D_+ \longrightarrow S_h H_+^2, \\ K \longrightarrow H_+^2 \ominus S_G H_+^2, \qquad U_t f \longrightarrow \left(F_- U_t F_-^{-1} \widetilde{f}_-\right)(\lambda) = e^{i\lambda t}\widetilde{f}_-(\lambda). \end{gathered} \tag{3.35}$$

The formulae (3.35) show that operator A_h is a unitarily equivalent to the model dissipative operator with the characteristic function $S_h(\lambda)$. Since the characteristic functions of unitary

equivalent dissipative operator coincide (see [26]), we have thus proved the following theorem.

Theorem 3.8. *The characteristic function of the maximal dissipative operator A_h coincides with the function $S_h(\lambda)$ defined in* (3.23).

Using characteristic function, the spectral properties of the maximal dissipative operator A_h can be investigated. The characteristic function of the maximal dissipative operator A_h is known to lead to information of completeness about the spectral properties of this operator. For instance, the absence of a singular factor $s(\lambda)$ of the characteristic function $S_h(\lambda)$ in the factorization $\det S_h(\lambda) = s(\lambda)B(\lambda)$, where $B(\lambda)$ is a Blaschke product, ensures completeness of the system of eigenvectors and associated vectors of the operator A_h in the space $L_q^2(0,a)$ (see [25]).

Theorem 3.9. *For all the values of h with* $\operatorname{Im} h > 0$, *except possibly for a single value $h = h_0$, the characteristic function $S_h(\lambda)$ of the maximal dissipative operator A_h is a Blaschke product. The spectrum of A_h is purely discrete and belongs to the open upper half-plane. The operator A_h has a countable number of isolated eigenvalues with finite multiplicity and limit points at infinity. The system of all eigenvectors and associated vectors of the operator A_h is complete in the space H.*

Proof. From (3.23), it is clear that $S_h(\lambda)$ is an inner function in the upper half-plane, and it is meromorphic in the whole complex λ-plane. Therefore, it can be factored in the form

$$S_h(\lambda) = e^{i\lambda c}B_h(\lambda), \quad c = c(h) \geq 0, \tag{3.36}$$

where $B_h(\lambda)$ is a Blaschke product. It follows from (3.36) that

$$|S_h(\lambda)| = \left|e^{i\lambda c}\right||B_h(\lambda)| \leq e^{-b(h)\operatorname{Im}\lambda}, \quad \operatorname{Im}\lambda \geq 0. \tag{3.37}$$

Further, for $m_a(\lambda)$ in terms of $S_h(\lambda)$, we find from (3.23) that

$$m_a(\lambda) = \frac{h - \overline{h}S_h(\lambda)}{S_h(\lambda) - 1}. \tag{3.38}$$

If $c(h) > 0$ for a given value $h(\operatorname{Im} h > 0)$, then (3.37) implies that $\lim_{t\to+\infty} S_h(it) = 0$, and then (3.24) gives us that $\lim_{t\to+\infty} m_a(it) = -G$. Since $m_a(\lambda)$ does not depend on h, this implies that $c(h)$ can be nonzero at not more than a single point $h = h_0$ (and further $h_0 = -\lim_{t\to+\infty} m_a(it)$). The theorem is proved.

Due to Theorem 2.4, since the eigenvalues of the boundary value problem (2.1)–(2.3) and eigenvalues of the operator A_h coincide, including their multiplicity and, furthermore, for the eigenfunctions and associated functions the boundary problems (2.1)–(2.3), then theorem is interpreted as follows. □

Corollary 3.10. *The spectrum of the boundary value problem* (2.1)–(2.3) *is purely discrete and belongs to the open upper half-plane. For all the values of h with* $\operatorname{Im}\lambda > 0$, *except possible for a single value $h = h_0$, the boundary value problem* (2.1)–(2.3) $(h \neq h_0)$ *has a countable number of isolated*

eigenvalues with finite multiplicity and limit points and infinity. The system of the eigenfunctions and associated functions of this problem $(h \neq h_0)$ *is complete in the space* $L_q^2(0, a)$.

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