**Research Article**

**A Characterization of Some Function Classes**

**M. T. Karaev**

*Isparta Vocational School, Suleyman Demirel University, 32260 Isparta, Turkey*

Correspondence should be addressed to M. T. Karaev, mubariztapdigoglu@sdu.edu.tr

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We give in terms of Berezin symbols a characterization of Hardy and Besov classes with a variable exponent.

1. **Introduction and Notations**

In his book [1, page 96], Pavlović proved the following characterization of functions belonging to the classical Hardy space:

\[ H^1 = H^1(\mathbb{D}) := \left\{ f \in \text{Hol}(\mathbb{D}) : \|f\|_1 = \sup_{0 < r < 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})| \, dt < \infty \right\}, \tag{1.1} \]

where \( \mathbb{D} := \{ z \in \mathbb{C} : |z| < 1 \} \) is the unit disc of the complex plain \( \mathbb{C} \).

**Theorem A.** For a function \( f \) analytic in \( \mathbb{D} \), the following assertions are equivalent:

(a) \( f \in H^1; \)

(b) \( \sup_n (1/a_n) \sum_{j=0}^n (1/(j+1)) \|s_j(f)\|_1 < \infty; \)

(c) \( \sup_n \|P_n f\|_1 < \infty. \)

Here, \( P_n f = (1/a_n) \sum_{j=0}^n (1/(j+1)) s_j(f), \) where \( a_n = \sum_{j=0}^n (1/(j+1)) (n = 0, 1, 2, \ldots) \) and \( s_j(f) \) are the partial sums of the Taylor series of \( f. \)

Recently, Popa [2] gave some generalization of this result by proving a similar characterization of upper triangular trace class matrices.
In the present paper, we give in terms of the so-called Berezin symbols a new characterization of analytic functions belonging to the Hardy class $H^p(\mathbb{D})$ and Besov class $B_p(\mathbb{D})$ with a variable exponent. Our results are new even for the usual Hardy and Besov spaces $H^p$ and $B_p$.

Recall that the Hardy space $H^p = H^p(\mathbb{D})$ $(1 \leq p < \infty)$ is the collection of holomorphic functions in $\mathbb{D}$ which satisfy the inequality

$$\|f\|_{H^p} := \left(\sup_{0 < r < 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p r \, d\theta\right)^{1/p} < \infty. \quad (1.2)$$

Let $dA(z)$ be the area measure on $\mathbb{D}$ normalized so that the area of $\mathbb{D}$ is 1. In rectangular and polar coordinates,

$$dA(z) = \frac{1}{\pi} dx \, dy = \frac{1}{\pi} r \, dr \, d\theta. \quad (1.3)$$

For $1 < p < +\infty$, the Besov space $B_p = B_p(\mathbb{D})$ is defined to be the space of analytic functions $f$ in such that

$$\|f\|_{B_p} := \left(\int_{\mathbb{D}} \left(1 + |z|^2\right)^p |f'(z)|^p \, d\lambda(z)\right)^{1/p} < \infty, \quad (1.4)$$

where

$$d\lambda(z) = \frac{dA(z)}{(1 - |z|^2)^2} \quad (1.5)$$

is the Möbius invariant measure on $\mathbb{D}$. We refer to Duren [3] and Zhu [4] for the theory of these spaces.

Let $T = \partial \mathbb{D}$, and let $p = p(t)$, $t \in T$, be a bounded, positive, measurable function defined on it. Following by Kokilashvili and Paatashvili [5, 6], we say that the analytic in the disc $\mathbb{D}$ function $f$ belongs to the Hardy class $H^{p(t)}$ if

$$\sup_{0 < r < 1} \int_0^{2\pi} |f(re^{i\theta})|^p r \, d\theta = C < +\infty, \quad (1.6)$$

where $p(\theta) = p(e^{i\theta})$, $\theta \in [0, 2\pi)$.

For $p(\theta) = p = \text{const} > 0$, the $H^{p(t)}$ class coincides with the classical Hardy class $H^p$.

Analogously, we say that the analytic in $\mathbb{D}$ function $f$ belongs to the Besov class $B_{p(t)}$ with a variable exponent if

$$\int_0^{2\pi} \int_0^1 \left(1 - r^2\right)^{p(t)} |f'(re^{i\theta})|^p r \, dr \frac{dt}{\pi} < +\infty, \quad (1.7)$$

where $p(t) = p(e^{i\theta})$, $t \in [0, 2\pi)$.

For $p(t) = p = \text{const} > 0$, the $B_{p(t)}$ class coincides with the Besov class $B_p$. 
Suppose that \( p := \inf_{t \in T} p(t) \), \( \overline{p} := \sup_{t \in T} p(t) \). If \( p > 0 \), then it is obvious that

\[
H^\overline{p} \subset H^{p(\cdot)} \subset H^p,
\]

\[
B^\overline{p} \subset B_{p(\cdot)} \subset B_p.
\]

Recall that for any bounded linear operator \( A \) acting in the functional Hilbert space \( H = H(Ω) \) over some set \( Ω \) with reproducing kernel \( k_1(z) \), its Berezin symbol \( \tilde{A} \) is defined by

\[
\tilde{A}(\lambda) := \left\langle A\tilde{k}_1, \tilde{k}_1 \right\rangle \quad (\lambda \in Ω),
\]

where \( \tilde{k}_1 := k_1/\|k_1\| \) is the normalized reproducing kernel of \( H \). (We mention [4, 7–11] as references for the Berezin symbols.)

2. On the Membership of Functions in Hardy and Besov Classes with a Variable Exponent

In this section, we characterize the function classes \( H^{p(\cdot)} \) and \( B_{p(\cdot)} \) in terms of the Berezin symbols.

For any bounded sequence \( \{a_n\}_{n \geq 0} \) of complex numbers \( a_n \), let \( D_{\{a_n\}} \) denote the associate diagonal operator acting in the Hardy space \( H^2 \) by the formula

\[
D_{\{a_n\}} z^n = a_n z^n, \quad n = 0, 1, 2, \ldots
\]

(2.1)

It is known that the reproducing kernel of the Hardy space \( H^2 \) has the form \( k_1(z) = 1/(1 - \overline{\lambda}z) \quad (\lambda \in \mathbb{D}) \). Then, it is easy to show that (see [11])

\[
\tilde{D}_{\{a_n\}}(\lambda) = \left(1 - |\lambda|^2\right)^\infty \sum_{k=0}^\infty a_n |\lambda|^{2k} \quad (\lambda \in \mathbb{D}),
\]

(2.2)

that is, the Berezin symbol of the diagonal operator \( D_{\{a_n\}} \) on the Hardy space \( H^2 \) is a radial function.

Note that the inequality \( |\hat{f}(n)| \leq \text{const}, n \geq 0 \), is the necessary condition for the function \( f(z) = \sum_{n=0}^\infty \hat{f}(n) z^n \) to be in the spaces \( H^p (1 \leq p \leq \infty) \). Also note that if \( f \in B_p \), then \( \tilde{f}(n) = O(n^{-1/p}) \quad (p \geq 1) \) (see, for instance, Duren [3] and Zhu [4]).

Our main result is the following theorem.

**Theorem 2.1.** Let \( f(z) = \sum_{n=0}^\infty \hat{f}(n) z^n \in \text{Hol}(\mathbb{D}) \) be a function with the bounded sequence \( \{\hat{f}(n)\}_{n \geq 0} \) of Taylor coefficients \( \hat{f}(n) = f^{(n)}/n! \quad (n = 0, 1, 2, \ldots) \). Then, the following are true:

(a) \( f \in H^{p(\cdot)} \) if and only if

\[
\sup_{0 < r < 1} \frac{1}{2\pi} \int_0^{2\pi} \frac{\left| \tilde{D}_{\{f(n)e^{it}\}}(\sqrt{r}) \right|^{p(\tau)}}{(1 - r)^{p(\tau)}} \, dt < +\infty;
\]

(2.3)
(b) If, in addition, \( f(n) = O(n^{-1}) \) as \( n \to \infty \), then \( f \in B_p(\ell) \) if and only if
\[
\int_0^{2\pi} \int_0^1 |\tilde{D}_{\{(n+1)\hat{f}(n+1)e^{int}\}}(\sqrt{r})|^p(t) \frac{(1 + r)^{p(t)-2}}{(1 - r)^2} r \, dr \, dt < +\infty. \tag{2.4}
\]

Proof. Indeed, by using the concept of Berezin symbols and formula (2.2), let us rewrite the function \( f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n \in \text{Hol}(\mathbb{D}) \) as follows:
\[
f(z) = f(re^{it}) = \sum_{n=0}^{\infty} \hat{f}(n)(re^{it})^n = \sum_{n=0}^{\infty} \hat{f}(n)e^{int}r^n
= \frac{(1 - r)\sum_{n=0}^{\infty} \hat{f}(n)e^{int}r^n}{1 - r} = \frac{\tilde{D}_{\{(n)\hat{f}(n)e^{int}\}}(\sqrt{r})}{1 - r},
\]
thus
\[
f(z) = \frac{\tilde{D}_{\{(n)\hat{f}(n)e^{int}\}}(\sqrt{r})}{1 - r} \tag{2.6}
\]
for every \( z = re^{it} \in \mathbb{D} \), where, as usual, \( r = |z| \) and \( t = \arg(z) \). Now, assertion (a) is immediate from the definition of considering space and formula (2.6).

Let us prove (b). Indeed, it follows from the condition \( \hat{f}(n) = O(n^{-1}) \) that the diagonal operator \( D_{\{(n+1)\hat{f}(n+1)e^{int}\}} \) is bounded in \( H^2 \) (and hence \( D_{\{(n+1)\hat{f}(n+1)e^{int}\}} \) is bounded for every fixed \( t \in [0,2\pi) \)). Then, we have
\[
f'(z) = \left( \sum_{n=0}^{\infty} \hat{f}(n)z^n \right)' = \sum_{n=1}^{\infty} n\hat{f}(n)z^{n-1}
= \sum_{n=0}^{\infty} (n + 1)\hat{f}(n+1)z^n = \sum_{n=0}^{\infty} (n + 1)\hat{f}(n+1)e^{int}r^n
= \frac{(1 - r)\sum_{n=0}^{\infty} (n + 1)\hat{f}(n+1)e^{int}r^n}{1 - r}
= \frac{\tilde{D}_{\{(n+1)\hat{f}(n+1)e^{int}\}}(\sqrt{r})}{1 - r},
\]
thus
\[
f'(z) = \frac{\tilde{D}_{\{(n+1)\hat{f}(n+1)e^{int}\}}(\sqrt{r})}{1 - r}. \tag{2.8}
\]
Therefore, by using formula (2.8), we have that
\[
\int_0 |1 - |z|^2|^{p(t)} |f'(z)|^{p(t)} \frac{dA(z)}{(1 - |z|^2)^2} < +\infty
\]
(2.9)
if and only if
\[
\int_0^{2\pi} \int_0^1 |\tilde{D}_{(n+1)f(n+1)e^{it}}(\sqrt{r})|^{p(t)} \frac{(1 - r^2)^{p(t)}}{(1 - r)^{p(t)}} \frac{r}{(1 - r^2)^2} dr dt < +\infty,
\]
(2.10)
that is,
\[
\int_0^{2\pi} \int_0^1 |\tilde{D}_{(n+1)f(n+1)e^{it}}(\sqrt{r})|^{p(t)} \frac{(1 + r)^{p(t) - 2}}{(1 - r)^2} r dr dt < +\infty,
\]
(2.11)
as desired. The theorem is proved. □

We remark that, in case of classical Hardy space, Theorem 2.1 shed some light on the following old problem for the Hardy space functions (see Privalov [12] and Duren [3]): how an $H^p$ function can be recognized by the behavior of its Taylor coefficients?

References


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