Research Article

Convexity Invariance of Fuzzy Sets under the Extension Principles

Dong Qiu¹ and Weiquan Zhang²

¹ College of Mathematics and Physics, Chongqing University of Posts and Telecommunications, Nan'an, Chongqing 400065, China

² School of Information Engineering, Guangdong Medical College, Guangdong, Dongguan 523808, China

Correspondence should be addressed to Dong Qiu, dongqiu@yahoo.cn

Received 28 April 2012; Accepted 10 September 2012

Academic Editor: Manuel Sanchis

Copyright © 2012 D. Qiu and W. Zhang. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We discuss the convexity invariance of fuzzy sets under the extension principles. Particularly, we give a necessary and sufficient condition for a mapping to be an inverse $*$ -convex transformation, and also obtain some sufficient conditions for a mapping to be an $*$ -convex transformation. Two applications are given to illustrate the obtained results. Finally, we give some applications of the main results to the hyperstructure convexity invariance of type 2 fuzzy sets under hyperalgebra operations, and to the convexity invariance of fuzzy numbers under basic arithmetic operations.

1. Introduction

As a suitable mathematical model to handle vagueness and uncertainty, fuzzy set theory is emerging as a powerful theory and has attracted the attention of many researchers and practitioners who contributed to its development and applications [1–12]. Convexity plays the most useful role in the theory and applications of fuzzy sets; and the research on convexity and generalized convexity is one of the most important aspects of fuzzy set theory [9, 10, 13–16].

Moreover, the extension principle for fuzzy sets is in essence a basic identity which allows the domain of the definition of a mapping or a relation to be extended from points in a set U to fuzzy subsets of U . This, in fact, is the underlying basis for many operations of the most basic concepts of fuzzy set theory such as arithmetic operations of fuzzy numbers, hyperalgebra operations of type 2 fuzzy sets, and synthetic operations of fuzzy relations.

Hence it is important and interesting to study the convexity invariance of concepts of fuzzy set theory under the extension principles, which has been explored by many

researchers [4, 5, 8, 16–22]. In this paper, we will discuss the convexity invariance of fuzzy sets under the extension principles in the general case. We hope that our results of the convexity invariance may lead to significant, new, and innovative results in those related fields [1, 3, 6, 8].

2. Preliminaries

In this section, some basic definitions of t -norms, extension principles, convex fuzzy sets, and type 2 fuzzy sets are reviewed. Throughout this paper, the letters $\mathbb{N}$ and $\mathbb{R}$ will denote the set of all positive integers and real numbers, respectively. A fuzzy set A in a universe of discourse X is characterized by a membership function $A(x)$ which associates with each point in X a real number in the interval $[0, 1]$, with the value of $A(x)$ at x representing the “grade of membership” of x in A [10]. The symbol $\mathcal{F}(X)$ denotes the family of all fuzzy subsets of a set X .

Definition 2.1 (see [1, 12]). For a fuzzy set A in a universe X and each $\lambda \in [0, 1]$, the strong λ -level set of A , denoted by $[A]_\lambda$, is defined as

$$[A]_\lambda = \{x \in X \mid A(x) > \lambda\}. \quad (2.1)$$

Specially, the set $[A]_0$ is called the support set of fuzzy set A .

According to Zadeh’s definition, a type 2 fuzzy set is a fuzzy set with a fuzzy membership function.

Definition 2.2 (see [21]). A fuzzy set of type-2, A , in a universe of discourse X is characterized by a fuzzy membership function μ_A as

$$\mu_A : X \longrightarrow [0, 1]^{[0, 1]}, \quad (2.2)$$

where the value $\mu_A(x)$ is a fuzzy grade and is a fuzzy set in the unit interval $[0, 1]$. A fuzzy grade $\mu_A(x)$ is represented by

$$\mu_A(x) = \int \frac{f(u)}{u}, \quad u \in [0, 1], \quad (2.3)$$

where f is a membership function for fuzzy grade $\mu_A(x)$ and is defined as

$$f : [0, 1] \longrightarrow [0, 1]. \quad (2.4)$$

Definition 2.3 (see [2]). A t -norm is a binary operation on the unit interval $[0, 1]$, that is, a function $*$: $[0, 1]^2 \rightarrow [0, 1]$, such that for all $a, b, c, d \in [0, 1]$, the following four axioms are satisfied:

(T-1) $a * 1 = a$, (boundary condition)

(T-2) $a * b \leq c * d$, whenever $a \leq c$ and $b \leq d$, (monotonicity)

(T-3) $a * b = b * a$, (commutativity)

(T-4) $a * (b * c) = (a * b) * c$ (associativity).

A t -norm $*$ is said to be continuous if it is a continuous function on $[0, 1]^2$.

Example 2.4 (see [2]). The following are the four basic t -norms $*_M$, $*_P$, $*_L$, and $*_D$ given by, respectively,

$$\begin{aligned} x *_M y &= \min(x, y), & (\text{minimum}) \\ x *_P y &= x \cdot y, & (\text{product}) \\ x *_L y &= \max(x + y - 1, 0), & (\text{Lukasiewicz } t\text{-norm}) \\ x *_D y &= \begin{cases} 0 & \text{if } (x, y) \in [0, 1)^2, \\ \min(x, y) & \text{otherwise.} \end{cases} & (\text{drastic product}) \end{aligned} \quad (2.5)$$

The t -norms $*_M$, $*_P$, and $*_L$ are continuous, but $*_D$ is not.

A t -norm $*$ can be extended (by associativity) in a unique way to a n -ary operation taking for $(x_1, \dots, x_n) \in [0, 1]^n$ ($n \geq 3$) the value $x_1 * \dots * x_n$ defined recurrently by

$$x_1 * \dots * x_n = (x_1 * \dots * x_{n-1}) * x_n, \quad \forall n \geq 3. \quad (2.6)$$

Definition 2.5 (see [14]). Let E be a real linear space. A fuzzy set A of E is said to be a convex fuzzy set if for all $x, y \in E$ and $\lambda \in [0, 1]$,

$$A(\lambda x + (1 - \lambda)y) \geq \min(A(x), A(y)). \quad (2.7)$$

Yuan and Lee [9] introduced the notion of t -norm convexity for fuzzy sets as follows.

Definition 2.6 (see [9]). Let E be a real linear space, and let $*$ be a t -norm. A fuzzy set A of E is said to be an $*$ -convex fuzzy set if for all $x, y \in E$ and $\lambda \in [0, 1]$,

$$A(\lambda x + (1 - \lambda)y) \geq A(x) * A(y). \quad (2.8)$$

A convex fuzzy set A is indeed an $*_M$ -convex fuzzy set according to Definition 2.6.

Definition 2.7 (see [1, 21, 23] (Zadeh's extension principle)). Let f be a mapping from a universe X to another universe Y . Two mappings $\tilde{f}$ from $\mathcal{F}(X)$ to $\mathcal{F}(Y)$ and $\tilde{f}^{-1}$ from $\mathcal{F}(Y)$ to $\mathcal{F}(X)$ can be induced by f as follows, respectively:

$$\begin{aligned} \tilde{f}(A)(v) &= \begin{cases} \sup_{f(u)=v} A(u) & \text{if } f^{-1}(v) \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases} \\ \tilde{f}^{-1}(B)(u) &= B(f(u)), \end{aligned} \quad (2.9)$$

for all $v \in Y$, $u \in X$, $A \in \mathcal{F}(X)$, and $B \in \mathcal{F}(Y)$, where $f^{-1}(v)$ is the inverse image set of v .

Definition 2.8 (see [1, 21, 23] (Zadeh's multivariable extension principle)). Let X be a Cartesian product of universes, $X = X_1 \times \cdots \times X_r$, and let f be a mapping from X to a universe Y such that $y = f(x_1, \dots, x_r)$. Then a mapping $\tilde{f}$ from $\mathcal{F}(X_1) \times \cdots \times \mathcal{F}(X_r)$ to $\mathcal{F}(Y)$ can be induced by f as follows:

$$\tilde{f}(A_1, \dots, A_r)(v) = \begin{cases} 0 & \text{if } f^{-1}(v) = \emptyset, \\ \sup_{f(u_1, \dots, u_r) = v} \min(A_1(u_1), \dots, A_r(u_r)), & \end{cases} \quad (2.10)$$

for all $v \in Y$ and all n -tuple of fuzzy sets $(A_1, \dots, A_r)$ which are fuzzy sets in $X_1, \dots, X_r$, respectively.

The multivariable extension principle as stated in Definition 2.8 can and has been generalized by using sup- $(t$ -norm) convolution rather than sup-min convolution in [17].

Definition 2.9 (see [1, 17] (generalized multivariable extension principle)). Let $*$ be a t -norm, let X be a Cartesian product of universes, $X = X_1 \times \cdots \times X_r$, and let f be a mapping from X to a universe Y such that $y = f(x_1, \dots, x_r)$. Then a mapping $\tilde{f}$ from $\mathcal{F}(X_1) \times \cdots \times \mathcal{F}(X_r)$ to $\mathcal{F}(Y)$ can be induced by f as follows:

$$\tilde{f}(A_1, \dots, A_r)(v) = \begin{cases} 0 & \text{if } f^{-1}(v) = \emptyset, \\ \sup_{f(u_1, \dots, u_r) = v} (A_1(u_1) * \cdots * A_r(u_r)), & \end{cases} \quad (2.11)$$

for all $v \in Y$ and all n -tuple of fuzzy sets $(A_1, \dots, A_r)$ which are fuzzy sets in $X_1, \dots, X_r$, respectively.

In set theory, a total order is a binary relation (here denoted by infix $\leq$) on some set X . The relation is transitive, antisymmetric, and total. A set paired with a total order is called a totally ordered set. If X is a totally ordered set, the order topology on X is generated by the subbase of "open rays"

$$\begin{aligned} (a, \infty) &= \{x \mid a < x\}, \\ (-\infty, b) &= \{x \mid x < b\}, \end{aligned} \quad (2.12)$$

for all a, b in X [24].

Remark 2.10. It should be noted that a fuzzy set A in a totally ordered set Y is an $*$ -convex fuzzy set if and only if for all $x, y, z \in Y$ with $x \leq y \leq z$,

$$\mu(y) \geq \mu(x) * \mu(z). \quad (2.13)$$

Definition 2.11. Let E be a real linear space, $*$ be a t -norm, and let f be a mapping from E to $(Y, \geq)$, a totally ordered set equipped with the order topology. The mapping f is said to be an $*$ -convex transformation from E to Y if the induced mapping $\tilde{f}$ (by Zadeh's extension

principle) transforms every $*$ -convex fuzzy set of E into an $*$ -convex fuzzy set of Y ; the mapping f is said to be an inverse $*$ -convex transformation from E to Y if the induced mapping $\tilde{f}^{-1}$ (by Zadeh's extension principle) transforms every $*$ -convex fuzzy set of Y into an $*$ -convex fuzzy set of E .

Definition 2.12. Let E be a Cartesian product of real linear topological spaces $E = E_1 \times \cdots \times E_r$, let $*$ be a t -norm, and let f be a mapping from E to $(Y, \geq)$, a totally ordered set equipped with the order topology. The mapping f is said to be a multivariable $*$ -convex transformation from E to Y if the induced mapping $\tilde{f}$ (by the generalized multivariable extension principle) transforms every n -tuple of $*$ -convex fuzzy sets $(A_1, \dots, A_r)$ of E into an $*$ -convex fuzzy set of Y , where $(A_1, \dots, A_r)$ are $*$ -convex fuzzy sets in $X_1, \dots, X_r$, respectively.

3. Main Results

Let E be a real linear topological space. For arbitrary two points $x, y \in E$, the line segment $\overline{xy}$ joining x and y is the set of all points of the form $tx + (1 - t)y$, $t \in [0, 1]$.

Theorem 3.1. *Let E be a real linear topological space, let $*$ be a continuous t -norm, and let f be a mapping from E to $(Y, \geq)$, a totally ordered set equipped with the order topology. If the restriction of f to every line segment $\overline{xy}$, $f|_{\overline{xy}}$, is continuous, then f is an $*$ -convex transformation.*

Proof. Let A be an arbitrary $*$ -convex fuzzy set of E . For any given three points $x, y, z \in Y$ with $x \leq y \leq z$, if either $f^{-1}(x)$ or $f^{-1}(z)$ is an empty set, then it is obvious that

$$\tilde{f}(A)(y) \geq \tilde{f}(A)(x) * \tilde{f}(A)(z) = 0. \quad (3.1)$$

Thus, without loss of generality, suppose that $f^{-1}(x)$ and $f^{-1}(z)$ are nonempty sets. For any two points $u \in f^{-1}(x)$ and $v \in f^{-1}(z)$, the restriction $f|_{\overline{uv}}$ is continuous. In addition, we have that $f(u) = x$ and $f(v) = z$. From the generalized intermediate value theorem [24], it follows that there is a point $w \in \overline{uv}$ with $f(w) = y$ which implies that $f^{-1}(y)$ is nonempty. Then by the $*$ -convexity of A , we have

$$\tilde{f}(A)(y) = \sup_{m \in f^{-1}(y)} A(m) \geq A(w) \geq A(u) * A(v). \quad (3.2)$$

Then the continuity and the monotonicity of $*$ and the arbitrariness of u in $f^{-1}(x)$ imply that

$$\begin{aligned} \tilde{f}(A)(y) &\geq \sup_{u \in f^{-1}(x)} (A(u) * A(v)) \\ &= \left(\sup_{u \in f^{-1}(x)} A(u) \right) * A(v) \\ &= (\tilde{f}(A)(x)) * A(v). \end{aligned} \quad (3.3)$$

Similarly from the continuity and the monotonicity of $*$ and the arbitrariness of v in $f^{-1}(z)$, it follows that

$$\begin{aligned}\tilde{f}(A)(y) &\geq \sup_{v \in f^{-1}(z)} (\tilde{f}(A)(x) * A(v)) \\ &= \tilde{f}(A)(x) * \left(\sup_{v \in f^{-1}(z)} A(v) \right) \\ &= (\tilde{f}(A)(x)) * (\tilde{f}(A)(z)),\end{aligned}\tag{3.4}$$

which implies that $\tilde{f}(A)$ is an $*$ -convex fuzzy set of Y . This completes the whole proof. $\square$

Corollary 3.2. *Let $*$ be a continuous t -norm. If f is a continuous mapping from $\mathbb{R}$ to $(Y, \geq)$, a totally ordered set equipped with the order topology, then f is an $*$ -convex transformation.*

Proof. It is obvious that f satisfies the conditions of Theorem 3.1. Then the desired result follows quickly from Theorem 3.1. $\square$

In Corollary 3.2, the continuity of f is not a necessary condition for a mapping f to be an $*$ -convex transformation. To give such a counterexample, we need some lemmas.

Lemma 3.3. *A fuzzy set A in a real linear space E is an $*_M$ -convex fuzzy set if and only if all of its strong λ -level sets, $[A]_\lambda$, are convex sets.*

Proof. Suppose that A is an $*_M$ -convex fuzzy set. For every $\lambda \in [0, 1)$ and any $x, y \in [A]_\lambda$ and $\alpha \in [0, 1]$, the inequalities

$$A(\alpha x + (1 - \alpha)y) \geq \min(A(x), A(y)) > \lambda\tag{3.5}$$

imply that the point $\alpha x + (1 - \alpha)y$ belongs to $[A]_\lambda$. Thus, $[A]_\lambda$ is convex.

Conversely, suppose that all of the strong λ -level sets, $[A]_\lambda$, are convex sets. For any $x, y \in E$ and $\alpha \in [0, 1]$, if either $A(x)$ or $A(y)$ is 0, then it is obvious that

$$A(\alpha x + (1 - \alpha)y) \geq \min(A(x), A(y)) = 0.\tag{3.6}$$

Thus, without loss of generality, suppose that $A(x)$ and $A(y)$ are not 0. Taking $\lambda = \min(A(x), A(y)) - \varepsilon$, where $\varepsilon \in (0, \min(A(x), A(y))]$, we have that x and y belong to the convex set $[A]_\lambda$, which implies that for every $\alpha \in [0, 1]$,

$$A(\alpha x + (1 - \alpha)y) > \lambda = \min(A(x), A(y)) - \varepsilon.\tag{3.7}$$

By the arbitrariness of ε , we obtain that for every $\alpha \in [0, 1]$,

$$A(\alpha x + (1 - \alpha)y) \geq \min(A(x), A(y)).\tag{3.8}$$

Thus, A is an $*_M$ -convex fuzzy set. We complete the whole proof here. $\square$

Lemma 3.4 (see [23]). *If f is a mapping from a universe X to another universe Y . Then for any $A \in \mathcal{F}(X)$, the following equation:*

$$\left[\tilde{f}(A) \right]_{\lambda} = f([A]_{\lambda}) \quad (3.9)$$

holds for all $\lambda \in [0, 1]$.

Example 3.5. Define the function $f : \mathbb{R} \rightarrow [-1, 1]$ by

$$f(x) = \begin{cases} \sin \frac{1}{x} & \text{if } x \neq 0, \\ 0 & x = 0. \end{cases} \quad (3.10)$$

This function is not continuous at $x = 0$ because the limit of $f(x)$ as x tends to 0 does not exist. However, it is an $*_M$ -convex transformation.

In order to show this, let A be an arbitrary $*_M$ -convex fuzzy set in $\mathbb{R}$. Thus, Lemma 3.3 implies that every strong λ -level set $[A]_{\lambda}$ is a convex set in $\mathbb{R}$. In addition, the convex sets in $\mathbb{R}$ are intervals. If the interval $[A]_{\lambda}$ does not contain 0, then $f([A]_{\lambda})$ is an interval because f is a continuous function on $[A]_{\lambda}$. If the interval $[A]_{\lambda}$ contains 0, then $f([A]_{\lambda}) = [-1, 1]$ because any neighborhood of 0 can always include an interval $[1/2(n+1)\pi, 1/2n\pi]$ or an interval $[-1/2n\pi, -1/2(n+1)\pi]$ for sufficiently large $n \in \mathbb{N}$.

Thus, by Lemma 3.4, we have proved that all of the strong λ -level sets, $[\tilde{f}(A)]_{\lambda}$, are convex sets, which shows that $\tilde{f}(A)$ is an $*_M$ -convex fuzzy set in $[-1, 1]$. Then f is an $*_M$ -convex transformation.

Definition 3.6. Let E be a real linear topological space, let $*$ be a t -norm, and let f be a mapping from E to $(Y, \geq)$, a totally ordered set equipped with the order topology. For a line segment $\overline{xy}$, define the mapping $g : [0, 1] \rightarrow Y$ by $g(t) = f(tx + (1-t)y)$. f is said to be monotonous on a line segment $\overline{xy}$, or $f|_{\overline{xy}}$ is said to be monotonous, if $g(t)$ is monotonous on $[0, 1]$ with respect to t .

Theorem 3.7. *Let E be a real linear topological space, $*$ be a t -norm, and f be a mapping from E to $(Y, \geq)$, a totally ordered set equipped with the order topology. f is an inverse $*$ -convex transformation if and only if the restriction of f to every line segment is monotonous.*

Proof. Suppose that the restriction of f to every line segment is monotonous. Let B be an arbitrary $*$ -convex fuzzy set in Y . For any $u, v \in X$ and $t \in [0, 1]$, we have that

$$f(u) \leq f(tu + (1-t)v) \leq f(v) \quad (3.11)$$

or

$$f(v) \leq f(tu + (1-t)v) \leq f(u), \quad (3.12)$$

because $f|_{\overline{uv}}$ is monotonous. Thus, from the $*$ -convexity of B , it follows that

$$\begin{aligned}\tilde{f}^{-1}(B)(tu + (1-t)v) &= B(f(tu + (1-t)v)) \\ &\geq B(f(u)) * B(f(v)) \\ &= (\tilde{f}^{-1}(B)(u)) * (\tilde{f}^{-1}(B)(v)),\end{aligned}\tag{3.13}$$

which implies that $\tilde{f}^{-1}(B)$ is an $*$ -convex fuzzy set. Therefore, f is an inverse $*$ -convex transformation.

Conversely, suppose that f is an inverse $*$ -convex transformation. If there is a line segment $\overline{uv}$ on which f is not monotonous. Then there is a $t_0 \in (0, 1)$ which satisfies $f(u) < f(t_0u + (1-t_0)v)$ but $f(t_0u + (1-t_0)v) > f(v)$ (or $f(u) > f(t_0u + (1-t_0)v)$ but $f(t_0u + (1-t_0)v) < f(v)$).

Now define the fuzzy set B in Y by

$$B(x) = \begin{cases} 1 & \text{if } x \in [\min(f(u), f(v)), \max(f(u), f(v))], \\ 0 & \text{otherwise.} \end{cases}\tag{3.14}$$

It is easy to check that B is an $*$ -convex fuzzy set. But $\tilde{f}^{-1}(B)$ is not an $*$ -convex fuzzy set in E because

$$\begin{aligned}\tilde{f}^{-1}(B)(t_0u + (1-t_0)v) &= B(f(t_0u + (1-t_0)v)) \\ &= 0 \\ &< 1 \\ &= B(f(u)) * B(f(v)) \\ &= (\tilde{f}^{-1}(B)(u)) * (\tilde{f}^{-1}(B)(v)).\end{aligned}\tag{3.15}$$

This is a contradiction. Thus, we can complete the whole proof here. $\square$

Corollary 3.8. *Let E be a real linear topological space, and let $*$ be a t -norm. If f is a real linear functional, then f is an inverse $*$ -convex transformation.*

Proof. For any line segment $\overline{xy}$, the function $g : [0, 1] \rightarrow \mathbb{R}$ defined as

$$g(t) = f(tx + (1-t)y) = tf(x) + (1-t)f(y)\tag{3.16}$$

satisfies

$$\min(g(0), g(1)) \leq g(t) \leq \max(g(0), g(1)),\tag{3.17}$$

for all $t \in [0, 1]$. From the arbitrariness of the line segment $\overline{xy}$ and the above inequalities, one can easily deduce that $g(t)$ is monotonous on $[0, 1]$. Then by Theorem 3.7, we get the desired result. $\square$

Theorem 3.9. Let E be a Cartesian product of real linear topological spaces $E = E_1 \times \cdots \times E_r$, let $*$ be a continuous t -norm, and let f be a mapping from E to $(Y, \geq)$, a totally ordered set equipped with the order topology. If the restriction of f to every line segment $\overline{xy}$, $f|_{\overline{xy}}$, is continuous, then f is a multi-variable $*$ -convex transformation.

Proof. Let $A_1, \dots, A_r$ be $*$ -convex fuzzy sets in $E_1, \dots, E_r$, respectively. For any given three points $x, y, z \in Y$ with $x \leq y \leq z$, if either $f^{-1}(x)$ or $f^{-1}(z)$ is an empty set, then it is obvious that

$$\tilde{f}(A_1, \dots, A_r)(y) \geq 0 = \tilde{f}(A_1, \dots, A_r)(x) * \tilde{f}(A_1, \dots, A_r)(z). \quad (3.18)$$

Thus, without loss of generality, suppose that $f^{-1}(x)$ and $f^{-1}(z)$ are nonempty sets. For any two points $u = (u_1, \dots, u_r) \in f^{-1}(x)$ and $v = (v_1, \dots, v_r) \in f^{-1}(z)$, define the mapping $g : [0, 1] \rightarrow Y$ by

$$\begin{aligned} g(t) &= f(tu + (1-t)v) \\ &= f(t(u_1, \dots, u_r) + (1-t)(v_1, \dots, v_r)) \\ &= f(tu_1 + (1-t)v_1, \dots, tu_r + (1-t)v_r). \end{aligned} \quad (3.19)$$

The mapping $g(t)$ is continuous on $[0, 1]$ with respect to t because the restriction $f|_{\overline{uv}}$ is continuous. Since $g(1) = x$ and $g(0) = z$, from the generalized intermediate value theorem [24], it follows that there is a $t_0 \in [0, 1]$ such that $g(t_0) = y$ which implies that $f^{-1}(y)$ is nonempty. Then by the commutativity and the associativity of $*$ and the $*$ -convexities of $A_1, \dots, A_r$, we have that

$$\begin{aligned} \tilde{f}(A_1, \dots, A_r)(y) &= \sup_{(w_1, \dots, w_r) \in f^{-1}(y)} (A_1(w_1) * \cdots * A_r(w_r)) \\ &\geq A_1(t_0 u_1 + (1-t_0)v_1) * \cdots * A_r(t_0 u_r + (1-t_0)v_r) \\ &\geq (A_1(u_1) * A_1(v_1)) * \cdots * (A_r(u_r) * A_r(v_r)) \\ &= (A_1(u_1) * \cdots * A_r(u_r)) * (A_1(v_1) * \cdots * A_r(v_r)). \end{aligned} \quad (3.20)$$

Combining with the continuity and the monotonicity of $*$ and the arbitrariness of u in $f^{-1}(x)$, the inequality (3.20) implies that

$$\begin{aligned} \tilde{f}(A_1, \dots, A_r)(y) &\geq \sup_{u \in f^{-1}(x)} (A_1(u_1) * \cdots * A_r(u_r) * A_1(v_1) * \cdots * A_r(v_r)) \\ &= \left(\sup_{u \in f^{-1}(x)} A_1(u_1) * \cdots * A_r(u_r) \right) * A_1(v_1) * \cdots * A_r(v_r) \\ &= \left(\tilde{f}(A_1, \dots, A_r)(x) \right) * (A_1(v_1) * \cdots * A_r(v_r)). \end{aligned} \quad (3.21)$$

Similarly from the continuity and the monotonicity of $*$ and the arbitrariness of v in $f^{-1}(z)$, it follows from the above inequality that

$$\begin{aligned}
 & \tilde{f}(A_1, \dots, A_r)(y) \\
 & \geq \sup_{v \in f^{-1}(z)} \left(\tilde{f}(A_1, \dots, A_r)(x) \right) * (A_1(v_1) * \dots * A_r(v_r)) \\
 & = \left(\tilde{f}(A_1, \dots, A_r)(x) \right) * \left(\sup_{v \in f^{-1}(z)} A_1(v_1) * \dots * A_r(v_r) \right) \\
 & = \left(\tilde{f}(A_1, \dots, A_r)(x) \right) * \left(\tilde{f}(A_1, \dots, A_r)(z) \right),
 \end{aligned} \tag{3.22}$$

which implies that $\tilde{f}(A_1, \dots, A_r)$ is an $*$ -convex fuzzy set of Y . We complete the whole proof here. $\square$

4. Applications and Examples

Now we give some applications of the main results to the hyperstructure convexity invariance of type-2 fuzzy sets under hyperalgebra operations, and to the convexity invariance of fuzzy numbers under basic arithmetic operations.

4.1. Convexity Invariance of Type-2 Fuzzy Sets under Set Operations

Let $*$ be a t -norm, and let $\mu_A(x)$ and μ_B be fuzzy grades for type-2 fuzzy sets, A and B , represented as

$$\begin{aligned}
 \mu_A(x) &= \int \frac{g(u)}{u}, \quad u \in [0, 1], \\
 \mu_B(x) &= \int \frac{h(v)}{v}, \quad v \in [0, 1].
 \end{aligned} \tag{4.1}$$

Then the hyperalgebra operations for type-2 fuzzy sets are expressed as follows by using the extension principles, generalized union:

$$\begin{aligned}
 A \cup_* B &\iff \mu_{A \cup_* B}(x) = \mu_A(x) \cup_* \mu_B(x) \\
 &= \left(\int \frac{g(u)}{u} \right) \cup_* \left(\int \frac{h(v)}{v} \right) \\
 &= \int \frac{\sup_{w=\max(u,v)} (g(u) * h(v))}{w}
 \end{aligned} \tag{4.2}$$

and generalized intersection:

$$\begin{aligned}
 A \cap_* B &\iff \mu_{A \cap_* B}(x) = \mu_A(x) \cap_* \mu_B(x) \\
 &= \left(\int \frac{g(u)}{u} \right) \cap_* \left(\int \frac{h(v)}{v} \right) \\
 &= \int \frac{\sup_{w=\min(u,v)} (g(u) * h(v))}{w}.
 \end{aligned} \tag{4.3}$$

It should be noted that a type-2 fuzzy set is $*$ -convex if all of its fuzzy grades are $*$ -convex fuzzy sets. The following theorem is a generalization of the results in [4, 8].

Theorem 4.1. *Let $*$ be a continuous t -norm. If A and B are two $*$ -convex type-2 fuzzy sets, then $A \cup_* B$, $A \cap_* B$ are $*$ -convex type-2 fuzzy sets.*

Proof. Define two mappings $f_{\cup_*}, f_{\cap_*} : [0, 1]^2 \rightarrow [0, 1]$ as $f(x, y) = \max(x, y)$ and $f_{\cap_*}(x, y) = \min(x, y)$ for $(x, y) \in [0, 1]^2$, respectively. It is easy to see that the two mappings are continuous on $[0, 1]^2$. Thus, by Theorem 3.9, we can get the desired results. $\square$

Corollary 4.2 (see [4, 8]). *If A and B are two convex type-2 fuzzy sets, then $A \cup_{*M} B$, $A \cap_{*M} B$ are convex type-2 fuzzy sets.*

Corollary 4.3. *If A and B are two $*_P$ -convex type-2 fuzzy sets, then $A \cup_{*P} B$, $A \cap_{*P} B$ are $*_P$ -convex type-2 fuzzy sets.*

Corollary 4.4. *If A and B are two $*_L$ -convex type-2 fuzzy sets, then $A \cup_{*L} B$, $A \cap_{*L} B$ are $*_L$ -convex type-2 fuzzy sets.*

Corollary 4.5. *If A and B are two $*_D$ -convex type-2 fuzzy sets, then $A \cup_{*D} B$, $A \cap_{*D} B$ are $*_D$ -convex type-2 fuzzy sets.*

4.2. Convexity Invariance of Fuzzy Numbers under Basic Arithmetic Operations

In the early literature, fuzzy number is defined as a convex fuzzy set in a real line $\mathbb{R}$ [1, 5, 11, 12, 22]. Although this definition is very often modified nowadays, the convexity is always one of the conditions for a fuzzy set to be a fuzzy number. The arithmetic operations of $+$, $-$, $\times$, $\div$ for fuzzy numbers can be defined by using the Zadeh's multivariable extension principle.

Define four mappings $f_+, f_-, f_\times$, and $f_\div : \mathbb{R}^2 \rightarrow \mathbb{R}$ by $f_+(x, y) = x + y$, $f_-(x, y) = x - y$, $f_\times(x, y) = x \times y$, and $f_\div(x, y) = x \div y$, for $(x, y) \in \mathbb{R}^2$, respectively. It is easy to see that the first three mappings are continuous on $\mathbb{R}^2$, and $f_\div$ is continuous on $\mathbb{R} \times (0, +\infty)$ (or $\mathbb{R} \times (-\infty, 0)$). Thus, let the t -norm be $*_M$, and let the mapping be one of the above four mappings in Theorem 3.9 in turn, one can get the following theorem.

Theorem 4.6 (see [1, 5, 12]). *If A and B are fuzzy numbers in a real line $\mathbb{R}$, then $A + B$, $A - B$, and $A \times B$ are fuzzy numbers. In addition, if 0 does not belong to the support set $[B]_0$ of B , then $A \div B$ is a fuzzy number.*

5. Conclusions

In this work, we have discussed the convexity invariance of fuzzy sets under the extension principles. Particularly, we have given a necessary and sufficient condition for a mapping to be an inverse $*$ -convex transformation and have also obtained some sufficient conditions for a mapping to be an $*$ -convex transformation. Finally, two applications are given to illustrate the obtained results. The properties of the introduced concept, $*$ -convex transformation, certainly deserve further investigation.

Acknowledgments

The authors thank the anonymous reviewers for their valuable comments. This work was supported by the National Natural Science Foundation of China (Grant no. 11201512) and the Natural Science Foundation Project of CQ CSTC (cstc2012jjA00001).

References

- [1] D. Dubois and H. Prade, *Fuzzy Sets and Systems: Theory and Applications*, vol. 144 of *Mathematics in Science and Engineering*, Academic Press, New York, NY, USA, 1980.
- [2] E. P. Klement, R. Mesiar, and E. Pap, *Triangular Norms*, vol. 8, Kluwer Academic, Dordrecht, The Netherlands, 2000.
- [3] J. Mendel, *Uncertain Rule-Based Fuzzy Logic Systems*, Prentice Hall, Englewood Cliffs, NJ, USA, 2001.
- [4] M. Mizumoto and K. Tanaka, "Some properties of fuzzy sets of type-2," *Information and Computation*, vol. 31, no. 4, pp. 312–340, 1976.
- [5] M. Mizumoto and K. Tanaka, "Algebraic properties of fuzzy numbers," in *Proceedings of the IEEE International Conference on Cybernetics and Society*, pp. 559–563, Washington, DC, USA, 1976.
- [6] H. T. Nguyen and E. A. Walker, *A First Course in Fuzzy Logic*, Chapman & Hall/CRC, Boca Raton, Fla, USA, 2nd edition, 2000.
- [7] D. Qiu, L. Shu, and Z. W. Mo, "Notes on fuzzy complex analysis," *Fuzzy Sets and Systems*, vol. 160, no. 11, pp. 1578–1589, 2009.
- [8] C. L. Walker and E. A. Walker, "The algebra of fuzzy truth values," *Fuzzy Sets and Systems*, vol. 149, no. 2, pp. 309–347, 2005.
- [9] X. H. Yuan and E. S. Lee, "The definition of convex fuzzy subset," *Computers & Mathematics with Applications*, vol. 47, no. 1, pp. 101–113, 2004.
- [10] L. A. Zadeh, "Fuzzy sets," *Information and Computation*, vol. 8, pp. 338–353, 1965.
- [11] L. A. Zadeh, "The concept of a linguistic variable and its application to approximate reasoning. III," *Information Sciences*, vol. 9, no. 1, pp. 43–80, 1975.
- [12] H. Zimmermann, *Fuzzy Set Theory and Its Applications*, Kluwer Academic, Dordrecht, The Netherlands, 2nd edition, 1991.
- [13] E. E. Ammar, "Some properties of convex fuzzy sets and convex fuzzy cones," *Fuzzy Sets and Systems*, vol. 106, no. 3, pp. 381–386, 1999.
- [14] R. Lowen, "Convex fuzzy sets," *Fuzzy Sets and Systems*, vol. 3, no. 3, pp. 291–310, 1980.
- [15] D. Qiu, L. Shu, and Z. W. Mo, "On starshaped fuzzy sets," *Fuzzy Sets and Systems*, vol. 160, no. 11, pp. 1563–1577, 2009.
- [16] D. Qiu, F. Yang, and L. Shu, "On convex fuzzy processes and their generalizations," *International Journal of Fuzzy Systems*, vol. 12, no. 3, pp. 267–273, 2010.
- [17] D. Dubois and H. Prade, "Additions of interactive fuzzy numbers," *IEEE Transactions on Automatic Control*, vol. 26, no. 4, pp. 926–936, 1981.
- [18] J. Harding, C. Walker, and E. Walker, "Lattices of convex normal functions," *Fuzzy Sets and Systems*, vol. 159, no. 9, pp. 1061–1071, 2008.
- [19] H. Tahayori and G. Degli Antoni, "Operations on concavoconvex type-2 fuzzy sets," *International Journal of Fuzzy Systems*, vol. 10, no. 4, pp. 276–286, 2008.
- [20] H. Tahayori, A. Tettamanzi, G. Antoni, A. Visconti, and M. Moharrer, "Concave type-2 fuzzy sets: properties and operations," *Soft Computing*, vol. 14, no. 7, pp. 749–756, 2010.

- [21] L. A. Zadeh, "The concept of a linguistic variable and its application to approximate reasoning. I," vol. 8, no. 3, pp. 199–249, 1975.
- [22] L. A. Zadeh, "The concept of a linguistic variable and its application to approximate reasoning. II," *Information Sciences*, vol. 8, pp. 301–357, 1975.
- [23] W. Zhang, *Introduction To Fuzzy Mathematics*, Xi'an Jiaotong University Press, Xi'an, China, 1991.
- [24] J. R. Munkres, *Topology: A First Course*, Prentice Hall, Englewood Cliffs, NJ, USA, 1975.