Research Article

Nontangential Limits for Modified Poisson Integrals of Boundary Functions in a Cone

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Received 17 May 2012; Accepted 8 July 2012

Academic Editor: Dachun Yang

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Our aim in this paper is to deal with non-tangential limits for modified Poisson integrals of boundary functions in a cone, which generalized results obtained by Brundin and Mizuta-Shimomura.

1. Introduction and Main Results

Let $\mathbf{R}$ and $\mathbf{R}_+$ be the set of all real numbers and the set of all positive real numbers, respectively. We denote by $\mathbf{R}^n$ ($n \geq 2$) the n -dimensional Euclidean space. A point in $\mathbf{R}^n$ is denoted by $P = (X, x_n)$, where $X = (x_1, x_2, \dots, x_{n-1})$. The Euclidean distance of two points P and Q in $\mathbf{R}^n$ is denoted by $|P - Q|$. Also $|P - O|$ with the origin O of $\mathbf{R}^n$ is simply denoted by $|P|$. The boundary, the closure, and the complement of a set S in $\mathbf{R}^n$ are denoted by ∂S , $\bar{S}$, and S^c , respectively.

We introduce a system of spherical coordinates (r, Θ) , $\Theta = (\theta_1, \theta_2, \dots, \theta_{n-1})$, in $\mathbf{R}^n$ which are related to cartesian coordinates $(x_1, x_2, \dots, x_{n-1}, x_n)$ by $x_n = r \cos \theta_1$.

For positive functions h_1 and h_2 , we say that $h_1 \lesssim h_2$ if $h_1 \leq Mh_2$ for some constant $M > 0$. If $h_1 \lesssim h_2$ and $h_2 \lesssim h_1$, we say that $h_1 \approx h_2$.

For $P \in \mathbf{R}^n$ and $R > 0$, let $B(P, R)$ denote the open ball with center at P and radius R in $\mathbf{R}^n$. The unit sphere and the upper half unit sphere are denoted by S^{n-1} and S_+^{n-1} , respectively. For simplicity, a point $(1, \Theta)$ on S^{n-1} and the set $\{\Theta; (1, \Theta) \in \Omega\}$ for a set $\Omega, \Omega \subset S^{n-1}$ are often identified with Θ and Ω , respectively. For two sets $\Xi \subset \mathbf{R}_+$ and $\Omega \subset S^{n-1}$, the set $\{(r, \Theta) \in \mathbf{R}^n; r \in \Xi, (1, \Theta) \in \Omega\}$ in $\mathbf{R}^n$ is simply denoted by $\Xi \times \Omega$. In particular, the half space $\mathbf{R}_+ \times S_+^{n-1} = \{(X, x_n) \in \mathbf{R}^n; x_n > 0\}$ will be denoted by T_n .

By $C_n(\Omega)$, we denote the set $\mathbf{R}_+ \times \Omega$ in $\mathbf{R}^n$ with the domain Ω on S^{n-1} . We call it a cone. Then T_n is a special cone obtained by putting $\Omega = S_+^{n-1}$. We denote the sets $I \times \Omega$ and $I \times \partial\Omega$

with an interval on $\mathbf{R}$ by $C_n(\Omega; I)$ and $S_n(\Omega; I)$. By $S_n(\Omega)$ we denote $S_n(\Omega; (0, +\infty))$ which is $\partial C_n(\Omega) - \{O\}$.

Let Ω be a domain on $\mathbf{S}^{n-1}$ with smooth boundary. Consider the Dirichlet problem:

$$\begin{aligned} (\Lambda_n + \lambda)\varphi &= 0 \quad \text{on } \Omega, \\ \varphi &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where Λ_n is the spherical part of the Laplace operator Δ_n :

$$\Delta_n = \frac{n-1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} + \frac{\Lambda_n}{r^2}. \tag{1.2}$$

We denote the least positive eigenvalue of this boundary value problem by λ_Ω and the normalized positive eigenfunction corresponding to λ_Ω by $\varphi_\Omega(\Theta)$,

$$\int_{\Omega} \varphi_\Omega^2(\Theta) d\sigma_\Theta = 1, \tag{1.3}$$

where $d\sigma_\Theta$ is the surface area on $\mathbf{S}^{n-1}$. We denote the solutions of the equation $t^2 + (n-2)t - \lambda_\Omega = 0$ by $\alpha_\Omega, -\beta_\Omega$ ($\alpha_\Omega, \beta_\Omega > 0$). If $\Omega = \mathbf{S}_+^{n-1}$, then $\alpha_\Omega = 1$, $\beta_\Omega = n-1$, and $\varphi_1(\Theta) = (2ns_n^{-1})^{1/2} \cos \theta_1$, where s_n is the surface area $2\pi^{n/2}(\Gamma(n/2))^{-1}$ of $\mathbf{S}_1$.

To simplify our consideration in the following, we will assume that if $n \geq 3$, then Ω is a $C^{2,\alpha}$ -domain ($0 < \alpha < 1$) on $\mathbf{S}^{n-1}$ surrounded by a finite number of mutually disjoint closed hypersurfaces (e.g., see ([1], pages 88-89) for the definition of $C^{2,\alpha}$ -domain). Then by modifying Miranda's method ([2], pages 7-8), we can prove the following inequality:

$$\varphi_\Omega(\Theta) \approx \text{dist}(\Theta, \partial\Omega) \quad (\Theta \in \Omega). \tag{1.4}$$

For any $(1, \Theta) \in \Omega$, we have (see [3])

$$\varphi_\Omega(\Theta) \approx \text{dist}((1, \Theta), \partial C_n(\Omega)), \tag{1.5}$$

which yields that

$$\delta(P) \approx r\varphi_\Omega(\Theta), \tag{1.6}$$

where $\delta(P) = \text{dist}(P, \partial C_n(\Omega))$ and $P = (r, \Theta) \in C_n(\Omega)$.

Let $G_\Omega(P, Q)$ ($P = (r, \Theta), Q = (t, \Phi) \in C_n(\Omega)$) be the Green function of $C_n(\Omega)$. We define the Poisson kernel $K_\Omega(P, Q)$ by

$$K_\Omega(P, Q) = \frac{1}{c_n} \frac{\partial}{\partial n_Q} G_\Omega(P, Q), \tag{1.7}$$

where

$$c_n = \begin{cases} 2\pi & n = 2, \\ (n-2)s_n & n \geq 3, \end{cases} \quad (1.8)$$

$Q \in S_n(\Omega)$ and $\partial/\partial n_Q$ denotes the differentiation at Q along the inward normal into $C_n(\Omega)$.

In this paper, we consider functions $f \in L^p(\partial C_n(\Omega))$, where $1 \leq p < \infty$. Then the Poisson integral $W_\Omega f(P)$ ($P \in C_n(\Omega)$) is defined by

$$W_\Omega f(P) = \int_{S_n(\Omega)} K_\Omega(P, Q) f(Q) d\sigma_Q, \quad (1.9)$$

where $d\sigma_Q$ is the surface area element on $S_n(\Omega)$.

Remark 1.1. Let $\Omega = \mathbf{S}_+^{n-1}$. Then

$$G_{\mathbf{S}_+^{n-1}}(P, Q) = \begin{cases} \log|P - Q^*| - \log|P - Q| & n = 2, \\ |P - Q|^{2-n} - |P - Q^*|^{2-n} & n \geq 3, \end{cases} \quad (1.10)$$

where $Q^* = (Y, -y_n)$, that is, Q^* is the mirror image of $Q = (Y, y_n)$ with respect to ∂T_n . Hence, for the two points $P = (X, x_n) \in T_n$ and $Q = (Y, y_n) \in \partial T_n$, we have

$$c_n K_{\mathbf{S}_+^{n-1}}(P, Q) = \frac{\partial}{\partial n_Q} G_{\mathbf{S}_+^{n-1}}(P, Q) = \begin{cases} 2|P - Q|^{-2} x_n & n = 2, \\ 2(n-2)|P - Q|^{-n} x_n & n \geq 3. \end{cases} \quad (1.11)$$

We fix an open, nonempty, and bounded set $G(\Omega) \subset \partial C_n(\Omega)$. In $C_n(\Omega)$, we normalise the extension, with respect to $G(\Omega)$, by

$$\rho \mathcal{J}_\Omega f(P) = \frac{W_\Omega f(P)}{W_\Omega \chi_{G(\Omega)}(P)}, \quad (1.12)$$

where $\chi_{G(\Omega)}$ denotes the characteristic function of $G(\Omega)$.

Let

$$\Gamma(\Omega, \zeta) = \{P = (r, \Theta) \in C_n(\Omega) : |(r, \Theta) - \zeta| \lesssim \delta(P)\} \quad (1.13)$$

be a nontangential cone in $C_n(\Omega)$ with vertex $\zeta \in \partial C_n(\Omega)$.

We define

$$\mathfrak{K}_p(f, l, P) = \left(\frac{1}{l^{n-1}} \int_{B(P, l)} |f(Q)|^p d\sigma_Q \right)^{1/p}, \quad (1.14)$$

$$\mathbb{E}_f^p(G(\Omega)) = \{P \in G(\Omega) : \mathfrak{K}_p(f - f(P), l, P) \longrightarrow 0 \text{ as } l \longrightarrow 0\}.$$

Note that, if $f \in L^p(\partial C_n(\Omega))$, then $|G(\Omega) \setminus \mathbb{E}_f^p(G(\Omega))| = 0$ (a.e. point is a Lebesgue point).

In T_n , the following conclusion was proved by Brundin (see ([4], pages 11–16)) and Mizuta and Shimomura (see ([5], Theorem 3)), respectively. In the unit disc, about related results, we refer the readers to the papers by Sjögren (see [6, 7]), Rönning (see [8]), and Brundin (see [9]).

Theorem A. For a.e. $\zeta \in G(\mathbf{S}_+^{n-1})$, $\rho \mathcal{D}_{\mathbf{S}_+^{n-1}} f(P) \rightarrow f(\zeta)$ (see Remark 1.1 for the definition of $\rho \mathcal{D}_{\mathbf{S}_+^{n-1}} f(P)$) as $P \rightarrow \zeta$ along $\Gamma(\mathbf{S}_+^{n-1}, \zeta)$.

Our aim is to generalize Theorem A to the conical case.

Theorem 1.2. For any $\zeta \in \mathbb{E}_f^p(G(\Omega))$ (in particular, for a.e. $\zeta \in G(\Omega)$) one has that $\rho \mathcal{D}_\Omega f(P) \rightarrow f(\zeta)$ as $P \rightarrow \zeta$ along $\Gamma(\Omega, \zeta)$.

2. Some Lemmas

Lemma 2.1. One has

$$\begin{aligned} K_\Omega(P, Q) &\approx r^{-\beta_\Omega} t^{\alpha_\Omega-1} \varphi_\Omega(\Theta), \\ \left(\text{resp. } K_\Omega(P, Q) &\approx r^{\alpha_\Omega} t^{-\beta_\Omega-1} \varphi_\Omega(\Theta) \right), \end{aligned} \quad (2.1)$$

for any $P = (r, \Theta) \in C_n(\Omega)$ and any $Q = (t, \Phi) \in S_n(\Omega)$ satisfying $0 < t/r \leq 4/5$ (resp. $0 < r/t \leq 4/5$);

$$K_\Omega(P, Q) \approx \frac{r \varphi_\Omega(\Theta)}{|P - Q|^n}, \quad (2.2)$$

for any $P = (r, \Theta) \in C_n(\Omega)$ and any $Q = (t, \Phi) \in S_n(\Omega; (4r/5, 5r/4))$.

Proof. These immediately follow from ([10], Lemma 2), ([11], Lemma 4 and Remark), and (1.4). $\square$

Lemma 2.2. One has

$$W_\Omega 1(P) = O(1) \quad \text{as } P \rightarrow \zeta \in G. \quad (2.3)$$

Proof. Write

$$\begin{aligned} W_\Omega 1(P) &= \int_{E_1} + \int_{E_2} + \int_{E_3} \\ &= U_1(P) + U_2(P) + U_3(P), \end{aligned} \quad (2.4)$$

where

$$E_1 = S_n\left(\Omega; \left(0, \frac{4}{5}r\right]\right), \quad E_2 = S_n\left(\Omega; \left[\frac{5}{4}r, \infty\right)\right), \quad E_3 = S_n\left(\Omega; \left(\frac{4}{5}r, \frac{5}{4}r\right)\right). \quad (2.5)$$

By (2.1), we have the following estimates

$$\begin{aligned} U_1(P) &\approx r^{-\beta_\Omega} \varphi_\Omega(\Theta) \int_{E_1} t^{\alpha_\Omega-1} d\sigma_Q \\ &\approx \frac{s_n}{\beta_\Omega} \left(\frac{4}{5}\right)^{\beta_\Omega} \varphi_\Omega(\Theta), \end{aligned} \quad (2.6)$$

$$U_2(P) \approx \frac{s_n}{\alpha_\Omega} \left(\frac{4}{5}\right)^{\alpha_\Omega} \varphi_\Omega(\Theta). \quad (2.7)$$

Next, we will estimate $U_3(P)$. Take a sufficiently small positive number k such that

$$E_3 \subset \bigcup_{P=(r,\Theta) \in \Lambda(k)} B\left(P, \frac{1}{2}r\right), \quad (2.8)$$

where

$$\Lambda(k) = \left\{ P = (r, \Theta) \in C_n(\Omega); \inf_{z \in \partial\Omega} |(1, \Theta) - (1, z)| < k, 0 < r < \infty \right\}. \quad (2.9)$$

Since $P \rightarrow \zeta \in G$, we only consider the case $P \in \Lambda(k)$. Now put

$$H_i(P) = \left\{ Q \in E_3; 2^{i-1}\delta(P) \leq |P - Q| < 2^i\delta(P) \right\}. \quad (2.10)$$

Since $S_n(\Omega) \cap \{Q \in \mathbf{R}^n : |P - Q| < \delta(P)\} = \emptyset$, we have

$$U_3(P) \approx \sum_{i=0}^{i(P)} \int_{H_i(P)} \frac{r\varphi_\Omega(\Theta)}{|P - Q|^n} d\sigma_Q, \quad (2.11)$$

where $i(P)$ is a positive integer satisfying $2^{i(P)-1}\delta(P) \leq r/2 < 2^{i(P)}\delta(P)$.

By (1.6) we have

$$\begin{aligned} \int_{H_i(P)} \frac{r\varphi_\Omega(\Theta)}{|P - Q|^n} d\sigma_Q &\approx r\varphi_\Omega(\Theta) \int_{H_i(P)} \frac{1}{\delta(P)} d\sigma_Q \\ &= \frac{r\varphi_\Omega(\Theta)}{\delta(P)} \frac{s_n}{2^{i(P)}} \approx \frac{s_n}{2^{i(P)}} \end{aligned} \quad (2.12)$$

for $i = 0, 1, 2, \dots, i(P)$.

So

$$U_3(P) \approx O(1). \quad (2.13)$$

Combining (2.6)–(2.13), Lemma 2.2 is proved. $\square$

Lemma 2.3. *One has*

$$W_\Omega \chi_{G(\Omega)}(P) = W_\Omega 1(P) + O(1) \quad \text{as } P \rightarrow \zeta \in G(\Omega). \quad (2.14)$$

Proof. In fact, we only need to prove

$$U_4(P) = \int_{S_n(\Omega) - G(\Omega)} K_\Omega(P, Q) d\sigma_Q \lesssim O(1). \quad (2.15)$$

Write

$$\begin{aligned} U_4(P) &= \int_{(S_n(\Omega) - G(\Omega)) \cap E_1} + \int_{(S_n(\Omega) - G(\Omega)) \cap E_2} + \int_{(S_n(\Omega) - G(\Omega)) \cap E_3} \\ &= U_5(P) + U_6(P) + U_7(P), \end{aligned} \quad (2.16)$$

where E_1 , E_2 , and E_3 are sets on $S_n(\Omega)$ used in Lemma 2.2.

Obviously,

$$U_5(P) \lesssim U_1(P) \approx O(1), \quad (2.17)$$

$$U_6(P) \lesssim U_2(P) \approx O(1). \quad (2.18)$$

Further, we have by (2.2)

$$\begin{aligned} U_7(P) &\approx r\varphi_\Omega(\Theta) \int_{(S_n(\Omega) - G(\Omega)) \cap E_3} \frac{1}{|P - Q|^n} d\sigma_Q \\ &\lesssim \frac{S_n}{d} |\zeta| \varphi_\Omega(\Theta) \quad (P \rightarrow \zeta \in G(\Omega)), \end{aligned} \quad (2.19)$$

where

$$d = \inf_{Q \in \partial C_n(\Omega) - G(\Omega)} |Q - \zeta|. \quad (2.20)$$

Combining (2.17)–(2.19), (2.15) holds which gives the conclusion. $\square$

3. Proof of the Theorem 1.2

As $P \rightarrow \zeta \in G(\Omega)$, $W_\Omega \chi_{G(\Omega)}(P) = O(1)$ from Lemmas 2.2 and 2.3.

Now, let $f \in L^p(\partial C_n(\Omega))$ and $\zeta \in \mathbb{E}_f^p(G(\Omega))$ be given. We may, without loss of generality, assume that $f(\zeta) = 0$. Furthermore, we assume that $P = (r, \Theta) \in \Gamma(\Omega, \zeta)$. For short, let $s = |(r, \Theta) - \zeta|$. We write

$$\begin{aligned} W_\Omega f(P) &= \int_{E_1} + \int_{E_2} + \int_{E_3 \cap B(\zeta, 2s)} + \int_{E_3 \cap B^c(\zeta, 2s)} \\ &= V_1 f(P) + V_2 f(P) + V_3 f(P) + V_4 f(P), \end{aligned} \quad (3.1)$$

where E_1 , E_2 , and E_3 are sets on $S_n(\Omega)$ used in Lemma 2.2.

By using Hölder's inequality, (2.1), we have the following estimates

$$\begin{aligned}
 |V_1 f(P)| &\lesssim r^{-\beta_\Omega} \varphi_\Omega(\Theta) \int_{E_1} t^{\alpha_\Omega-1} f(Q) d\sigma_Q \\
 &\lesssim r^{(1-n)/p} \|f\|_{p'} \\
 |V_2 f(P)| &\lesssim r^{(1-n)/p} \|f\|_p.
 \end{aligned} \tag{3.2}$$

Similar to the estimate of $U_3(P)$ in Lemma 2.2, we only consider the following inequality by (1.6)

$$\begin{aligned}
 \int_{H_i(P)} \frac{r\varphi_\Omega(\Theta)}{|P-Q|^n} d\sigma_Q &\approx r\varphi_\Omega(\Theta) \int_{H_i(P)} \frac{1}{\{2^{i-1}\delta(P)\}^n} d\sigma_Q \\
 &\lesssim r^{\alpha_\Omega} \varphi_\Omega(\Theta) \int_{E_2} t^{-\beta_\Omega-1} |f(Q)| d\sigma_Q \\
 &\lesssim r^{(1-n)/p} \|f\|_p
 \end{aligned} \tag{3.3}$$

for $i = 0, 1, 2, \dots, i(P)$, which is similar to the estimate of $V_2 f(P)$.

So

$$|V_3 f(P)| \lesssim r^{(1-n)/p} \|f\|_p. \tag{3.4}$$

Notice that $|P-Q| > (1/2)|\zeta-Q|$ in the case $Q \in E_3 \cap B^c(\zeta, 2s)$. By (1.6) and (2.2), we have

$$\begin{aligned}
 |V_4 f(P)| &\lesssim \delta(P) \int_{E_3 \cap B^c(\zeta, 2s)} \frac{|f(Q)|}{|P-Q|^n} d\sigma_Q \\
 &\lesssim \delta(P) \sum_{i=1}^{\infty} \int_{E_3 \cap (B(\zeta, 2^{i+1}s) \setminus B(\zeta, 2^i s))} \frac{|f(Q)|}{|\zeta-Q|^n} d\sigma_Q \\
 &\lesssim \delta(P) \sum_{i=1}^{\infty} \left(\frac{1}{2^i s}\right)^n \int_{E_3 \cap B(\zeta, 2^{i+1}s)} |f(Q)| d\sigma_Q \\
 &\lesssim \delta(P) \sum_{i=1}^{\infty} \mathfrak{K}_1(f, 2^{i+1}s, \zeta) \\
 &\lesssim \delta(P) \sum_{i=1}^{\infty} \int_{2^{i+1}s}^{2^{i+2}s} \frac{\mathfrak{K}_1(f, l, \zeta)}{l} dl \\
 &\lesssim \delta(P) \int_s^\infty \frac{\mathfrak{K}_1(f, l, \zeta)}{l} dl \\
 &\lesssim \delta(P) \int_{\delta(P)}^\infty \frac{\mathfrak{K}_1(f, l, \zeta)}{l} dl.
 \end{aligned} \tag{3.5}$$

Thus, it follows that

$$\begin{aligned} |\mathcal{P}\mathcal{J}_\Omega f(P)| &\lesssim \frac{1}{O(1)} [|V_1 f(P)| + |V_2 f(P)| + |V_3 f(P)| + |V_4 f(P)|] \\ &\lesssim r^{(1-n)/p} \|f\|_p + \delta(P) \int_{\delta(P)}^\infty \frac{\mathfrak{K}_1(f, l, \zeta)}{l} dl. \end{aligned} \quad (3.6)$$

Using the fact that $s \lesssim \delta(P) \lesssim r\varphi_\Omega(\Theta)$, we get

$$|\mathcal{P}\mathcal{J}_\Omega f(P)| \lesssim \mathfrak{K}_1(f, 2s, \zeta) + \delta(P) \int_{\delta(P)}^\infty \frac{\mathfrak{K}_1(f, l, \zeta)}{l} dl. \quad (3.7)$$

It is clear that

$$\int_{\delta(P)}^\infty \frac{\mathfrak{K}_1(f, l, \zeta)}{l} dl \quad (3.8)$$

is a convergent integral, since

$$\begin{aligned} \frac{\mathfrak{K}_1(f, l, \zeta)}{l} &\lesssim s^{-1-n} s^{n/q} \|f\|_p \\ &\lesssim s^{-1-(n/p)} \|f\|_p \end{aligned} \quad (3.9)$$

from the Hölder's inequality.

Now, as $\delta(P) \rightarrow 0$, we also have $s \rightarrow 0$. Since $f(\zeta) = 0$ and since we have assumed that $\zeta \in \mathbb{E}_f^p(G(\Omega))$ (and thus that $\zeta \in \mathbb{E}_f^1(G(\Omega))$), it follows that $\mathcal{P}\mathcal{J}_\Omega f(P) \rightarrow 0 = f(\zeta)$ as $P = (r, \Theta) \rightarrow \zeta$ along $\Gamma(\Omega, \zeta)$. This concludes the proof.

Acknowledgments

This paper is supported by SRFDP (no. 20100003110004) and NSF of China (no. 11071020).

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