Research Article

On the Existence of Variational Principles for a Class of the Evolutionary Differential-Difference Equations

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Received 10 March 2011; Accepted 18 October 2011

Academic Editor: V. Stepanov

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Necessary and sufficient conditions for the existence of variational principles for a given wide class of evolutionary differential-difference operator equation are obtained. The theoretic results are illustrated by two examples.

1. Introduction.

We consider the equation:

$$N(u) = 0, \quad u \in D(N), \quad (1.1)$$

where $D(N)$ is a domain of the definition of the operator $N : D(N) \subseteq U \rightarrow V$, and U, V are normed linear spaces over the field of real numbers $\mathbb{R}$.

Later, we will assume that at every point $u \in D(N)$, there exists the Gâteaux derivative N'_u of N defined by the formula $(d/de)N(u + \epsilon h)|_{\epsilon=0} = \delta N(u, h) = N'_u h$.

The operator $N : D(N) \rightarrow V$ is said to be potential [1] on the set $D(N)$ relative to a given bilinear form $\Phi(\cdot, \cdot) : V \times U \rightarrow R$, if there exists a functional $F_N : D(F_N) = D(N) \rightarrow R$ such that

$$\delta F_N[u, h] = \Phi(N(u), h), \quad \forall u \in D(N), \quad \forall h \in D(N'_u). \quad (1.2)$$

In that case, we also say that the given equation admits the direct variational formulation.

A problem of the construction of the functional F_N upon the given operator N is known as the classical inverse problem of the calculus of variations [1]. Note that practically no one has been solving inverse problem of the calculus of variations for partial differential-difference operators until recently [2–4]. Let us also note that for a wide classes of partial differential equations, there has been developed the problem of recognition of variationality upon the structure of corresponding operators [5, 6]. There is a theoretical and practical interest in the extension of these results on partial differential-difference equations [7–9].

For what follows, we suppose that $D(N)$ is a convex set, and we also need the following potentiality criterion [1]:

$$\Phi(N'_u h, g) = \Phi(N'_u g, h), \quad \forall u \in D(N), \quad \forall h, g \in D(N'_u). \quad (1.3)$$

Under this condition, the potential F_N is given by

$$F_N[u] = \int_0^1 \Phi(N(u_0 + \lambda(u - u_0)), u - u_0) d\lambda + \text{const}, \quad (1.4)$$

where u_0 is a fixed element of $D(N)$.

The functional F_N is called the potential of the operator N , and in turn the operator N is called the gradient of the functional F_N .

2. Statement of the Problem.

Let us consider the following differential-difference operator equation:

$$N(u) \equiv \sum_{\lambda=-1}^1 P_\lambda(t) u_t(t + \lambda\tau) + Q(t, u(t + \lambda\tau)) = 0, \quad u \in D(N), \quad t \in [t_0, t_1] \subset \mathbb{R}. \quad (2.1)$$

Here, $P_\lambda : U_1 \rightarrow V_1$ ($\lambda = -1, 0, 1$) are linear operators depending on t ; $Q : [t_0 - \tau, t_1 + \tau] \times U_1^3 \rightarrow V_1$ is in general a nonlinear operator; $N : D(N) \subseteq U \rightarrow V$; $U = C^1([t_0 - \tau, t_1 + \tau]; U_1)$; $V = C([t_0 - \tau, t_1 + \tau]; V_1)$, where U_1, V_1 are real normed linear spaces, $U_1 \subseteq V_1$.

The domain of definition $D(N)$ is given by the equality:

$$D(N) = \{u \in U : u(t) = \varphi_1(t), \quad t \in [t_0 - \tau, t_0], \quad u(t) = \varphi_2(t), \quad t \in [t_1, t_1 + \tau]\}, \quad (2.2)$$

where φ_i , ($i = 1, 2$) are given functions.

Under the solution of (2.1), we mean a function $u \in D(N)$ satisfying the identity:

$$N(u) \equiv \sum_{\lambda=-1}^1 P_\lambda(t) u_t(t + \lambda\tau) + Q(t, u(t + \lambda\tau)) = 0, \quad \forall t \in [t_0, t_1] \subset \mathbb{R}. \quad (2.3)$$

Let us give the following bilinear form:

$$\Phi(\cdot, \cdot) \equiv \int_{t_0}^{t_1} \langle \cdot, \cdot \rangle dt : V \times U \longrightarrow \mathbb{R}, \quad (2.4)$$

where the bilinear mapping $\Phi_1 \equiv \langle \cdot, \cdot \rangle$ satisfies the following conditions:

$$\begin{aligned} \langle v_1, v_2 \rangle &= \langle v_2, v_1 \rangle, \quad \forall v_1, v_2 \in V_1, \\ D_t \langle v, g \rangle &= \langle D_t v, g \rangle + \langle v, D_t g \rangle, \quad \forall v, g \in V. \end{aligned} \quad (2.5)$$

Our aim is to define the structure of operators P_λ ($\lambda = -1, 0, 1$) and Q under which (2.1) allows the solution of the inverse problem of the calculus of variations relative to the bilinear form (2.4) such that $D_t = d/dt$ is the antisymmetric operator on $D(N'_u)$, that is,

$$\Phi(D_t h_1, h_2) = -\Phi(D_t h_2, h_1), \quad \forall h_1, h_2 \in D(N'_u). \quad (2.6)$$

3. Conditions of Potentiality and the Structure of (2.1) in the Case of Its Variationality.

We denote by K^* the operator adjoint to K .

Theorem 3.1. *If $D_t^* = -D_t$ on the set $D(N'_u)$, then for the existence of the direct variational formulation for the operator (2.1) on the set $D(N)$ relative to (2.4), it is necessary and sufficient that the following conditions hold on the set $D(N'_u)$:*

$$\begin{aligned} P_{-\lambda} + P_\lambda^*|_{t \rightarrow t-\lambda\tau} &= 0, \quad \frac{\partial P_\lambda^*}{\partial t} \Big|_{t \rightarrow t-\lambda\tau} + Q'_{u(t-\lambda\tau)} - Q'_{u(t+\lambda\tau)}^* \Big|_{t \rightarrow t-\lambda\tau} = 0, \\ \lambda &= -1, 0, 1, \quad \forall u \in D(N), \quad \forall t \in (t_0, t_1). \end{aligned} \quad (3.1)$$

Proof. Taking into account formula (2.1), we get

$$N'_u h = \sum_{\lambda=-1}^1 P_\lambda h_t(t + \lambda\tau) + \sum_{\lambda=-1}^1 Q'_{u(t+\lambda\tau)} h(t + \lambda\tau). \quad (3.2)$$

The criterion of potentiality takes the following form:

$$\begin{aligned} & \int_{t_0}^{t_1} \left\langle \sum_{\lambda=-1}^1 \{ P_\lambda h_t(t + \lambda\tau) + Q'_{u(t+\lambda\tau)} h(t + \lambda\tau) \}, g(t) \right\rangle dt \\ &= \int_{t_0}^{t_1} \left\langle \sum_{\lambda=-1}^1 \{ P_\lambda g_t(t + \lambda\tau) + Q'_{u(t+\lambda\tau)} g(t + \lambda\tau) \}, h(t) \right\rangle dt, \end{aligned} \quad (3.3)$$

or

$$\begin{aligned} \int_{t_0}^{t_1} \left[\left\langle \sum_{\lambda=-1}^1 \{P_\lambda h_t(t+\lambda\tau) + Q'_{u(t+\lambda\tau)} h(t+\lambda\tau)\}, g(t) \right\rangle \right. \\ \left. - \left\langle \sum_{\lambda=-1}^1 (P_\lambda g_t(t+\lambda\tau) + Q'_{u(t+\lambda\tau)} g(t+\lambda\tau)), h(t) \right\rangle \right] dt = 0, \quad \forall u \in D(N), \quad \forall g, h \in D(N'_u). \end{aligned} \quad (3.4)$$

Bearing into account the condition $D_t^* = -D_t$ on the set $D(N'_u)$, from (3.4), we get

$$\int_{t_0}^{t_1} \left[\left\langle \sum_{\lambda=-1}^1 (P_\lambda D_t + Q'_{u(t+\lambda\tau)}) h(t+\lambda\tau), g(t) \right\rangle - \sum_{\lambda=-1}^1 \left\langle [-D_t(P_\lambda^* h(t)) + Q'_{u(t+\lambda\tau)}]^* h(t), g(t+\lambda\tau) \right\rangle \right] dt = 0, \quad (3.5)$$

or

$$\begin{aligned} \int_{t_0}^{t_1} \left[\left\langle \sum_{\lambda=-1}^1 (P_{-\lambda} D_t + Q'_{u(t-\lambda\tau)}) h(t-\lambda\tau), g(t) \right\rangle \right. \\ \left. - \left\langle \sum_{\lambda=-1}^1 -\frac{\partial P_\lambda^*}{\partial t} - P_\lambda^* D_t + Q_u'^* \right|_{t \rightarrow t-\lambda\tau} h(t-\lambda\tau), g(t) \right] dt = 0. \end{aligned} \quad (3.6)$$

Thus, condition (3.4) can be reduced to the following form:

$$\begin{aligned} \int_{t_0}^{t_1} \sum_{\lambda=-1}^1 \left\langle (P_{-\lambda} D_t + Q'_{u(t-\lambda\tau)}) h(t-\lambda\tau) - \left(-\frac{\partial P_\lambda^*}{\partial t} - P_\lambda^* D_t + Q_u'^* \right) \right|_{t \rightarrow t-\lambda\tau} h(t-\lambda\tau), g(t) \right\rangle dt = 0, \\ \forall u \in D(N), \quad \forall g, h \in D(N'_u). \end{aligned} \quad (3.7)$$

This equality is fulfilled identically if and only if

$$\sum_{\lambda=-1}^1 \left[(P_{-\lambda} + P_\lambda^*|_{t \rightarrow t-\lambda\tau}) D_t + \frac{\partial P_\lambda^*}{\partial t} \right|_{t \rightarrow t-\lambda\tau} + Q'_{u(t-\lambda\tau)} - Q'_{u(t+\lambda\tau)}|^*|_{t \rightarrow t-\lambda\tau} \cdot h(t-\lambda\tau) = 0, \quad (3.8)$$

for all $u \in D(N)$. Thus, it is necessary and sufficient that conditions (3.1) hold. $\square$

Theorem 3.2. *Conditions (3.1) are held if and only if (2.1) has the following form:*

$$N_1(u) \equiv \sum_{\lambda=-1}^1 (R_\lambda^*|_{t \rightarrow t-\lambda\tau} - R_{-\lambda})u_t(t-\lambda\tau) + \left(\text{grad}_\Phi B[u] - \sum_{\lambda=-1}^1 \frac{\partial R_\lambda}{\partial t} u(t+\lambda\tau) \right) = 0, \quad (3.9)$$

$$\forall u \in D(N_1), \quad t \in [t_0, t_1] \subset \mathbb{R}.$$

The operators R_λ and B depend on $P_\lambda(t)$ and $Q(t, u(t+\lambda\tau))$.

Proof. If $D_t^* = -D_t$ on the set $D(N'_u)$ and conditions (3.1) are held, then according to Theorem 3.1, operator (2.1) is potential on the set $D(N)$ relative to a given bilinear form (2.4).

Let us consider the following functional:

$$F_N[u] = \int_{t_0}^{t_1} \left[\left\langle \sum_{\lambda=-1}^1 R_\lambda(t)u(t+\lambda\tau), u_t(t) \right\rangle + B[u] \right] dt + F_N[u_0]. \quad (3.10)$$

It is easy to check that

$$\begin{aligned} \delta F_N[u, h] &= \int_{t_0}^{t_1} \left\langle \sum_{\lambda=-1}^1 R_\lambda h(t+\lambda\tau), u_t(t) \right\rangle + \left\langle \sum_{\lambda=-1}^1 R_\lambda u(t+\lambda\tau), h_t(t) \right\rangle + \langle \text{grad}_\Phi B[u], h(t) \rangle dt \\ &= \int_{t_0}^{t_1} \left\langle \sum_{\lambda=-1}^1 R_\lambda^*|_{t \rightarrow t-\lambda\tau} u_t(t-\lambda\tau), h(t) \right\rangle - \left\langle \sum_{\lambda=-1}^1 D_t(R_\lambda u(t+\lambda\tau)), h(t) \right\rangle \\ &\quad + \langle \text{grad}_\Phi B[u], h(t) \rangle dt \\ &= \int_{t_0}^{t_1} \left\langle \sum_{\lambda=-1}^1 R_\lambda^*|_{t \rightarrow t-\lambda\tau} u_t(t-\lambda\tau), h(t) \right\rangle - \left\langle \sum_{\lambda=-1}^1 \frac{\partial R_\lambda}{\partial t} u(t+\lambda\tau), h(t) \right\rangle \\ &\quad - \left\langle \sum_{\lambda=-1}^1 R_{-\lambda} u_t(t-\lambda\tau), h(t) \right\rangle + \langle \text{grad}_\Phi B[u], h \rangle dt \\ &= \int_{t_0}^{t_1} \left\langle \sum_{\lambda=-1}^1 (R_\lambda^*|_{t \rightarrow t-\lambda\tau} - R_{-\lambda})u_t(t-\lambda\tau) \right. \\ &\quad \left. + \left(\text{grad}_\Phi B[u] - \sum_{\lambda=-1}^1 \frac{\partial R_\lambda}{\partial t} u(t+\lambda\tau) \right), h(t) \right\rangle dt \\ &= \int_{t_0}^{t_1} \langle N(u), h \rangle dt, \quad \forall u \in D(N), \quad \forall h \in D(N'_u). \end{aligned} \quad (3.11)$$

Then functional (3.10) is a potential of evolutionary operator (2.1).

If $D_t^* = -D_t$ on $D(N'_u)$, then

$$\begin{aligned} P_{-\lambda}(t) &= R_\lambda^*|_{t \rightarrow t-\lambda\tau} - R_{-\lambda}, \\ Q(t, u(t + \lambda\tau)) &= - \sum_{\lambda=-1}^1 \frac{\partial R_\lambda}{\partial t} u(t + \lambda\tau) + \text{grad}_\Phi B[u]. \end{aligned} \quad (3.12)$$

This theorem shows the structure of the given kind of differential-difference operator, which admits the solution of the inverse problem of the calculus of variations. $\square$

4. Examples

Example 4.1. Let us consider the evolutionary differential-difference equation with partial derivatives in the following form:

$$N_1(u) \equiv \sum_{\lambda=-1}^1 \left(a_\lambda(x, t) \frac{\partial u}{\partial t}(x, t + \lambda\tau) - b_\lambda^{ij}(x, t) \frac{\partial^2 u}{\partial x_i \partial x_j}(x, t + \lambda\tau) \right) = 0, \quad (4.1)$$

where $(x, t) \in Q = \Omega \times (t_0, t_1)$, $t_1 - t_0 > 2\tau$, $i, j = \overline{1, n}$, $a_\lambda(x, t) \in C_{x,t}^{0,1}(\overline{Q})$, $b_\lambda^{ij}(x, t) \in C_{x,t}^{2,0}(\overline{Q})$.

Ω is a bounded domain in $\mathbb{R}^n$ with piecewise smooth boundary $\partial\Omega$.

The domain of definition $D(N_1)$ is given by the equality:

$$\begin{aligned} D(N_1) = \left\{ u \in U = C_{x,t}^{2,1}(\overline{\Omega} \times [t_0 - \tau, t_1 + \tau]) : u(x, t) = \varphi_1(x, t), (x, t) \in E_1 = \overline{\Omega} \times [t_0 - \tau, t_0], \right. \\ \left. u(x, t) = \varphi_2(x, t), (x, t) \in E_2 = \overline{\Omega} \times [t_1, t_1 + \tau], \frac{\partial^\mu u}{\partial n_x^\mu} \Big|_{\partial\Gamma_\tau} = \psi_\mu(x, t), \mu = 0, 1 \right\}, \end{aligned} \quad (4.2)$$

where $\Gamma_\tau = \partial\Omega \times [t_0 - \tau, t_1 + \tau]$, $\varphi_i \in C(E_i)$, $(i = 0, 1)$, $\psi_\mu \in C(\Gamma_\tau)$, $(\mu = 0, 1)$ are given functions.

We investigate the existence of variational principle for (4.1) relative to a given bilinear form (2.4).

For (4.1), we get

$$N'_{1u} h = \sum_{\lambda=-1}^1 (a_\lambda(x, t) h_t(x, t + \lambda\tau) - b_\lambda^{ij}(x, t) h_{x_i x_j}(x, t + \lambda\tau)). \quad (4.3)$$

Necessary and sufficient conditions of potentiality take the form:

$$\begin{aligned}
& \Phi(N'_{1u}h, g) - \Phi(N'_{1u}g, h) \\
& \equiv - \int_{t_0}^{t_1} \int_{\Omega} \left\{ \left(\sum_{\lambda=-1}^1 D_t(a_{\lambda}(x, t)g(x, t)) \right) \Big|_{t \rightarrow t-\lambda\tau} \right. \\
& \quad \left. + \left(\sum_{\lambda=-1}^1 D_{x_i x_j} b_{\lambda}^{ij}(x, t)g(x, t) \right) \Big|_{t \rightarrow t-\lambda\tau} h(x, t) \right\} dx dt \\
& - \int_{t_0}^{t_1} \int_{\Omega} \sum_{\lambda=-1}^1 \left\{ a_{\lambda}(x, t)g_t(x, t + \lambda\tau) - b_{\lambda}^{ij}(x, t)g_{x_i x_j}(x, t + \lambda\tau) \right\} \\
& \quad \cdot h(x, t) dx dt = 0, \quad \forall u \in D(N_1), \quad \forall g, h \in D(N'_{1u}).
\end{aligned} \tag{4.4}$$

From that, we come to the following:

$$\begin{aligned}
- \sum_{\lambda=-1}^1 \left(D_t(a_{\lambda}g(x, t)) - D_{x_i x_j}(b_{\lambda}^{ij}g(x, t)) \right) \Big|_{t \rightarrow t-\lambda\tau} &= \sum_{\lambda=-1}^1 \left\{ a_{\lambda}g_t(x, t + \lambda\tau) - b_{\lambda}^{ij}g_{x_i x_j}(x, t + \lambda\tau) \right\}, \\
& \forall g \in D(N'_{1u}).
\end{aligned} \tag{4.5}$$

That is true if and only if

$$\begin{aligned}
-a_{-\lambda}(x) &= a_{\lambda}(x), \quad \lambda = 1, -1, \quad a_0(x) = 0, \quad b_{\lambda}^{ij}(t) \Big|_{t \rightarrow t-\lambda\tau} = b_{-\lambda}^{ij}(t), \quad \lambda = -1, 0, 1, \\
i, j &= \overline{1, n}, \quad \forall (x, t) \in Q.
\end{aligned} \tag{4.6}$$

Under the fulfilment of that conditions, the corresponding functional is given by

$$\begin{aligned}
F_{N_1}[u] &= \frac{1}{2} \int_{t_0}^{t_1} \int_{\Omega} \left\{ a_1(x)u_t(x, t - \tau)u(x, t) - a_1(x)u_t(x, t + \tau)u(x, t) + b_1^{ij}(t - \tau)u_{x_i}(x, t - \tau) \right. \\
& \quad \left. \cdot u_{x_j}(x, t) + b_0^{ij}(t)u_{x_i}(x, t)u_{x_j}(x, t) + b_{-1}^{ij}(t + \tau)u_{x_i}(x, t + \tau)u_{x_j}(x, t) \right\} dx dt + \text{const.}
\end{aligned} \tag{4.7}$$

Let us consider an example when this criterion of potentiality fails.

Example 4.2. Consider the equation:

$$\begin{aligned}
N_2(u) &= u_t(t, x) - 2u(t, x + 2\tau)u_x(t, x + 2\tau) - 2u_x(t, x)u(t, x - 2\tau) - 2u(t, x)u_x(t, x - 2\tau) \\
& + \frac{1}{2}u_{xxx}(t, x + 2\tau) + \frac{1}{2}u_{xxx}(t, x - 2\tau) = 0, \quad (t, x) \in Q_T = (0, T) \times (-\infty, +\infty).
\end{aligned} \tag{4.8}$$

Let us note that this equation is a Korteweg-de Vries' equation $\tau = 0$.

We denote

$$D(N_2) = \left\{ u \in U : C_{t,x}^{1,3}((0,T) \times (-\infty, +\infty)) : u|_{t=0} = u_0(x), u|_{t=T} = u_1(x), \lim_{|x| \rightarrow +\infty} D_x^n u = 0, (n = \overline{0,3}) \right\}. \quad (4.9)$$

It is easy to be convinced that operator (4.8) is not potential on set (4.9) relative to the bilinear form:

$$\Phi(v, g) = \int_0^T \int_{-\infty}^{+\infty} v(t, x) \cdot g(t, x) dx dt, \quad v \in V, g \in U. \quad (4.10)$$

Here, $V = \{v \in C_{t,x}^{0,3}((0,T) \times (-\infty, +\infty)) : \lim_{|x| \rightarrow +\infty} D_x^n v = 0 (n = \overline{0,3})\}$.

We define the integrating operator M as $Mv(t, x) = \int_{-\infty}^x v(t, y) dy$.

Then, the operator,

$$\begin{aligned} \widetilde{N}(u) \equiv MN_2(u) &= \int_{-\infty}^x u_t(t, y) dy - u^2(t, x + 2\tau) - 2u(t, x)u(t, x - 2\tau) \\ &\quad + \frac{1}{2}u_{xx}(t, x + 2\tau) + \frac{1}{2}u_{xx}(t, x - 2\tau)dy \end{aligned} \quad (4.11)$$

is potential on set (4.9) relative to bilinear form (4.10). The corresponding functional $F_{N_2}[u]$ has the form:

$$\begin{aligned} F_{N_2}[u] &= \frac{1}{2} \int_0^T \int_{-\infty}^{+\infty} \left\{ u(t, x) \int_{-\infty}^x u_t(t, y) dy - u_x(t, x - \tau)u_x(t, x + \tau) - 2u^2(t, x + \tau)u(t, x - \tau) \right. \\ &\quad \left. - u^2(t, x - 2\tau)dy \right\} dx dt. \end{aligned} \quad (4.12)$$

Indeed, using (4.12), we find that

$$\begin{aligned} \delta F_{N_2}[u, h] &= \int_0^T \int_{-\infty}^{+\infty} \left\{ \int_{-\infty}^x u_t(t, y) dy + \frac{1}{2}u_{xx}(t, x + 2\tau) + \frac{1}{2}u_{xx}(t, x - 2\tau) - 2u(t, x)u(t, x - 2\tau) \right. \\ &\quad \left. - u^2(t, x + 2\tau)dy \right\} h(t, x) dx dt, \quad \forall u \in D(N), \forall h \in D(N'_u). \end{aligned} \quad (4.13)$$

From the condition $\delta F_N[u, h] = 0$, $u \in D(N)$, for all $h \in D(N'_u)$, we obtain

$$\begin{aligned} \widetilde{N}(u) \equiv MN_2(u) &\equiv \int_{-\infty}^x u_t(t, y) dy - u^2(t, x + 2\tau) - 2u(t, x)u(t, x - 2\tau) + \frac{1}{2}u_{xx}(t, x + 2\tau) \\ &\quad + \frac{1}{2}u_{xx}(t, x - 2\tau)dy = 0, \quad u \in D(N), \end{aligned} \quad (4.14)$$

This equation is equivalent to (4.8).

Let us note that the formula $I[u] = \int_{-\infty}^{+\infty} (1/2)u_x(t, x - \tau)u_x(t, x + \tau) + u^2(t, x + \tau)u(t, x - \tau)dx$ defines the first integral $I[u] = \text{const}$ of (4.8).

Acknowledgment

The work of V. M. Sovchin was partially supported by the Russian Foundation for Basic Research (Projects 09-01-00093 and 09-01-00586).

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