Research Article

Carleson Measure Theorems for Large Hardy-Orlicz and Bergman-Orlicz Spaces

Stéphane Charpentier¹ and Benoît Sehba²

¹ Département de Mathématiques, Université Paris-Sud, Bâtiment 425, 91405 Orsay, France

² School of Mathematics, Trinity College, Dublin 2, Ireland

Correspondence should be addressed to Benoît Sehba, sehbab@tcd.ie

Received 24 October 2011; Accepted 20 February 2012

Academic Editor: Miroslav Engliš

Copyright © 2012 S. Charpentier and B. Sehba. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We characterize those measures μ for which the Hardy-Orlicz (resp., weighted Bergman-Orlicz) space H^{Ψ_1} (resp., $A_{\alpha}^{\Psi_1}$) of the unit ball of $\mathbb{C}^N$ embeds boundedly or compactly into the Orlicz space $L^{\Psi_2}(\mathbb{B}_N, \mu)$ (resp., $L^{\Psi_2}(\mathbb{B}_N, \mu)$), when the defining functions Ψ_1 and Ψ_2 are growth functions such that $L^1 \subset L^{\Psi_j}$ for $j \in \{1, 2\}$, and such that Ψ_2/Ψ_1 is nondecreasing. We apply our result to the characterization of the boundedness and compactness of composition operators from H^{Ψ_1} (resp., $A_{\alpha}^{\Psi_1}$) into H^{Ψ_2} (resp., $A_{\alpha}^{\Psi_2}$).

1. Introduction and Preliminaries

1.1. Introduction

Let $\mathbb{B}_N = \{z = (z_1, \dots, z_N) \in \mathbb{C}^N : |z|^2 = \sum_{i=1}^N |z_i|^2 < 1\}$ and $\mathbb{S}_N = \partial\mathbb{B}_N$, $N \geq 1$, denoting, respectively, the unit ball and the unit sphere of $\mathbb{C}^N$. For $N = 1$, we denote by $\mathbb{D}$ the unit disc of the complex plane.

For a large class of spaces X of holomorphic functions in the unit disc or the unit ball, characterizations of the boundedness and compactness of the canonical embedding $X \hookrightarrow L^p(\mu)$ have been given and applied to different areas, for example, interpolation, multipliers, integral operators, composition operators, and so forth. These results are known as Carleson's type theorems.

First, when $X = H^p(\mathbb{D})$, Carleson [1] proved that $H^p(\mathbb{D}) \hookrightarrow L^p(\mu)$ if and only if the finite positive Borel measure μ on $\mathbb{D}$ (or $\mathbb{D}$) is a so-called *Carleson measure*. This result was extended to the unit ball by Hörmander [2], whose proof was simplified by Power [3]. Duren [4] characterized those measures μ such that $H^p(\mathbb{D}) \hookrightarrow L^q(\mu)$, with $0 < p \leq q < \infty$, in terms of

(q/p)-Carleson measures. For the unweighted and weighted Bergman spaces $A_\alpha^p(\mathbb{B}_N)$, $N \geq 1$, similar results were obtained by Cima and Wogen [5], Luecking [6], Ueki [7]. We recall also that the compactness of $X \hookrightarrow L^p(\mu)$ was also characterized in the previous cases, in terms of *vanishing* Carleson's type measures. It is usual to assume that when the measure μ is defined on $\overline{\mathbb{B}_N}$, then $\mu|_{\mathbb{S}_N}$ is absolutely continuous with respect to $\mathbb{S}_N$. This assumption will be done, without mentioning it any further.

Some observations may be done. First, it appears that the characterizations of both boundedness and compactness of $H^p(\mathbb{B}_N) \hookrightarrow L^q(\mu)$ are not always satisfied and just depend on the ratio q/p for $0 < p \leq q < \infty$; in particular, when $p = q$, they are independent of p . Now, since the restriction to $\mathbb{S}_N$ of the finite positive Borel μ is assumed to be absolutely continuous with respect to the Lebesgue measure, then it is trivial that $H^\infty \hookrightarrow L^\infty(\mu)$ always holds. On the contrary, the compactness of this inclusion implies strong condition on μ . This suggests to think about this question when the space X is intercalated between every H^p and H^∞ .

This motivation is reinforced by a second observation: if the measure μ_ϕ is the pull-back measure of the invariant-rotation measure on $\mathbb{S}_N$ under a holomorphic map $\phi : \mathbb{B}_N \rightarrow \mathbb{B}_N$, $N \geq 1$, then μ_ϕ is always a Carleson measure when $N = 1$ (this is the Littlewood Subordination Principle, see [8]), but it is not systematic for $N > 1$. For the compactness, there is still a big gap between H^p and H^∞ for any $N \geq 1$. This observation is directly connected to the study of composition operators C_ϕ on H^p , which are defined by $C_\phi(f) = f \circ \phi$ for $f \in H(\mathbb{B}_N)$, and which may be seen as the embedding operators $H^p \hookrightarrow L^p(\mu_\phi)$. By the way, this leads the authors of [9, 10] to state Carleson theorems for Hardy-Orlicz and Bergman-Orlicz spaces (resp., denoted by $H^\Psi(\mathbb{D})$ and $A^\Psi(\mathbb{D})$) in the unit disc, when the defining function Ψ is an *Orlicz function*. These spaces appear as good candidates for generalizing H^p and A^p spaces, $1 \leq p < \infty$, and for covering the gap with H^∞ . This fact was still more pointed out in [11, 12], where the author gave Carleson theorems in the unit ball, under some mild conditions on the defining function Ψ . Indeed, he showed that, when $\mu = \mu_\phi$ is the pull-back measure under a holomorphic self-map ϕ of $\mathbb{B}_N$, then H^Ψ always embeds into $L^\Psi(\mu)$ whenever Ψ satisfied a fast growth condition (namely, the Δ^2 -Condition, that implies $\Psi(x) \geq e^x$ for large value of x and means that we are *close to* H^∞). Similarly, it was shown in [13] that if H^Ψ compactly embeds into $L^\Psi(\mu)$ for *every* Ψ , then $H^\infty \hookrightarrow L^\infty(\mu)$ compactly (the converse easily holds). We mention that there is no Ψ such that $H^\Psi \hookrightarrow L^\Psi(\mu)$ if and only if $H^\infty \hookrightarrow L^\infty(\mu)$ [14], and that such previous results are not true for arbitrary measure μ . Yet, a link has been made between the involved Carleson conditions and the type of growth of the Orlicz function Ψ . This is also strengthened by the fact that if the Orlicz function Ψ is *dominated* by a power function (exactly Ψ satisfies the Δ_2 -Condition), then $H^\Psi \hookrightarrow L^\Psi(\mu)$ boundedly (resp., compactly) if and only if $H^p \hookrightarrow L^p(\mu)$, that is, if and only if μ is a classical Carleson measure (resp., a vanishing Carleson measure).

Let us note that similar results hold for Bergman-Orlicz spaces.

The purpose of the present paper is to deal with the same kind of question on the opposite side, that is, for Hardy-Orlicz and Bergman-Orlicz spaces which are larger than $H^1(\mathbb{B}_N)$. It seems that nothing has been done in this direction, except for explicit functions Ψ . In particular, the second author gave a necessary and sufficient condition for the inclusion $H^\Psi(\mathbb{B}_N) \hookrightarrow L^q(\mu)$ to be bounded, when $\Psi(t) = (t/\log(e+t))^s$, $0 < s \leq 1$, and $1 \leq q < \infty$ [15]. Moreover, [16] characterized Carleson measures for $A_\alpha^\Psi(\mathbb{D})$, where $\Psi = \log_+^p$, $1 \leq p < \infty$, that is, when $A_\alpha^\Psi(\mathbb{D})$ is the area Nevanlinna space. These measures reveal to be those μ for which $A_\alpha^p \hookrightarrow L^p(\mu)$ holds, which are Bergman-Carleson measures.

A first difficulty when dealing with large Hardy-Orlicz or Bergman-Orlicz spaces is that we do not have normed spaces any more, and we need to exhibit the *good properties* of

the function Ψ , in order to define spaces with which it is *reasonable* to work. For that, we were inspired by [17, 18] and the references therein. Then, we obtain more generally a complete characterization of those finite positive Borel measures μ such that $H^{\Psi_1}(\mathbb{B}_N) \hookrightarrow L^{\Psi_2}(\mathbb{B}_N, \mu)$ (resp., $A_\alpha^{\Psi_1}(\mathbb{B}_N) \hookrightarrow L^{\Psi_2}(\mathbb{B}_N, \mu)$) is bounded or compact, when Ψ_1 and Ψ_2 are two *growth* functions (i.e., for which $H^1 \subset H^{\Psi_i} \subset H^p$ (resp., $A_\alpha^1 \subset A_\alpha^{\Psi_i} \subset A_\alpha^p$) for some $p, i, j \in \{1, 2\}$), and such that Ψ_2 *grows faster* than Ψ_1 . It appears that if $\Psi_1 = \Psi_2$, then such measures are exactly those which are Carleson (resp., Bergman-Carleson) measures for some $0 < p < 1$. For the Bergman-Orlicz case, these results let one think that, between the area Nevanlinna class [16] and A_α^1 , there is no difference regarding to Carleson theorems, whenever the defining functions Ψ share some natural properties. This is in contrast with what happens between A_α^1 and H^∞ .

The paper is organized as follows. In the next subsection, we introduce the Hardy-Orlicz and Bergman-Orlicz spaces under further considerations, proving or generalizing some useful and classical results. The second section consists of the statements and the proofs of our Carleson theorems for these spaces. A third and last part is an immediate application of our result to composition operators.

Notation. Given two points $z, w \in \mathbb{C}^N$, the euclidean inner product of z and w will be denoted by $\langle z, w \rangle$, that is, $\langle z, w \rangle = \sum_{i=1}^N z_i \overline{w_i}$; the notation $|\cdot|$ will stand for the associated norm, as well as for the modulus of a complex number.

σ will stand for the invariant-rotation measure on the unit sphere. For $\alpha > -1$, v_α will be the measure on $\mathbb{B}_N$ defined by $dv_\alpha = c_\alpha(1 - |z|)^\alpha dv$, where v is the Lebesgue measure on $\mathbb{B}_N$ and c_α is the constant of normalization.

We will use the notations $\lesssim$ and $\gtrsim$ for one-sided estimates up to an absolute constant, and the notation $\approx$ for two-sided estimates up to an absolute constant.

Without possible confusions, we will write H^Ψ (resp., A_α^Ψ) instead of $H^\Psi(\mathbb{B}_N)$ (resp., $A_\alpha^\Psi(\mathbb{B}_N)$).

1.2. Preliminaries: Menagerie of Spaces

Let $\Psi : [0, \infty) \rightarrow [0, \infty)$ be a continuous nondecreasing function which vanishes and is continuous at 0. Given a probabilistic space $(\Omega, \mathbb{P})$, we define the Orlicz class $L^\Psi(\Omega, \mathbb{P})$ as the set of all (equivalence classes of) measurable functions f on Ω such that $\int_\Omega \Psi(|f|/C) d\mathbb{P} < \infty$ for some $0 < C < \infty$. We use to define the Morse-Transue space $M^\Psi(\Omega, \mathbb{P})$ by

$$M^\Psi(\Omega, \mathbb{P}) = \left\{ f : \Omega \rightarrow \mathbb{C} \text{ measurable; } \int_\Omega \Psi\left(\frac{|f|}{C}\right) < \infty \text{ for any } C > 0 \right\}, \quad (1.1)$$

and we also introduce the following set:

$$\mathcal{L}^\Psi(\Omega, \mathbb{P}) = \left\{ f : \Omega \rightarrow \mathbb{C} \text{ measurable; } \int_\Omega \Psi(|f|) < \infty \right\}. \quad (1.2)$$

In general, these three sets are not vector spaces and do not coincide, but we trivially have

$$M^\Psi(\Omega, \mathbb{P}) \subset \mathcal{L}^\Psi(\Omega, \mathbb{P}) \subset L^\Psi(\Omega, \mathbb{P}). \quad (1.3)$$

We also define the *Luxembourg gauge* on $L^\Psi(\Omega, \mathbb{P})$ by

$$\|f\|_\Psi = \inf \left(\lambda > 0, \int_\Omega \Psi \left(\frac{|f|}{\lambda} \right) d\mathbb{P} \leq 1 \right). \quad (1.4)$$

This functional is homogeneous and is 0 if and only if $f = 0$ $\mathbb{P}$ -a.e., but it is not subadditive a priori.

We say that two functions Ψ_1 and Ψ_2 as above are equivalent if there exists some constant c such that

$$c\Psi_1(cx) \leq \Psi_2(x) \leq c^{-1}\Psi_1(c^{-1}x), \quad (1.5)$$

for any x large enough. Two equivalent functions define the same Orlicz class with equivalent Luxembourg functionals.

In order to define a good topology on $L^\Psi(\Omega, \mathbb{P})$ and to get properties convenient for our purpose, we will assume that Ψ satisfies the following definition.

Definition 1.1. Let $0 < p \leq 1$. We say that $\Psi : [0, \infty) \rightarrow [0, \infty)$ is a growth function of order p if it satisfies the two following conditions:

- (1) Ψ is of lower type p , that is, $\Psi(yx) \leq y^p \Psi(x)$ for any $0 < y \leq 1$ and at least for x large enough;
- (2) $x \mapsto \Psi(x)/x$ is nonincreasing, at least for every x large enough.

We shall say that ψ is a growth function if it is a growth function of order p for some $0 < p \leq 1$.

In particular, a growth function Ψ is equivalent to the function $x \mapsto \int_0^x (\Psi(s)/s) ds$ which is concave (see [17]). Now, for such a concave growth function of order p ,

$$L^1(\Omega, \mathbb{P}) \subset M^\Psi(\Omega, \mathbb{P}) = \mathcal{L}^\Psi(\Omega, \mathbb{P}) = L^\Psi(\Omega, \mathbb{P}) \subset L^p(\Omega, \mathbb{P}), \quad (1.6)$$

$$\|f\|_\Psi \lesssim \min \left\{ \int_\Omega \Psi(|f|) d\mathbb{P}, \left(\int_\Omega \Psi(|f|) d\mathbb{P} \right)^p \right\}, \quad (1.7)$$

while

$$\|f\|^\Psi := \int_\Omega \Psi(|f|) d\mathbb{P} \lesssim \max \left\{ \|f\|_\Psi, \|f\|_\Psi^p \right\}. \quad (1.8)$$

Moreover, if we define $d_\Psi(f, g) = \|f - g\|_\Psi$ (resp., $d^\Psi(f, g) = \|f - g\|^\Psi$), then d_Ψ and d^Ψ (that we will simply denote by $\|\cdot\|_\Psi$ and $\|\cdot\|^\Psi$) are two equivalent metrics on $L^\Psi(\Omega, \mathbb{P})$ for which it is complete. Without loss of generality, we then assume that every growth function that we consider further is concave and diffeomorphic.

Remark 1.2. Note that every concave function which vanishes at 0 satisfies the Δ_2 -condition, that is, $\Psi(2x) \leq K\Psi(x)$ for some $K > 1$ and x large enough. This condition is very classical

when the function Ψ is an Orlicz function, that is, a nondecreasing continuous convex function (see [19, 20]). For large Orlicz class, this condition is natural in order to have vector space.

When we deal with spaces of holomorphic functions, it is very natural to require subharmonicity. Then we will assume that Ψ is such that $\Psi(|f|)$ is subharmonic when f is holomorphic. We will refer to such a function as a subharmonic-preserving function.

Here are some examples of (concave) growth function that we may consider further.

Example 1.3. (1) $\Psi_1(x) = x^p$ for $0 < p \leq 1$ and any $x \geq x_0$.

(2) $\Psi_2(x) = x^p \log^q(C + x)$ (for $x \geq x_0$) with $C > 0$ large enough, $0 < p \leq 1$ and $q \geq 1$.

(3) $\Psi_3(x) = \Phi^p(x)$ at least for any x large, where Φ is an Orlicz function (see Remark 1.2) and $p > 0$ is such that $\Phi^p(x)/x$ is nonincreasing. Note that in this case, $\Psi(|f|)$ is subharmonic whenever f is holomorphic, because we have the following.

Proposition 1.4. *Let Ψ be as in (3) above. There exists a convex function ψ such that $\Psi(x) = \psi(x^p)$ for any $x > 0$ large enough.*

Proof. Let $p > 0$ and Φ be an Orlicz function such that $\Psi(x) = \Phi^p(x)$ for any $x \geq 0$. Since Ψ is (assumed to be) bijective, we can define the function ψ by $\psi(x^p) = \Phi^p(x)$ for every x . Now, using that Φ is convex, ψ is convex since

$$\begin{aligned} \frac{\psi(x)}{x} &= \left(\frac{\Phi(x^{1/p})}{x^{1/p}} \right)^p \leq \left(\frac{\Phi(y^{1/p})}{y^{1/p}} \right)^p \\ &= \frac{\psi(y)}{y}, \end{aligned} \quad (1.9)$$

for any $0 \leq x \leq y$. □

It follows that $\Psi(|f|) = \psi(|f|^p)$ is subharmonic (for f holomorphic).

The following lemma gives an upper estimate of the Luxembourg norm of a function in $L^\infty(\Omega, \mathbb{P})$.

Lemma 1.5. *Let $(\Omega, \mathbb{P})$ be a probabilistic space and let Ψ be a growth function of order p . For any $f \in L^\infty(\Omega, \mathbb{P})$, one has*

$$\|f\|_\Psi \leq \frac{\|f\|_\infty}{\Psi^{-1}(\|f\|_\infty / \|f\|_p^p)}. \quad (1.10)$$

Proof. It is quite identical to that of [10, Lemma 3.9], but we prefer to give the details. Without loss of generality, we may assume that $\|f\|_\infty = 1$. For every $C > 0$, one has the following, using that Ψ is of lower type p :

$$\int_\Omega \Psi\left(\frac{|f|}{C}\right) d\mathbb{P} \leq \int_\Omega |f|^p \Psi\left(\frac{1}{C}\right) d\mathbb{P} \leq \|f\|_p^p \Psi\left(\frac{1}{C}\right). \quad (1.11)$$

Then the last expression is less than or equal to 1 if and only if $C \geq 1/\Psi^{-1}(1/\|f\|_p^p)$. □

1.2.1. Large Bergman-Orlicz Spaces

For Ψ a subharmonic-preserving growth function and $\alpha > -1$, the weighted Bergman-Orlicz space A_α^Ψ of the ball consists of those holomorphic functions on $\mathbb{B}_N$ which belongs to the Orlicz space $L^\Psi(\mathbb{B}_N, v_\alpha)$. To avoid further confusion, we will denote by $\|\cdot\|_{\alpha, \Psi}$ the corresponding Luxembourg (quasi)norm and by $\|\cdot\|_\alpha^\Psi$ the quantity $\int_{\mathbb{B}_N} \Psi(|f|) dv_\alpha$. A_α^Ψ is metric space for the distance $d_{\alpha, \Psi}$ or d_α^Ψ defined by, respectively, $d_{\alpha, \Psi}(f, g) = \|f - g\|_{\alpha, \Psi}$ and $d_\alpha^\Psi(f, g) = \|f - g\|_\alpha^\Psi$. If $\Psi(t) = t^p$, then we recover the usual weighted Bergman space A_α^p . One checks that we have the followings inclusions:

$$A_\alpha^1 \subset A_\alpha^\Psi \subset A_\alpha^p, \quad (1.12)$$

whenever Ψ is a growth function of order p .

It seems important to us to mention that a linear operator T from $A_\alpha^{\Psi_1}$ to X with $X = L^{\Psi_2}(\mathbb{B}_N, v_\alpha)$ or $X = A_\alpha^{\Psi_2}$, where Ψ_1 and Ψ_2 are two growth functions, is continuous if and only if it maps a bounded set into a bounded set, or equivalently if and only if there exists a constant $C > 0$ such that

$$\max\left(\|T(f)\|_{\alpha, \Psi_2}, \|T(f)\|_\alpha^{\Psi_2}\right) \leq C, \quad (1.13)$$

for any $f \in A_\alpha^{\Psi_1}$ such that $\min(\|f\|_{\alpha, \Psi_1}, \|f\|_\alpha^{\Psi_1}) \leq 1$. If $X = \mathbb{C}$, then T is bounded if and only if $|T(f)| \leq C$ for f as previously.

The next proposition says that the point evaluation functionals are continuous on A_α^Ψ .

Proposition 1.6. *Let $\alpha > -1$ and let Ψ be a subharmonic-preserving growth function. For any $a \in \mathbb{B}_N$ and any $f \in A_\alpha^\Psi$, one has*

$$|f(a)| \leq \Psi^{-1}\left(\left(\frac{2}{1-|a|}\right)^{(N+\alpha+1)}\right) \|f\|_{\alpha, \Psi}. \quad (1.14)$$

The proof is the same as that of [11, Proposition 1.9] and so is omitted (still use the hypothesis that $\Psi(|f|)$ is subharmonic). We easily deduce from this and the completeness of L^Ψ the following result.

Corollary 1.7. *A_α^Ψ , endowed with $d_{\alpha, \Psi}$ or d_α^Ψ , is a complete metric space.*

Let $p > 0$ and $\alpha > -1$. For $a \in \mathbb{B}_N$, we introduce the following “test” function $f_{a, \alpha, p}$ defined by

$$f_{a, \alpha, p} = \left(\frac{1 - |a|^2}{(1 - \langle z, a \rangle)^2} \right)^{(N+\alpha+1)/p}. \quad (1.15)$$

$|f_{a,\alpha,p}|^p$ is nothing but the Berezin kernel, hence $\|f_{a,\alpha,p}\|_\infty = ((1+|a|)/(1-|a|))^{(N+\alpha+1)/p}$ while $\|f_{a,\alpha,p}\|_p = 1$. Then, as a consequence of Lemma 1.5, we have

$$\|f_{a,\alpha,p}\|_{\alpha,\Psi} \leq \left(\frac{2}{1-|a|} \right)^{(N+\alpha+1)/p} \frac{1}{\Psi^{-1}\left((2/1-|a|)^{(N+\alpha+1)}\right)}, \quad (1.16)$$

whenever Ψ is a growth function of order p . These functions will be of interest to us later, when proving Carleson theorem for large Bergman-Orlicz spaces.

We now define a maximal operator which was introduced in [11] and that will be bounded on A_α^Ψ . The definition needs to introduce the sets $Q(\zeta, h)$, defined by

$$Q(\zeta, h) = \{z \in \mathbb{S}_N, |1 - \langle z, \zeta \rangle| < h\}, \quad (1.17)$$

and requires the construction of convenient sets based on the following lemma ([11, Lemma 2.1]; we also refer to the forthcoming Section 2.1).

Lemma 1.8. *There exists an integer $M > 0$ such that for any $0 < r < 1$, one can find a finite sequence $\{\xi_k\}_{k=1}^m$ (m depending on r) in $\mathbb{S}_N$ with the following properties.*

- (1) $\mathbb{S}_N = \bigcup_k Q(\xi_k, r)$.
- (2) The sets $Q(\xi_k, r/4)$ are mutually disjoint.
- (3) Each point of $\mathbb{S}_N$ belongs to at most M of the sets $Q(\xi_k, 4r)$.

From now on, M denotes the constant involved in the previous lemma. Let $n \geq 0$ be an integer and let C_n be the corona

$$C_n = \left\{ z \in \mathbb{B}_N, 1 - \frac{1}{2^n} \leq |z| < 1 - \frac{1}{2^{n+1}} \right\}. \quad (1.18)$$

For any $n \geq 0$, let $(\xi_{n,k})_k \subset \mathbb{S}_N$ be given by Lemma 1.8 putting $r = 1/2^n$. For $k \geq 0$, we set

$$T_{0,k} = \left\{ z \in \mathbb{B}_N \setminus \{0\}, \frac{z}{|z|} \in Q(\xi_{0,k}, 1) \right\} \cup \{0\}. \quad (1.19)$$

Then we define the sets $T_{n,k}$, for $n \geq 1$ and $k \geq 0$, by

$$T_{n,k} = \left\{ z \in \mathbb{B}_N \setminus \{0\}, \frac{z}{|z|} \in Q\left(\xi_{n,k}, \frac{1}{2^n}\right) \right\}. \quad (1.20)$$

We have both

$$\begin{aligned} \bigcup_{n \geq 0} C_n &= \mathbb{B}_N, \\ \bigcup_{k \geq 0} T_{0,k} &= \mathbb{B}_N, \quad \bigcup_{k \geq 0} T_{n,k} = \mathbb{B}_N \setminus \{0\}, \quad n \geq 1. \end{aligned} \quad (1.21)$$

For $(n, k) \in \mathbb{N}^2$, we finally define the subset $\Delta_{(n,k)}$ of $\mathbb{B}_N$ by $\Delta_{(n,k)} = C_n \cap T_{n,k}$. These sets have good covering properties that we do not recall here (we refer to [11]). Anyway, we define the following maximal function Λ_f for $f \in A_\alpha^\Psi(\mathbb{B}_N)$ by

$$\Lambda_f = \sum_{n,k \geq 0} \sup_{\Delta_{(n,k)}} (|f(z)|) \chi_{\Delta_{(n,k)}}, \quad (1.22)$$

where $\chi_{\Delta_{(n,k)}}$ is the characteristic function of $\Delta_{(n,k)}$. Now we may easily adapt the proof of [11, Proposition 2.2] (which only relies on the subharmonicity of $\Psi(|f|)$) to get the following.

Proposition 1.9. *Let Ψ be a subharmonic-preserving growth function and let $\alpha > -1$. Then the maximal operator Λ , which carries f to Λ_f , is bounded from A_α^Ψ to $L^\Psi(\mathbb{B}_N, v_\alpha)$. More precisely, there exists $B \geq 1$ such that for every $f \in A_\alpha^\Psi$, one has*

$$\|\Lambda_f\|_{L^\Psi(\mathbb{B}_N, v_\alpha)} \leq B \|f\|_{\alpha, \Psi}. \quad (1.23)$$

In particular, a holomorphic function f belongs to A_α^Ψ if and only if Λ_f belongs to $L^\Psi(\mathbb{B}_N, v_\alpha)$.

1.2.2. Large Hardy-Orlicz Spaces

Let Ψ still be a (concave) subharmonic-preserving growth function. With the notations of Section 1.2, let $(\Omega, \mathbb{P}) = (\mathbb{S}_N, \sigma)$. The Hardy-Orlicz space H^Ψ of the ball consists of all holomorphic functions f on $\mathbb{B}_N$ such that

$$\|f\|_{H^\Psi} := \sup_{0 < r < 1} \|f_r\|_\Psi < \infty, \quad (1.24)$$

where $f_r(z) = f(rz)$, and where $\|\cdot\|_\Psi$ is the Luxembourg norm on the Orlicz space $L^\Psi(\mathbb{S}_N, \sigma)$. Note that we can replace $\sup_{0 < r < 1} \|f_r\|_\Psi < \infty$ by $\lim_{r \rightarrow 1} \|f_r\|_\Psi < \infty$ thanks to the subharmonicity of $\Psi(|f|)$. Because Ψ is supposed to be a growth function, we have the following inclusion:

$$H^1 \subset H^\Psi \subset H^p \quad (1.25)$$

for a growth function Ψ of order p . In particular, every $f \in H^\Psi$ admits a boundary radial limit, denoted by f^* , σ -almost everywhere on $\mathbb{S}_N$. Let us also note that if Ψ is a growth function, then $f \in H^\Psi$ if and only if

$$\|f\|_{H^\Psi}^\Psi := \sup_{0 < r < 1} \|f_r\|^\Psi = \lim_{r \rightarrow 1} \|f_r\|^\Psi < \infty \quad (\text{Inequalities (1.7) and (1.8)}). \quad (1.26)$$

Without possible confusion, we will write $\|\cdot\|_\Psi$ instead of $\|\cdot\|_{H^\Psi}$ (resp., $\|\cdot\|^\Psi$ instead of $\|\cdot\|_{H^\Psi}^\Psi$). As for Bergman-Orlicz spaces, a linear operator T from H^{Ψ_1} to some $L^{\Psi_2}(\mathbb{S}_N, \sigma)$ or H^{Ψ_2} where Ψ_1 and Ψ_2 are two growth functions is continuous (bounded) if and only if there exists a constant $C > 0$ such that

$$\max\left(\|T(f)\|_{\Psi_2}, \|T(f)\|_{\Psi_2}^{\Psi_2}\right) \leq C \quad (1.27)$$

for any $f \in H^{\Psi_1}$ such that $\min(\|f\|_{\Psi_1}, \|f\|_{\Psi_1}^{\Psi_1}) \leq 1$.

In addition, it is clear that, for any $\alpha > -1$, $A_\alpha^\Psi \supset H^\Psi$ and $\|f\|_{\alpha,\Psi} \leq \|f\|_\Psi$ for any $f \in H^\Psi$. Therefore, letting α tend to -1 in Proposition 1.6, we get the following.

Proposition 1.10. *Let Ψ be a subharmonic-preserving growth function. For any $a \in \mathbb{B}_N$ and any $f \in H^\Psi$, one has*

$$|f(a)| \leq \Psi^{-1} \left(\left(\frac{1}{1-|a|} \right)^N \right) \|f\|_\Psi. \quad (1.28)$$

As a corollary, we have the following.

Corollary 1.11. *Let Ψ be a subharmonic-preserving growth function. H^Ψ is a complete metric space (with the equivalent distances induced by $\|\cdot\|_\Psi$ and $\|\cdot\|^\Psi$, as usual).*

For $a \in \mathbb{B}_N$ and $p > 0$, we introduce the “test” function $f_{a,p}$ defined for any $z \in \overline{\mathbb{B}_N}$ by

$$f_{a,p}(z) = \left(\frac{1-|a|}{1-\langle z, a \rangle} \right)^{2N/p}. \quad (1.29)$$

It is easily seen that $f_{a,p} \in H^\infty$ with $\|f_{a,p}\|_\infty = 1$ and that $|f_{a,p}(z)| \leq 1$ for any $z \in \mathbb{S}_N$. Moreover, let us observe that

$$|f_{a,p}(z)|^p = \left(\frac{1-|a|}{1+|a|} \right)^N P \left(|a|z, \frac{a}{|a|} \right), \quad (1.30)$$

so that $\|f_{a,p}\|_p = ((1-|a|)/(1+|a|))^{N/p}$. Therefore, if Ψ is a growth function of order p , we have, by Lemma 1.5,

$$\|f_{a,p}\|_\Psi \leq \frac{1}{\Psi^{-1}(1/\|f_{a,p}\|_p)} \leq \frac{1}{\Psi^{-1}((1/(1-|a|))^{N/p})}. \quad (1.31)$$

It is very convenient to see H^Ψ as a closed subspace of $L^\Psi(\mathbb{S}_N)$. When Ψ is an Orlicz function, this is possible thanks to the representation of any function in H^1 by the Poisson integral of its boundary values. This does not work any more in H^p with $0 < p < 1$, even in this case. However, using a radial maximal function, we can still see H^p as a subspace of L^p . We are going to extend this to H^Ψ for Ψ a growth function which preserves the subharmonicity. To this purpose, we recall the definition of the nonisotropic distance on $\overline{\mathbb{B}_N}$: for $(z, w) \in \overline{\mathbb{B}_N}$,

$$d(z, w) = \sqrt{|1 - \langle z, w \rangle|}. \quad (1.32)$$

It is well known that d is a distance on $\mathbb{S}_N$ and a pseudodistance on $\mathbb{B}_N$ [21, Paragraph 5.1]. It permits to define the Korányi approach region $\Gamma(\zeta)$ for $\zeta \in \mathbb{S}_N$:

$$\Gamma(\zeta) = \left\{ z \in \mathbb{B}_N, d(z, \zeta)^2 < 1 - |z|^2 \right\}. \quad (1.33)$$

Then the maximal function N_f of f , associated to Korányi approach region, is given by

$$N_f(\zeta) = \sup_{z \in \Gamma(\zeta)} |f(z)| \quad (1.34)$$

for any $\zeta \in \mathbb{S}_N$. [17, Theorem 1.3] will be very useful.

Theorem 1.12. *Let Ψ be a growth function. Then, for any $f \in H^\Psi$,*

$$\int_{\mathbb{S}_N} \Psi(|N_f|) d\sigma \lesssim \|f\|_\Psi. \quad (1.35)$$

In particular, a holomorphic function f belongs to H^Ψ if and only if N_f belongs to $L^\Psi(\mathbb{S}_N, \sigma)$.

From this theorem, we deduce the following one.

Theorem 1.13. *Let Ψ be a subharmonic-preserving growth function. Then for every $f \in H^\Psi$, one has*

- (1) $\lim_{r \rightarrow 1} \int_{\mathbb{S}_N} \Psi(|f^* - f_r|) d\sigma = 0$;
- (2) $\|f^*\|^\Psi = \|f\|_{H^\Psi}^\Psi$;
- (3) H^Ψ is separable. More precisely, the polynomials are dense in H^Ψ .

Proof. Let $M_{\text{rad}}f(\zeta) = \sup_{0 < r < 1} |f(r\zeta)|$ for $\zeta \in \mathbb{S}_N$. Obviously, $M_{\text{rad}}f \leq N_f$, hence $\int_{\mathbb{S}_N} \Psi(M_{\text{rad}}f) d\sigma \lesssim \|f\|_\Psi$ (Theorem 1.12). Since Ψ is concave and vanishes at 0, we have $\Psi(|f^* - f_r|) \leq \Psi(|f^*|) + \Psi(|f_r|) \leq 2\Psi(M_{\text{rad}}f)$. Now $\Psi(|f^* - f_r|(\zeta))$ tends to 0 as r goes to 1 for σ -almost every $\zeta \in \mathbb{S}_N$. By the dominated convergence theorem, (1) follows.

Then $\|f^*\|^\Psi = \lim_{r \rightarrow 1} \|f_r\|^\Psi$ and (2) comes from the subharmonicity of $\Psi(|f|)$.

We proved in (1) that f_r tends to f^* in H^Ψ for $\|\cdot\|^\Psi$ (hence for $\|\cdot\|_\Psi$ also). We approach every f_r uniformly on $\overline{\mathbb{B}_N}$ by its Taylor series to get the third assertion. $\square$

2. Carleson Embedding Theorems

2.1. Statements of the Results

For $\zeta \in \overline{\mathbb{B}_N}$ and $h \in]0, 1]$, we define the nonisotropic “ball” of $\mathbb{B}_N$ by

$$S(\zeta, h) = \left\{ z \in \mathbb{B}_N, d(\zeta, z)^2 < h \right\} \quad (2.1)$$

and its analogue in $\overline{\mathbb{B}_N}$ by

$$\mathcal{S}(\zeta, h) = \left\{ z \in \overline{\mathbb{B}_N}, d(\zeta, z)^2 < h \right\}. \quad (2.2)$$

Let us also denote by

$$Q(\zeta, h) = \mathcal{S}(\zeta, h) \cap \mathbb{S}_N, \quad (2.3)$$

the “true” balls in $\mathbb{S}_N$. We have $\sigma(Q(\zeta, h)) \approx h^N$ and $v_\alpha(S(\zeta, h)) \approx h^{N+\alpha+1}$ [22].

Let μ be a positive Borel measure on $\overline{\mathbb{B}_N}$ whose restriction to $\mathbb{S}_N$ is absolutely continuous with respect to σ and let $0 < p \leq q < \infty$. By definition, μ is a (q/p) -Carleson measure if $\mu(\mathcal{S}(\zeta, h)) \leq Ch^{Nq/p}$, while it is a vanishing (q/p) -Carleson measure if $\mu(\mathcal{S}(\zeta, h)) = o(h^{Nq/p})$ when h goes to 0. A variant of the well-known Carleson theorem for Hardy spaces [1, 3] ensures that the embedding $H^p \hookrightarrow L^q(\overline{\mathbb{B}_N}, \mu)$ is bounded (resp., compact) if and only if μ is a (q/p) -Carleson measure (resp., a vanishing (q/p) -Carleson measure).

Similarly, we define the $(\alpha, (q/p))$ -Bergman Carleson measures (resp., vanishing $(\alpha, (q/p))$ -Bergman Carleson measures) for weighted Bergman spaces by $\mu(\mathcal{S}(\zeta, h)) \leq Ch^{(N+\alpha+1)q/p}$ (resp., $\mu(\mathcal{S}(\zeta, h)) = o(h^{(N+\alpha+1)q/p})$). When $p = q$, we just speak about α -Bergman Carleson measures (resp., vanishing α -Bergman Carleson measures). Ueki [7] showed that $A_\alpha^p \hookrightarrow L^q(\mathbb{B}_N, \mu)$ is bounded (resp., compact) if and only if μ is a $(\alpha, (q/p))$ -Bergman Carleson measure (resp., a vanishing $(\alpha, (q/p))$ -Carleson measure).

In the context of Hardy-Orlicz spaces (resp., weighted Bergman-Orlicz spaces) smaller than H^1 (resp., A_α^1) (i.e., when the defining function Ψ is an Orlicz function), much general results were obtained in [9, 10] in the unit disc, and in [11, 12] in the unit ball.

For Hardy-Orlicz (resp., weighted Bergman-Orlicz) spaces larger than H^1 (resp., A_α^1), we state that the characterizations of the boundedness and compactness of $H^{\Psi_1} \hookrightarrow L^{\Psi_2}(\overline{\mathbb{B}_N}, \mu)$ (resp., $A_\alpha^{\Psi_1} \hookrightarrow L^{\Psi_2}(\mathbb{B}_N, \mu)$), where Ψ_1 and Ψ_2 are two growth functions such that $x \mapsto \Psi_2(x)/\Psi_1(x)$ is nondecreasing at least for large values of x (or equivalently $x \mapsto \Psi_2 \circ \Psi_1^{-1}(x)/x$ nondecreasing, since Ψ_1 is increasing and does vanish except in 0), only depend on the growth of $\Psi_2 \circ \Psi_1^{-1}$ at infinity.

Note that if $\Psi_i(x) = x^{p_i}$, $\Psi_i = \Phi^{p_i}$ (with Φ an Orlicz function), or $\Psi_i(x) = x^{p_i} \log^{q_i}(C+x)$ with $p_2 \geq p_1$, then $x \mapsto \Psi_2 \circ \Psi_1^{-1}(x)/x$ is nondecreasing.

Theorem 2.1. *Let Ψ_1 and Ψ_2 be two subharmonic-preserving growth functions such that $x \mapsto \Psi_2(x)/\Psi_1(x)$ is nondecreasing. Let μ be a finite positive Borel measure on $\overline{\mathbb{B}_N}$ (whose restriction to $\mathbb{S}_N$ is absolutely continuous with respect to σ). Then,*

- (1) H^{Ψ_1} embeds into $L^{\Psi_2}(\overline{\mathbb{B}_N}, \mu)$ if and only if there exists some $h_0 \in (0, 1)$ such that, for any $h \in (0, h_0)$,

$$\mu(\mathcal{S}(\zeta, h)) \lesssim \frac{1}{\Psi_2 \circ \Psi_1^{-1}(1/h^N)} \quad (2.4)$$

uniformly in $\zeta \in \mathbb{S}_N$,

- (2) the embedding $H^{\Psi_1} \hookrightarrow L^{\Psi_2}(\overline{\mathbb{B}_N}, \mu)$ is compact if and only if

$$\mu(\mathcal{S}(\zeta, h)) = o_{h \rightarrow 0} \left(\frac{1}{\Psi_2 \circ \Psi_1^{-1}(1/h^N)} \right) \quad (2.5)$$

uniformly in $\zeta \in \mathbb{S}_N$.

A measure μ which satisfies (2.4) (resp., (2.5)) will be called a (Ψ_1, Ψ_2) -Carleson measure (resp., a vanishing (Ψ_1, Ψ_2) -Carleson measure).

For big Bergman-Orlicz spaces, we have the following.

Theorem 2.2. Let Ψ_1 and Ψ_2 be two subharmonic-preserving growth functions such that $x \mapsto \Psi_2(x)/\Psi_1(x)$ is nondecreasing, and let $\alpha > -1$. Let also μ be a finite positive Borel measure on $\mathbb{B}_N$. Then,

- (1) $A_\alpha^{\Psi_1}$ embeds into $L^{\Psi_2}(\mathbb{B}_N, \mu)$ if and only if there exists $h_0 \in (0, 1)$ such that, for any $h \in (0, h_0)$,

$$\mu(S(\zeta, h)) \lesssim \frac{1}{\Psi_2 \circ \Psi_1^{-1}(1/h^{N+\alpha+1})}, \quad (2.6)$$

uniformly in $\zeta \in \mathbb{S}_N$,

- (2) the embedding $A_\alpha^{\Psi_1} \hookrightarrow L^{\Psi_2}(\mathbb{B}_N, \mu)$ is compact if and only if

$$\mu(S(\zeta, h)) \leq o_{h \rightarrow 0} \left(\frac{1}{\Psi_2 \circ \Psi_1^{-1}(1/h^{N+\alpha+1})} \right), \quad (2.7)$$

uniformly in $\zeta \in \mathbb{S}_N$.

A measure μ which satisfies (2.6) (resp., (2.7)) will be called a (α, Ψ_1, Ψ_2) -Bergman-Carleson measure (resp., a vanishing (α, Ψ_1, Ψ_2) -Bergman-Carleson measure).

Remark 2.3. By the closed graph theorem, the above embeddings are bounded as soon as they exist.

We immediately deduce from the previous theorems the following corollaries.

Corollary 2.4. Let $\alpha > -1$ and let $0 < p \leq q < 1$ and $\Psi_1 = \Phi^p$, $\Psi_2 = \Phi^q$ as in (3) of Example 1.3. Let μ be a finite positive Borel measure on $\overline{\mathbb{B}_N}$ (whose restriction to $\mathbb{S}_N$ is absolutely continuous with respect to σ) (resp., on $\mathbb{B}_N$). Then,

- (1) H^{Φ^p} (resp., $A_\alpha^{\Phi^p}$) embeds into $L^{\Phi^q}(\overline{\mathbb{B}_N}, \mu)$ (resp., $L^{\Phi^q}(\mathbb{B}_N, \mu)$) if and only if μ is a (q/p) -Carleson measure (resp., a $(\alpha, (q/p))$ -Bergman-Carleson measure),
- (2) the embedding $H^{\Phi^p} \hookrightarrow L^{\Phi^q}(\overline{\mathbb{B}_N}, \mu)$ (resp., $A_\alpha^{\Phi^p} \hookrightarrow L^{\Phi^q}(\mathbb{B}_N, \mu)$) is compact if and only if μ is a vanishing (q/p) -Carleson measure (resp., a vanishing $(\alpha, (q/p))$ Bergman-Carleson measure).

If $\Psi_1 = \Psi_2$ (or equivalently if Ψ_1 and Ψ_2 are equivalent), we have the following.

Corollary 2.5. Let $\alpha > -1$, and let Ψ be subharmonic-preserving growth function. Let μ be a finite positive Borel measure on $\overline{\mathbb{B}_N}$ (whose restriction to $\mathbb{S}_N$ is absolutely continuous with respect to σ) (resp., on $\mathbb{B}_N$). Then,

- (1) H^Ψ (resp., A_α^Ψ) embeds into $L^\Psi(\overline{\mathbb{B}_N}, \mu)$ (resp., $L^\Psi(\mathbb{B}_N, \mu)$) if and only if μ is a Carleson measure (resp., a α -Bergman-Carleson measure),
- (2) the embedding $H^\Psi \hookrightarrow L^\Psi(\overline{\mathbb{B}_N}, \mu)$ (resp., $A_\alpha^\Psi \hookrightarrow L^\Psi(\mathbb{B}_N, \mu)$) is compact if and only if μ is a vanishing Carleson measure (resp., a vanishing α -Bergman-Carleson measure).

2.2. Proofs of Theorems 2.1 and 2.2

For the compactness parts, we will use a criterion given in [12, Proposition 2.11] and [11, Proposition 2.8]. Its proof is easy to adapt as soon as we have checked that the convergence in H^Ψ (resp., A_α^Ψ), for Ψ a subharmonic-preserving growth function, implies the convergence on every compact subset of $\mathbb{B}_N$, but it stems from Proposition 1.10 (resp., Proposition 1.6).

Proposition 2.6. *Let $\alpha > -1$, let Ψ_1 and Ψ_2 be two subharmonic-preserving growth functions and let μ (resp., μ) be a finite positive Borel measure on $\overline{\mathbb{B}_N}$ (resp., $\mathbb{B}_N$) whose restriction to $\mathbb{S}_N$ is absolutely continuous with respect to σ . One assume that $j_\mu : H^{\Psi_1} \hookrightarrow L^{\Psi_2}(\overline{\mathbb{B}_N}, \mu)$ (resp., $j_{\mu_\alpha} : A_\alpha^{\Psi_1} \hookrightarrow L^{\Psi_2}(\mathbb{B}_N, \mu)$) is well defined (hence bounded).*

(1) *The two following assertions are equivalent:*

- (a) *the canonical embedding j_μ (resp., j_{μ_α}) is compact;*
- (b) *every sequence in the unit ball of H^{Ψ_1} (resp., $A_\alpha^{\Psi_1}$), which is convergent to 0 uniformly on every compact subset of $\mathbb{B}_N$, is convergent to 0 in $L^{\Psi_2}(\mu)$ (resp., $L^{\Psi_2}(\mu)$).*

(2) *If $\lim_{r \rightarrow 1^-} \|I_r\| = 0$ (resp., $\lim_{r \rightarrow 1^-} \|I_{r,\alpha}\| = 0$), where $I_r(f) = f \cdot \chi_{\overline{\mathbb{B}_N} \setminus r\mathbb{B}_N}$ (resp., $I_r(f) = f \cdot \chi_{\mathbb{B}_N \setminus r\mathbb{B}_N}$), then the canonical embedding j_μ (resp., j_{μ_α}) is compact.*

2.2.1. Proof of Theorem 2.1

We assume that the hypothesis of Theorem 2.1 is fulfilled. The proof will be based on two lemmas, whose proofs follow that of Theorem 2.4 and Lemma 2.6 of [12]. These results are refinement of Carleson theorem and are the key to deal with different Hardy-Orlicz spaces which are not classical Hardy spaces. We need to introduce the function K_μ , associated to μ by

$$K_\mu(h) = \sup_{0 < t < h} \frac{\sup_{\zeta \in \mathbb{S}_N} \mu(\mathcal{S}(\zeta, t))}{t^N}, \quad \text{for } h \in (0, 1), \quad (2.8)$$

so that μ is a Carleson measure if and only if $K_\mu(h)$ is finite for some $h \in (0, 1)$.

Theorem 2.7 (see [12, Theorem 2.4]). *There exist two constants $\tilde{C} > 0$ and $C > 1$ such that, for every f continuous on $\mathbb{B}_N$ which admits boundary values almost everywhere on $\mathbb{S}_N$, and every finite positive Borel measure μ on $\mathbb{B}_N$, one has*

$$\mu\left(\left\{z \in \overline{\mathbb{B}_N}, |z| > 1 - h, |f(z)| > t\right\}\right) \leq \tilde{C} K_\mu(Ch) \sigma(\{N_f > t\}), \quad (2.9)$$

for every $h \in (0, 1/C)$ and every $t > 0$.

The next lemma is a the technical key to obtain Carleson theorems in Hardy-Orlicz context. Its proof closely follows that of [12, Lemma 2.6], but we prefer to give the details.

Lemma 2.8. *Let μ be a finite positive Borel measure on $\overline{\mathbb{B}_N}$ and let Ψ_1 and Ψ_2 be two subharmonic-preserving growth functions. Let $C \geq 1$ be the constant appearing in Theorem 2.7. Assume that there exist $A > 0$, $\eta > 0$ and $h_A \in (0, 1/C)$ such that*

$$K_\mu(h) \leq \eta \frac{1/h^N}{\Psi_2(A\Psi_1^{-1}(1/h^N))}, \quad (2.10)$$

for every $h \in (0, h_A)$. Then, for every $f \in H^{\Psi_1}(\mathbb{B}_N)$ such that $\|f\|_{\Psi_1} \leq 1$ and every Borel subset E of $\overline{\mathbb{B}_N}$,

$$\int_E \Psi_2(|f|) d\mu \lesssim \mu(E) + \eta \int_{\mathbb{S}_N} \Psi_1(N_f) d\sigma, \quad (2.11)$$

where $\lesssim$ involves a constant which is independent of f , η , and E .

Proof. Let $f \in H^{\Psi_1}(\mathbb{B}_N)$ with $\|f\|_{\Psi_1} \leq 1$. With the notations of the statement of the lemma and using Proposition 1.10, the proof of [12, Lemma 2.6] directly yields.

$$\begin{aligned} \int_E \Psi_2\left(\frac{A}{2^{N+1}(C+1)C^{N-1}}|f|\right) d\mu &\leq \int_0^{x_A} \Psi_2'(s) \mu(E) ds \\ &+ \frac{\eta \tilde{C}}{C^N} \int_{x_A}^\infty \Psi_2'(s) \frac{\Psi_1(((C+1)C^{N-1})/A)S)}{\Psi_2(((C+1)/C)S)} \\ &\times \sigma\left(\left\{N_f > \frac{2^{N+1}(C+1)C^{N-1}}{A}s\right\}\right) ds, \end{aligned} \quad (2.12)$$

for some constant x_A which depends only on A . Now, since Ψ_1 and Ψ_2 are two concave growth functions, we have

$$\Psi_i(s) \lesssim \Psi_i(Ks) \lesssim \Psi_i(s), \quad (2.13)$$

where $\lesssim$ involve constants which only depend on K and Ψ_i . Therefore, (2.12) becomes

$$\int_E \Psi_2(|f|) d\mu \lesssim \mu(E) + \eta \int_{x_A}^\infty \Psi_2'(s) \frac{\Psi_1(s)}{\Psi_2(s)} \sigma\left(\left\{N_f > \frac{2^{N+1}(C+1)C^{N-1}}{A}s\right\}\right) ds. \quad (2.14)$$

Since $x \mapsto \Psi_2(x)/\Psi_1(x)$ is nondecreasing and since Ψ_1 and Ψ_2 and their derivatives do not vanish except in 0, we have, for any $s > 0$,

$$\frac{\Psi_1'(s)}{\Psi_2'(s)} \leq \frac{\Psi_1(s)}{\Psi_2(s)}. \quad (2.15)$$

It follows that

$$\begin{aligned}
\int_E \Psi_2(|f|) d\mu &\lesssim \mu(E) + \eta \int_{x_A}^{\infty} \Psi'_1(s) \sigma \left(\left\{ N_f > \frac{2^{N+1}(C+1)C^{N-1}}{A} s \right\} \right) ds \\
&\lesssim \mu(E) + \eta \int_0^{+\infty} \Psi_1 \left(\left\{ \frac{AN_f}{2^{N+1}(C+1)C^{N-1}} > s \right\} \right) ds \\
&= \mu(E) + \eta \int_{\mathbb{S}_N} \Psi_1 \left(\frac{AN_f}{2^{N+1}(C+1)C^{N-1}} \right) d\sigma \\
&\lesssim \mu(E) + \eta \int_{\mathbb{S}_N} \Psi_1(N_f) d\sigma.
\end{aligned} \tag{2.16}$$

□

We now can prove Theorem 2.1.

Proof of Theorem 2.1. (1) We assume that μ is a (Ψ_1, Ψ_2) -Carleson measure, and we intend to show that $j_\mu : H^{\Psi_1} \hookrightarrow L^{\Psi_2}(\overline{\mathbb{B}_N}, \mu)$ is well defined (hence bounded). We observe that (2.10) is satisfied for $A = \eta = 1$. Indeed we have, for any $h \in (0, h_0)$,

$$\frac{\mu(\mathcal{S}(\zeta, h))}{h^N} \lesssim \frac{1/h^N}{\Psi_2 \circ \Psi_1^{-1}(1/h^N)}, \tag{2.17}$$

uniformly in $\zeta \in \mathbb{S}_N$. Now, $x \mapsto \Psi_2(x)/\Psi_1(x)$ is nondecreasing (at least for large x) so that we can find $h_1 \in (0, \min(1/C, h_0))$ such that, for any $h \in (0, h_1)$,

$$K_\mu(h) \lesssim \frac{1/h^N}{\Psi_2 \circ \Psi_1^{-1}(1/h^N)}. \tag{2.18}$$

Therefore, we may and will apply Lemma 2.8 with $E = \overline{\mathbb{B}_N}$ to get

$$\int_{\overline{\mathbb{B}_N}} \Psi_2(|f|) d\mu \lesssim \mu(E) + \int_{\mathbb{S}_N} \Psi_1(N_f) d\sigma. \tag{2.19}$$

We finish the proof of (1) using Theorem 1.12 and Inequality (1.7):

$$\int_{\overline{\mathbb{B}_N}} \Psi_2(|f|) d\mu \lesssim \mu(E) + \|f\|_{\Psi_1}. \tag{2.20}$$

For the converse, let $g_{a,p} = \Phi^{-1}((1/(1-|a|))^{N/p})f_{a,p}$, where $f_{a,p}$ is the test function introduced at the end of Section 1.2.2, with p such that Ψ_1 is a growth function of order p . According to (1.31), $g_{a,p}$ lies in the unit ball of H^{Ψ_1} . Let C be the (finite) norm of j_μ . Using a

classical computation which gives that $|1 - \langle z, a \rangle| \leq 2(1 - |a|)$ for any $z \in \mathcal{S}(a/|a|, 1 - |a|)$, we get $|f_{a,p}(z)| \geq 1/4^{N/p}$ for such z , hence

$$1 \geq \int_{\overline{\mathbb{B}_N}} \Psi_2 \left(\frac{|g_{a,p}|}{C} \right) d\mu \geq \mu \left(\mathcal{S} \left(\frac{a}{|a|}, 1 - |a| \right) \right) \Psi_2 \left(\frac{1}{4^{N/p} C} \Psi_1^{-1} \left(\left(\frac{1}{1 - |a|} \right)^{N/p} \right) \right) \quad (2.21)$$

for any $a \in \mathbb{B}_N$. We may assume that $C \geq 1$ and since Ψ_2 is concave and vanishes at 0, it follows that

$$\mu \left(\mathcal{S} \left(\frac{a}{|a|}, 1 - |a| \right) \right) \leq 4^{N/p} C \frac{1}{\Psi_2 \circ \Psi_1^{-1} (1/(1 - |a|)^N)}. \quad (2.22)$$

(2) We turn to the compactness part. We first prove the sufficient part. Let us assume that $x \mapsto \Psi_2(x)/\Psi_1(x)$ is nondecreasing and that μ is a vanishing (Ψ_1, Ψ_2) -Carleson measure. In particular, μ is a (Ψ_1, Ψ_2) -Carleson measure and we may apply Proposition 2.6. Then it is sufficient to prove that for every $\varepsilon > 0$, there exists r close enough to 1, such that $\|I_r\| < \varepsilon$ (where $I_r : H^{\Psi_1} \rightarrow L^{\Psi_2}(\overline{\mathbb{B}_N} \setminus r\mathbb{B}_N, \mu)$). We fix $\varepsilon > 0$ and let f be in the unit ball of H^{Ψ_2} . As in the proof of the sufficient part of (1), since $x \mapsto \Psi_2(x)/\Psi_1(x)$ is nondecreasing, μ is a vanishing (Ψ_1, Ψ_2) -Carleson measure implying that

$$K_\mu(h) \lesssim \varepsilon \frac{1/h^N}{\Psi_2 \circ \Psi_1^{-1}(1/h^N)}, \quad (2.23)$$

for h small enough. Then, we apply Lemma 2.8 with $\eta = \varepsilon$ and $E = \overline{\mathbb{B}_N} \setminus r\mathbb{B}_N$ to get the existence of r close enough to 1 (independent of f), such that

$$\begin{aligned} \|I_r(f)\|_{\Psi_2} &\lesssim \mu(\overline{\mathbb{B}_N} \setminus r\mathbb{B}_N) + \varepsilon \int_{\mathbb{S}_N} \Psi_1(N_f) d\sigma \\ &\lesssim \mu(\overline{\mathbb{B}_N} \setminus r\mathbb{B}_N) + \varepsilon \|f\|_{\Psi_1}, \end{aligned} \quad (2.24)$$

for any $\varepsilon > 0$. Now, we may argue as in [12, Lemma 2.13] to prove that, under the assumption that μ is a (Ψ_1, Ψ_2) -Carleson measure, $\mu(\mathbb{S}_N) = 0$. Hence

$$\limsup_{r \rightarrow 1} \|I_r(f)\|_{\Psi_2} \lesssim \varepsilon \|f\|_{\Psi_1}, \quad (2.25)$$

for any $\varepsilon > 0$, which gives (2).

For the converse, we assume that j_μ is compact but that μ is not a vanishing (Ψ_1, Ψ_2) -Carleson measure, that is there exist $\varepsilon_0 > 0$, a sequence $(h_n)_n$ decreasing to 0 and a sequence $(\zeta_n)_n \subset \mathbb{S}_N$ such that

$$\mu(\mathcal{S}(\zeta_n, h_n)) \geq \varepsilon_0 \frac{1}{\Psi_2 \circ \Psi_1^{-1}(1/h_n^N)}. \quad (2.26)$$

Now, let $g_{a_n,p} = \Phi^{-1}((1/(1-|a_n|))^{N/p})f_{a_n,p}$ be as in the proof of the necessity in the boundedness part with $a_n = (1-h_n)\zeta_n$ and p such that Ψ_1 is growth function of order p . Since $(g_{a_n,p})_n$ is bounded in H^{Ψ_1} and converges uniformly on every compact subset of $\mathbb{B}_N$, then we must have $\|j_\mu(g_{a_n,p})\|_{\Psi_2} \xrightarrow{n \rightarrow \infty} 0$, because of Proposition 2.6. But, for $z \in \mathcal{S}(\zeta_n, h_n)$, we saw that

$$|g_{a_n,p}(z)| \geq \frac{1}{4^{N/p}} \Psi_1^{-1}\left(\frac{1}{h_n^N}\right). \quad (2.27)$$

Since Ψ_2 is a growth function of order, let say q , we have, for any $t \in (0, 1)$, $\Psi_2(y/t) \geq \Psi_2(y)/t^q$. Hence, we get

$$\begin{aligned} \int_{\mathbb{B}_N} \Psi_2\left(\frac{4^{Nq/p}|g_{a_n,p}|}{\varepsilon_0^{1/q}}\right) d\mu &\geq \Psi_2\left(\frac{1}{\varepsilon_0^{1/q}} \Psi_1^{-1}\left(1/h_n^N\right)\right) \mu(\mathcal{S}(\zeta_n, h_n)) \\ &\geq \frac{\Psi_2 \circ \Psi_1^{-1}(1/h_n^N)}{\varepsilon_0} \varepsilon_0 \frac{1}{\Psi_2 \circ \Psi_1^{-1}(1/h_n^N)} \\ &= 1. \end{aligned} \quad (2.28)$$

Therefore, $\|j_\mu(g_{a_n,p})\|_{\Psi_2} \geq \varepsilon_0^{1/q}/4^{Nq/p}$, which is a contradiction. $\square$

2.2.2. Proof of Theorem 2.2

We introduce the following function:

$$K_{\mu,\alpha}(h) = \sup_{0 < t < h} \frac{\sup_{\zeta \in \mathbb{S}_N} \mu(\mathcal{S}(\zeta, t))}{t^{N+\alpha+1}}. \quad (2.29)$$

As for Hardy-Orlicz spaces, we need a refinement of Carleson theorem which allows to deal with weighted Bergman-Orlicz spaces associated to different growth functions. It is [11, Theorem 2.3].

Theorem 2.9. *There exists a constant $\tilde{C} > 0$ such that, for every f continuous on $\mathbb{B}_N$ and every positive finite Borel measure μ on $\mathbb{B}_N$, one has*

$$\mu(\{z \in \mathbb{B}_N, |z| > 1-h, |f(z)| > t\}) \leq \tilde{C} K_{\mu,\alpha}(2h) v_\alpha(\{\Lambda_f > t\}), \quad (2.30)$$

for every $h \in (0, 1/2)$ and every $t > 0$.

Our Carleson theorem for large weighted Bergman-Orlicz spaces is a consequence of the next technical lemma. Its proof is an easy adaptation and combination of the proofs of Lemma 2.8 and [11, Lemma 2.4], so we omit it.

Lemma 2.10. *Let μ be a finite positive Borel measure on $\mathbb{B}_N$ and let Ψ_1 and Ψ_2 be two subharmonic-preserving growth functions. Assume that there exist $A > 0$, $\eta > 0$, and $h_A \in (0, 1/2)$ such that*

$$K_{\mu,\alpha}(h) \leq \eta \frac{1/h^{N+1+\alpha}}{\Psi_2(A\Psi_1^{-1}(1/h^{N+1+\alpha}))}, \quad (2.31)$$

for every $h \in (0, h_A)$. Then, for every $f \in A_\alpha^{\Psi_1}(\mathbb{B}_N)$ with $\|f\|_{A_\alpha^{\Psi_1}} \leq 1$ and every Borel subset E of $\mathbb{B}_N$, one has

$$\int_E \Psi_2(|f|) d\mu \lesssim \mu(E) + \eta \int_{\mathbb{B}_N} \Psi_1(\Lambda_f) dv_\alpha, \quad (2.32)$$

where $\lesssim$ involves constants which are independent of f , η , and E .

We now prove Theorem 2.2.

Proof of Theorem 2.2. Let Ψ_1 and Ψ_2 be two subharmonic-preserving growth functions such that $x \mapsto \Psi_2(x)/\Psi_1(x)$ is nondecreasing.

(1) We first assume that μ is a (α, Ψ_1, Ψ_2) -Bergman Carleson measure, which implies, as in the beginning of the proof of (1) of Theorem 2.1, that condition (2.31) with $A = \eta = 1$ is fulfilled. Applying (2.31) to f , with $E = \mathbb{B}_N$, gives

$$\int_{\mathbb{B}_N} \Psi_2(|f|) d\mu \lesssim \mu(E) + \int_{\mathbb{B}_N} \Psi_1(\Lambda_f) dv_\alpha. \quad (2.33)$$

We conclude the proof using Proposition 1.9 together with (1.7).

To prove the converse, we argue as in the proof of the corresponding part of Theorem 2.1, using the test function $g_{a,\alpha,p} = \Phi^{-1}((1/(1-|a|))^{(N+\alpha+1)/p})f_{a,\alpha,p}$, where $f_{a,\alpha,p}$ has been introduced in Section 1.2.1, with p such that Ψ_1 is a growth function of order p .

(2) The proof of the compactness part of Theorem 2.2 is still similar to that for Hardy-Orlicz spaces: we apply Lemma 2.10 to $f \in A_\alpha^{\Psi_1}$, with $\eta = \varepsilon$ and $E = \mathbb{B}_N \setminus r\mathbb{B}_N$ to show that $\|f \cdot \chi_{\mathbb{B}_N \setminus r\mathbb{B}_N}\|_{\Psi_2}$ tends to 0, as r tends to 1. Proposition 2.6 then shows that $A_\alpha^{\Psi_1} \hookrightarrow L^{\Psi_2}(\mu)$ is compact.

For the converse, we proceed as for Theorem 2.1 using the test function introduced in the boundedness part above. $\square$

3. Applications to Composition Operators

Theorems 2.1 and 2.2 will be used to characterize both boundedness and compactness of composition operators on H^Ψ and A_α^Ψ . As usual, for $\phi : \mathbb{B}_N \rightarrow \mathbb{B}_N$ holomorphic, C_ϕ will be seen as an embedding operator. To do so, we define the pullback measures μ_ϕ and $\mu_{\phi,\alpha}$ of, respectively, σ and v_α under ϕ :

$$\mu_\phi(E) = \sigma(\phi^{*-1}(E) \cap \mathbb{S}_N), \quad (3.1)$$

where ϕ^* is the radial limit almost everywhere of ϕ and E is any Borel subset of $\overline{\mathbb{B}_N}$. And

$$\mu_{\phi,\alpha}(E) = v_\alpha(\phi^{-1}(E)) \quad (3.2)$$

for any Borel subset E of $\mathbb{B}_N$.

By a classical formula for pull-back measures,

$$\|j_{\mu_\phi}(f)\|_\Psi = \|C_\phi(f)\|_\Psi \left(\text{resp.}, \|j_{\mu_{\phi,\alpha}}(f)\|_{\alpha,\Psi} = \|C_\phi(f)\|_{\alpha,\Psi} \right) \quad (3.3)$$

for any growth function Ψ and every polynomial f (resp., for any $f \in A_\alpha^\Psi$). Then we extend this equality to the whole space H^Ψ by density of polynomials in H^Ψ (Theorem 1.13) and using Cauchy's formula.

Remark 3.1. Observe that it is not true that $j_{\mu_\phi}(f) = C_\phi(f)$ for any $f \in H^\Psi$ but it is for any polynomial f . Such an equality holds for every function in A_α^Ψ .

Moreover, we have the following criterion for compactness of composition operators.

Proposition 3.2. *Let $\alpha > -1$ and let Ψ_1 and Ψ_2 be as in Proposition 2.6. Let also $\phi : \mathbb{B}_N \rightarrow \mathbb{B}_N$ be holomorphic. C_ϕ is compact from H^{Ψ_1} into H^{Ψ_2} (resp., $A_\alpha^{\Psi_1}$ into $A_\alpha^{\Psi_2}$) if and only if, for every bounded sequence $(f_n)_n \subset H^{\Psi_1}$ (resp., $(f_n)_n \subset A_\alpha^{\Psi_1}$) which converges to 0 on every compact of $\mathbb{B}_N$, $\|C_\phi(f)\|_{\Psi_2}$ (resp., $\|C_\phi(f)\|_{\Psi_2,\alpha}$) tends to 0.*

The proof of this proposition is quite similar to that of Proposition 2.6.

Therefore, Theorems 2.1 and 2.2 together with Propositions 2.6 and 3.2 give the following characterizations of the boundedness and compactness of C_ϕ on H^Ψ and A_α^Ψ .

Theorem 3.3. *Let Ψ_1 and Ψ_2 be two subharmonic-preserving growth functions such that $x \mapsto \Psi_2(x)/\Psi_1(x)$ is nondecreasing. Let also $\phi : \mathbb{B}_N \rightarrow \mathbb{B}_N$ be holomorphic. Then,*

- (1) C_ϕ is bounded from H^{Ψ_1} into H^{Ψ_2} if and only if μ_ϕ is a (Ψ_1, Ψ_2) -Carleson measure,
- (2) C_ϕ is compact from H^{Ψ_1} into H^{Ψ_2} if and only if μ_ϕ is a vanishing (Ψ_1, Ψ_2) -Carleson measure.

For Bergman-Orlicz spaces, we have the following.

Theorem 3.4. *Let $\alpha > -1$ and let Ψ_1 and Ψ_2 be two subharmonic-preserving growth functions such that $x \mapsto \Psi_2(x)/\Psi_1(x)$ is nondecreasing. Let also $\phi : \mathbb{B}_N \rightarrow \mathbb{B}_N$ be holomorphic. Then,*

- (1) C_ϕ is bounded from $A_\alpha^{\Psi_1}$ into $A_\alpha^{\Psi_2}$ if and only if $\mu_{\phi,\alpha}$ is a (α, Ψ_1, Ψ_2) -Bergman Carleson measure,
- (2) C_ϕ is compact from $A_\alpha^{\Psi_1}$ into $A_\alpha^{\Psi_2}$ if and only if $\mu_{\phi,\alpha}$ is a vanishing (α, Ψ_1, Ψ_2) -Bergman Carleson measure.

We immediately deduce the following corollary, which reminds a similar result when the function Ψ is an Orlicz function satisfying the Δ_2 -Condition (see [11, 12]).

Corollary 3.5. *Let $\alpha > -1$ and let Ψ be a subharmonic preserving growth function. Let also $\phi : \mathbb{B}_N \rightarrow \mathbb{B}_N$ be holomorphic. Then,*

- (1) C_ϕ is bounded (resp., compact) on H^Ψ if and only if it is bounded (resp., compact) on one (or equivalently every) H^p , if and only if μ_ϕ is a Carleson measure (resp., a vanishing Carleson measure),
- (2) C_ϕ is bounded (resp., compact) on A_α^Ψ if and only if it is bounded (resp., compact) on one (or equivalently every) A_α^p , if and only if μ_ϕ is a Bergman-Carleson measure (resp., a vanishing Bergman-Carleson measure).

The following corollary shows that the behavior of composition operators between certain classes of different Hardy-Orlicz or Bergman-Orlicz spaces is still the same as that in the classical cases (see [23]).

Corollary 3.6. *Let $\alpha > -1$, let $0 < p \leq q < \infty$ and $\Psi_1 = \Phi^p$, and $\Psi_2 = \Phi^q$ as in (3) of Example 1.3. Let also $\phi : \mathbb{B}_N \rightarrow \mathbb{B}_N$ be holomorphic. Then,*

- (1) C_ϕ is bounded (resp., compact) from H^{Ψ_1} to H^{Ψ_2} if and only if it is bounded (resp., compact) from H^p to H^q , if and only if μ_ϕ is a q/p -Carleson measure (resp., a vanishing q/p -Carleson measure),
- (2) C_ϕ is bounded (resp., compact) from $A_\alpha^{\Psi_1}$ to $A_\alpha^{\Psi_2}$ if and only if it is bounded (resp., compact) from A_α^p to A_α^q , if and only if μ_ϕ is a q/p -Bergman-Carleson measure (resp., a vanishing q/p -Bergman-Carleson measure).

Acknowledgments

The authors would like to thank Ueki for providing his paper [7]. The second author acknowledges support from the Irish Research Council for Science, Engineering and Technology.

References

- [1] L. Carleson, "Interpolations by bounded analytic functions and the corona problem," *Annals of Mathematics. Second Series*, vol. 76, pp. 547–559, 1962.
- [2] L. Hörmander, " L^p estimates for (pluri-) subharmonic functions," *Mathematica Scandinavica*, vol. 20, pp. 65–78, 1967.
- [3] S. C. Power, "Hörmander's Carleson theorem for the ball," *Glasgow Mathematical Journal*, vol. 26, no. 1, pp. 13–17, 1985.
- [4] P. L. Duren, "Extension of a theorem of Carleson," *Bulletin of the American Mathematical Society*, vol. 75, pp. 143–146, 1969.
- [5] J. A. Cima and W. R. Wogen, "A Carleson measure theorem for the Bergman space on the ball," *Journal of Operator Theory*, vol. 7, no. 1, pp. 157–165, 1982.
- [6] D. Luecking, "A technique for characterizing Carleson measures on Bergman spaces," *Proceedings of the American Mathematical Society*, vol. 87, no. 4, pp. 656–660, 1983.
- [7] S.-I. Ueki, "Weighted composition operators between weighted Bergman spaces in the unit ball of $\mathbb{C}^n$," *Nihonkai Mathematical Journal*, vol. 16, no. 1, pp. 31–48, 2005.
- [8] J. H. Shapiro, *Composition Operators and Classical Function Theory*, Universitext: Tracts in Mathematics, Springer, New York, NY, USA, 1993.
- [9] P. Lefèvre, D. Li, H. Queffélec, and L. Rodríguez-Piazza, "Composition operators on Bergman-Orlicz spaces," hal-00426831, 2009.
- [10] P. Lefèvre, D. Li, H. Queffélec, and L. Rodríguez-Piazza, "Composition operators on Hardy-Orlicz spaces," *Memoirs of the American Mathematical Society*, vol. 207, no. 974, 2010.

- [11] S. Charpentier, "Composition operators on weighted Bergman-Orlicz spaces on the ball," to appear in *Complex Analysis and Operator Theory*.
- [12] S. Charpentier, "Composition operators on Hardy-Orlicz spaces on the ball," *Integral Equations and Operator Theory*, vol. 70, no. 3, pp. 429–450, 2011.
- [13] S. Charpentier, "Compact composition operators on the Hardy-Orlicz and weighted Bergman-Orlicz spaces on the ball," to appear in *The Journal of Operator Theory*.
- [14] P. Lefèvre, D. Li, H. Queffélec, and L. Rodríguez-Piazza, "Some revisited results about composition operators on Hardy spaces," *Revista Matemática Iberoamericana*, vol. 28, no. 1, pp. 57–76, 2012.
- [15] B. F. Sehba, " p -Carleson measures for a class of Hardy-Orlicz spaces," *International Journal of Mathematics and Mathematical Sciences*, vol. 2011, Article ID 926527, 15 pages, 2011.
- [16] B. R. Choe, H. Koo, and W. Smith, "Carleson measures for the area Nevanlinna spaces and applications," *Journal d'Analyse Mathématique*, vol. 104, pp. 207–233, 2008.
- [17] A. Bonami and S. Grellier, "Hankel operators and weak factorization for Hardy-Orlicz spaces," *Colloquium Mathematicum*, vol. 118, no. 1, pp. 107–132, 2010.
- [18] A. Hartmann, "Pointwise multipliers in Hardy-Orlicz spaces, and interpolation," *Mathematica Scandinavica*, vol. 106, no. 1, pp. 107–140, 2010.
- [19] M. A. Krasnosel'kiĭ and Ja. B. Rutickiĭ, *Convex Functions and Orlicz spaces*, P. Noordhoff Ltd., Groningen, The Netherlands, 1961.
- [20] M. M. Rao and Z. D. Ren, *Theory of Orlicz Spaces*, vol. 146 of *Monographs and Textbooks in Pure and Applied Mathematics*, Marcel Dekker, New York, NY, USA, 1991.
- [21] W. Rudin, *Function Theory in the Unit Ball of $\mathbb{C}^n$* , Classics in Mathematics, Springer, Berlin, Germany, 2008.
- [22] K. Zhu, *Spaces of Holomorphic Functions in the Unit Ball*, vol. 226 of *Graduate Texts in Mathematics*, Springer, New York, NY, USA, 2005.
- [23] C. C. Cowen and B. D. MacCluer, *Composition Operators on Spaces of Analytic Functions*, Studies in Advanced Mathematics, CRC Press, Boca Raton, Fla, USA, 1995.