Research Article

On Some Applications of a Special Integro-differential Operators

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Received 11 January 2010; Accepted 19 October 2010

Academic Editor: Nicolae Popa

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Let $C^{(n)}(\overline{\mathbb{D} \times \mathbb{D}})$ be a Banach space of complex-valued functions $f(x, y)$ that are continuous on $\overline{\mathbb{D} \times \mathbb{D}}$, where $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is the unit disc in the complex plane $\mathbb{C}$, and have n th partial derivatives in $\mathbb{D} \times \mathbb{D}$ which can be extended to functions continuous on $\overline{\mathbb{D} \times \mathbb{D}}$, and let $C_A^{(n)} = C_A^{(n)}(\mathbb{D} \times \mathbb{D})$ denote the subspace of functions in $C^{(n)}(\overline{\mathbb{D} \times \mathbb{D}})$ which are analytic in $\mathbb{D} \times \mathbb{D}$ (i.e., $C_A^{(n)} = C^{(n)}(\overline{\mathbb{D} \times \mathbb{D}}) \cap \mathcal{H}ol(\mathbb{D} \times \mathbb{D})$). The double integration operator is defined in $C_A^{(n)}$ by the formula $Wf(z, w) = \int_0^z \int_0^w f(u, v) dv du$. By using the method of Duhamel product for the functions in two variables, we describe the commutant of the restricted operator $W|_{E_{zw}}$, where $E_{zw} = \{f \in C_A^{(n)} : f(z, w) = f(zw)\}$ is an invariant subspace of W , and study its properties. We also study invertibility of the elements in $C_A^{(n)}$ with respect to the Duhamel product.

1. Introduction and Backgrounds

Let $\mathcal{H}ol(\mathbb{D} \times \mathbb{D})$ denote the Fréchet space of functions $f(z, w)$ that are holomorphic in the bidisc $\mathbb{D} \times \mathbb{D} = \{(z, w) \in \mathbb{C} \times \mathbb{C} : |z| < 1 \text{ and } |w| < 1\}$. The product we define on this space is

$$(f \circledast g)(z, w) := \frac{\partial^2}{\partial z \partial w} \int_0^z \int_0^w f(z-u, w-v) g(u, v) dv du, \quad (1.1)$$

which obviously defines an integrodifferential operator $\mathfrak{D}_f, \mathfrak{D}_f g := f \circledast g$. This product is a natural extension of the Duhamel product on $\mathcal{H}ol(\mathbb{D})$ [1]:

$$(f \circledast g)(z) := \frac{d}{dz} \int_0^z f(z-t) g(t) dt = \int_0^z f'(z-t) g(t) dt + f(0) g(z), \quad (1.2)$$

where the integrals are taken over the segment joining the points 0 and z .

Note that the Duhamel product is widely applied in various questions of analysis, for example, in the theory of differential equations and in solution of boundary value problems of mathematical physics. Wigley [2] showed that, for $p \geq 1$, the Hardy space $H^p(\mathbb{D})$ (which is the space of all holomorphic functions on the open unit disc $\mathbb{D}$ for which the norm

$$\|f\|_{H^p} = \sup_{0 < r < 1} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{1/p} \quad (1.3)$$

is finite) is a Banach algebra under the Duhamel product $\circledast$.

The Hardy space of the polydisc, $H^p(\mathbb{D}^n)$, is defined as those functions analytic on $\mathbb{D}^n := \mathbb{D} \times \cdots \times \mathbb{D}$ for which the following norm is finite:

$$\|f\|_{H^p} := \sup_{r_1 < 1} \cdots \sup_{r_n < 1} \left(\frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} |f(r_1 e^{i\theta_1}, \dots, r_n e^{i\theta_n})|^p d\theta_1 \cdots d\theta_n \right)^{1/p}. \quad (1.4)$$

If $p \geq 1$, this is a Banach space, and if $0 < r < 1$, this is a Fréchet space [3]. In [3], Merryfield and Watson proved that for $p \geq 1$ $H^p(\mathbb{D}^n)$ is a Banach algebra with respect to the product (1.1).

In the present paper we prove that the space E_{zw} can be given a Banach algebra structure under the Duhamel product (1.1); in particular, we describe the maximal ideal space of the Banach algebra $(E_{zw}, \circledast)$, where

$$E_{zw} := \left\{ f \in C_A^{(n)} : f(z, w) = f(zw) \right\}, \quad (1.5)$$

$$(f \circledast g)(zw) = \frac{\partial^2}{\partial z \partial w} \int_0^z \int_0^w f((z-u)(w-v)) g(uv) dv du. \quad (1.6)$$

By using product (1.6) we also describe commutant of the operator $W_{zw} := W|_{E_{zw}}$, that is, the set of bounded linear operators on E_{zw} commuting with W_{zw} . Moreover, we describe the set of cyclic vectors of the double integration operator W_{zw} acting on the closed subspace

$$H_{zw}^p := \left\{ f(z, w) \in H^p(\mathbb{D}^2) : f(z, w) = f(zw) \right\}. \quad (1.7)$$

We recall that a vector $x \in X$ is called cyclic vector for the operator $A \in \mathcal{L}(X)$ (Banach algebra of all bounded linear operators on a Banach space X) if

$$\text{span}\{x, Ax, A^2x, \dots\} = X, \quad (1.8)$$

where $\text{span}\{x, Ax, A^2x, \dots\}$ denotes the closure of the linear hull of the set $\{x, Ax, A^2x, \dots\}$.

2. Description of $\{W_{zw}\}'$

For any operator $A \in \mathcal{L}(X)$ its commutant $\{A\}'$ is defined by

$$\{A\}' := \{B \in \mathcal{L}(X) : BA = AB\}. \quad (2.1)$$

The study of commutant of the concrete operator $A \in \mathcal{L}(X)$ is one of the important, but generally, not easy problem of operator theory. For this, it is enough to remember the famous Lomonosov's theorem on the existence of nontrivial hyperinvariant subspace of compact operator $\mathcal{K}$ on a Banach space X (recall that a closed subspace $E \subset X$ is called hyperinvariant subspace for the operator $A \in \mathcal{L}(X)$, if it is invariant for any operator $B \in \{A\}'$). Note that many papers are devoted to the evident description of commutant (and, more generally, the set of so-called extended eigenvectors [4–6]) for some special operator classes (see, e.g., [7–14]). In this section we describe in terms of the Duhamel operators the commutant of the operator W_{zw} on the closed subspace E_{zw} of the space $C_A^{(n)}$. First, we prove the following lemma, which shows that E_{zw} is a Banach algebra under the Duhamel product $\circledast$ given by formula (1.6).

Lemma 2.1. $(E_{zw}, \circledast)$ is a Banach algebra.

Proof. Indeed, let $f, g \in E_{zw}$ be two functions. The norm in E_{zw} is defined by

$$\|f\|_n := \max \left\{ \max_{(z,w) \in \mathbb{D}^2} \left| \frac{\partial^{|\alpha|} f(zw)}{\partial z^{\alpha_1} \partial w^{\alpha_2}} \right| : |\alpha| = \alpha_1 + \alpha_2 = 0, 1, \dots, n \right\}. \quad (2.2)$$

Using (1.6), (2.2) and the Leibnitz formula for the partial derivatives of the product $f \circledast g$, it can be proved (which is omitted) that (see e.g., the method of the paper [15, 16])

$$\|f \circledast g\|_n \leq C_n \|f\|_n \|g\|_n \quad (2.3)$$

for some constant $C_n > 0$, which proves the lemma. □

The main result of this section is the following theorem.

Theorem 2.2. Let $T \in \mathcal{L}(E_{zw})$ be an operator. Then $T \in \{W_{zw}\}'$ if and only if there exists a function $\varphi \in E_{zw}$ such that $T = \mathfrak{D}_\varphi$, where $\mathfrak{D}_\varphi$ is the Duhamel operator defined by

$$\mathfrak{D}_\varphi f(zw) = (\varphi \circledast f)(zw) = \frac{\partial^2}{\partial z \partial w} \int_0^z \int_0^w \varphi((z-u)(w-v)) f(uv) dv du. \quad (2.4)$$

Proof. Let $T \in \{W_{zw}\}'$, that is,

$$TW_{zw} = W_{zw}T \quad (2.5)$$

Then we have that

$$TW_{zw}(zw)^k = W_{zw}T(zw)^k, \quad (2.6)$$

for all $k = 0, 1, \dots$, whence by computing $W_{zw}(zw)^k$ we have

$$\begin{aligned} T\left(\int_0^z \int_0^w (uv)^k dv du\right) &= T\left(\int_0^z u^k \left(\int_0^w v^k dv\right) du\right) = T\int_0^z u^k \frac{w^{k+1}}{k+1} du \\ &= T\left(\frac{z^{k+1}w^{k+1}}{(k+1)^2}\right) = \frac{1}{(k+1)^2}T(zw)^{k+1}, \end{aligned} \quad (2.7)$$

or

$$T(zw)^{k+1} = (k+1)^2 W_{zw}T(zw)^k, \quad (2.8)$$

for all $k = 0, 1, \dots$

From (2.8) we have by induction that

$$T(zw)^k = W_{zw}^k T1 \prod_{s=1}^k s^2 \quad (k = 1, 2, \dots). \quad (2.9)$$

Indeed, for $k = 1$ we have from (2.8) $T(zw) = W_{zw}T1$, as desired.

Assume for $k = n$ that

$$T(zw)^n = W_{zw}^n T1 \prod_{s=1}^n s^2. \quad (2.10)$$

For $k = n + 1$ we have from (2.8) that

$$T(zw)^{n+1} = (n+1)^2 W_{zw}T(zw)^n. \quad (2.11)$$

Now, by considering (2.10) from the latter equality we have

$$\begin{aligned} T(zw)^{n+1} &= (n+1)^2 W_{zw} \left(W_{zw}^n T1 \prod_{s=1}^n s^2 \right) \\ &= W_{zw}^{n+1} T1 (n+1)^2 \prod_{s=1}^n s^2 = W_{zw}^{n+1} T1 \prod_{s=1}^{n+1} s^2, \end{aligned} \quad (2.12)$$

which proves (2.9).

Now, let us show that

$$(W_{zw}^k f)(zw) = \int_0^z \int_0^w \frac{[(z-u)(w-v)]^{k-1}}{[(k-1)!]^2} f(uv) dv du. \quad (2.13)$$

For this purpose, first show that

$$(W_{zw}^k f)(zw) = \frac{(zw)^k}{[k!]^2} \circledast f(zw), \quad (2.14)$$

for all $k \geq 0$. Indeed, it follows directly from (1.6) that 1 is the unit with respect to the Duhamel product $\circledast$ in E_{zw} , and $W_{zw}f = zw \circledast f(zw)$ for every $f \in E_{zw}$. From this by induction we have equality (2.14) (we omit details).

Then we have

$$\begin{aligned} (W_{zw}^k f)(zw) &= \frac{(zw)^k}{[k!]^2} \circledast f(zw) = \frac{\partial^2}{\partial z \partial w} \int_0^z \int_0^w \frac{[(z-u)(w-v)]^k}{[(k!)]^2} f(uv) dv du \\ &= \frac{1}{(k!)^2} \int_0^z \int_0^w k^2 [(z-u)(w-v)]^{k-1} f(uv) dv du \\ &= \frac{k^2}{k^2 [(k-1)!]^2} \int_0^z \int_0^w [(z-u)(w-v)]^{k-1} f(uv) dv du \\ &= \int_0^z \int_0^w \frac{[(z-u)(w-v)]^{k-1}}{[(k-1)!]^2} f(uv) dv du, \end{aligned} \quad (2.15)$$

which proves (2.13).

Now by combining (2.9) and (2.13) we have

$$T(zw)^k = \prod_{s=1}^k s^2 \int_0^z \int_0^w \frac{[(z-u)(w-v)]^{k-1}}{[(k-1)!]^2} T1 dv du, \quad (2.16)$$

for all $k \geq 0$, which means that

$$T(zw)^k = (zw)^k \circledast T1 \quad (k \geq 0), \quad (2.17)$$

and hence

$$Tp(zw) = p(zw) \circledast T1, \quad (2.18)$$

for all polynomials p . Thus, by Lemma 2.1 and Weierstrass approximation theorem, we deduce that

$$\begin{aligned} (Tf)(zw) &= T1 \circledast f(zw) = \frac{\partial^2}{\partial z \partial w} \int_0^z \int_0^w f((z-u)(w-v))(T1)(uv) dv du \\ &= \frac{\partial^2}{\partial z \partial w} \int_0^z \int_0^w (T1)((z-u)(w-v)) f(uv) dv du \end{aligned}$$

$$\begin{aligned}
&= \int_0^z \int_0^w \frac{\partial^2}{\partial z \partial w} (T1)((z-u)(w-v)) f(uv) dv du \\
&= \int_0^z \int_0^w [(z-u)(w-v)(T1)_{zw}((z-u)(w-v)) \\
&\quad + (T1)_w((z-u)(w-v))] f(uv) dv du + (T1)(0) f(zw) \\
&= \int_0^z \int_0^w [\varphi_w((z-u)(w-v)) + (z-u)(w-v)\varphi_{zw}((z-u)(w-v))] f(uv) dv du \\
&\quad + \varphi_0 f(zw),
\end{aligned} \tag{2.19}$$

where

$$\varphi := T1 \in E_{zw}. \tag{2.20}$$

Thus,

$$(Tf)(zw) = \varphi(zw) \circledast f(zw) = \mathfrak{D}_\varphi f(zw) \tag{2.21}$$

for all $f \in E_{zw}$ and some $\varphi \in E_{zw}$. Conversely, if $\varphi \in E_{zw}$, then $\mathfrak{D}_\varphi$ commutes with W_{zw} . Since by Lemma 2.1 E_{zw} is a Banach algebra with respect to the Duhamel product $\circledast$, $\mathfrak{D}_\varphi$ is bounded operator on E_{zw} . The theorem is proved. $\square$

Corollary 2.3. *For a function φ in E_{zw} there exists a unique commutant T of the operator W_{zw} such that $T1 = \varphi(zw)$.*

Corollary 2.4. *One has $\{W_{zw}\}'' = \{W_{zw}\}'$, where $\{W_{zw}\}''$ stands for the bicommutant of the operator W_{zw} .*

Proof. It suffices to prove that $T_1 T_2 = T_2 T_1$ for every T_1, T_2 in $\{W_{zw}\}'$. Indeed, by Theorem 2.2, there exist $\varphi, \psi \in E_{zw}$ such that

$$\begin{aligned}
(T_1 f)(zw) &= \varphi(0) f(zw) + \int_0^z \int_0^w [\varphi_w((z-u)(w-v)) \\
&\quad + (z-u)(w-v)\varphi_{zw}((z-u)(w-v))] f(uv) dv du \\
&= (\varphi(0)I + \mathcal{K}_\Phi) f(zw),
\end{aligned} \tag{2.22}$$

$$\begin{aligned}
(T_2 f)(zw) &= \psi(0) f(zw) + \int_0^z \int_0^w [\psi_w((z-u)(w-v)) \\
&\quad + (z-u)(w-v)\psi_{zw}((z-u)(w-v))] f(uv) dv du \\
&= (\psi(0)I + \mathcal{K}_\Psi) f(zw)
\end{aligned} \tag{2.23}$$

for all $f \in E_{zw}$, where

$$\begin{aligned}
\Phi(zw) &:= \frac{\partial^2}{\partial z \partial w} \varphi(zw), \\
\Psi(zw) &:= \frac{\partial^2}{\partial z \partial w} \psi(zw), \\
\mathcal{K}_\Phi f(zw) &= (\Phi * f)(zw) := \int_0^z \int_0^w \frac{\partial^2}{\partial z \partial w} \varphi((z-u)(w-v)) f(uv) dv du \\
&= \int_0^z \int_0^w [\varphi_w((z-u)(w-v)) + (z-u)(w-v) \varphi_{zw}((z-u)(w-v))] f(uv) dv du, \\
\mathcal{K}_\Psi f(zw) &= (\Psi * f)(zw) := \int_0^z \int_0^w \frac{\partial^2}{\partial z \partial w} \psi((z-u)(w-v)) f(uv) dv du \\
&= \int_0^z \int_0^w [\psi_w((z-u)(w-v)) + (z-u)(w-v) \psi_{zw}((z-u)(w-v))] f(uv) dv du.
\end{aligned} \tag{2.24}$$

Since the usual convolution operators $\mathcal{K}_\Phi$ and $\mathcal{K}_\Psi$ are commuting operators, we have

$$T_1 T_2 = (\varphi(0)I + \mathcal{K}_\Phi)(\varphi(0)I + \mathcal{K}_\Psi) = (\varphi(0)I + \mathcal{K}_\Psi)(\varphi(0)I + \mathcal{K}_\Phi) = T_2 T_1, \tag{2.25}$$

which proves the corollary. $\square$

Theorem 2.5. *An operator $T \in \mathcal{L}(E_{zw})$ will be an isomorphism of the space E_{zw} into itself and commutes with W_{zw} if and only if it can be written in the form*

$$\begin{aligned}
(Tf)(zw) &= \varphi(0) f(zw) + \int_0^z \int_0^w [\varphi_w((z-u)(w-v)) \\
&\quad + (z-u)(w-v) \varphi_{zw}((z-u)(w-v))] f(uv) dv du,
\end{aligned} \tag{2.26}$$

and $\varphi(0) = (T1)|_{zw=0} \neq 0$.

Proof. If $T \in \mathcal{L}(E_{zw})$ is an isomorphism of the space E_{zw} into itself and commutes with W_{zw} , then by Theorem 2.2 we have for T representation (2.26) with $TW_{zw} = W_{zw}T$. Clearly, it follows from this equality and (2.26) that $\varphi(0) = (T1)|_{zw=0} \neq 0$.

Conversely, suppose that T has the form (2.26) with $\varphi(0) = (T1)|_{zw=0} \neq 0$, and prove then that $T \in \{W_{zw}\}'$ and T is an isomorphism on E_{zw} . Indeed, the inclusion $T \in \{W_{zw}\}'$ follows directly from Theorem 2.2. On the other hand, it is easy to see from the representation (2.26) that

$$T = \mathfrak{D}_\varphi = \varphi(0)I + \mathcal{K}_\Phi, \tag{2.27}$$

where $\mathcal{K}_\Phi$,

$$\mathcal{K}_\Phi f = \int_0^z \int_0^w [\varphi_w((z-u)(w-v)) + (z-u)(w-v)\varphi_{zw}((z-u)(w-v))] f(uv) dv du, \quad (2.28)$$

is the usual convolution operator on E_{zw} . It is not difficult to see that $\mathcal{K}_\Phi$ is a compact operator on E_{zw} .

Let us show that $\ker \mathfrak{D}_\varphi = \{0\}$. Indeed, let $\mathfrak{D}_\varphi f = 0$, where $f \in E_{zw}$. Then, $(\varphi(0)I + \mathcal{K}_\Phi)f = 0$, that is,

$$\frac{\partial^2}{\partial z \partial w} \int_0^z \int_0^w \varphi((z-u)(w-v)) f(uv) dv du = 0. \quad (2.29)$$

By standard calculation, we obtain from (2.29) that

$$\int_0^z \int_0^w \varphi((z-u)(w-v)) f(uv) dv du = c_1 z + c_2, \quad (2.30)$$

where c_1, c_2 are constants. Since

$$0 = \left(\int_0^z \int_0^w \varphi((z-u)(w-v)) f(uv) dv du \right) \Big|_{z=0} = (c_1 z + c_2) \Big|_{z=0} = c_2, \quad (2.31)$$

we have that $c_2 = 0$. On the other hand, since

$$0 = \left(\int_0^z \int_0^w \varphi((z-u)(w-v)) f(uv) dv du \right) \Big|_{w=0} = c_1 z, \quad (2.32)$$

for all $z \in \mathbb{D}$, we obtain that $c_1 = 0$. Thus,

$$\int_0^z \int_0^w \varphi((z-u)(w-v)) f(uv) dv du = 0, \quad (2.33)$$

for all $z \in \mathbb{D}$ and $w \in \mathbb{D}$. Now, by considering that $\varphi(0) \neq 0$ and φ is a continuous function on $\mathbb{D} \times \mathbb{D}$, by the Titchmarsh Convolution Theorem [17] for functions of several variables we deduce from (2.33) that $f(zw) = 0$ for all $z, w \in \mathbb{D}$, that is, $\ker T = \{0\}$. Since $\mathcal{K}_\Phi$ is compact, it follows from Fredholm alternative that T is invertible in E_{zw} , that is, T is an isomorphism. The theorem is proved. $\square$

From Theorem 2.5 and Corollary 2.3 we obtain the following.

Corollary 2.6. *For any function φ belonging to E_{zw} and satisfying $\varphi(0) \neq 0$, there exists a unique isomorphism T of the space E_{zw} such that T commutes with W_{zw} and $T1 = \varphi(zw)$.*

Corollary 2.7. *If $\varphi \in E_{zw}$ and $\varphi(0) \neq 0$, then the integrodifferential equation*

$$\begin{aligned} \varphi(0)x(zw) + \int_0^z \int_0^w [\varphi_w((z-u)(w-v)) + (z-u)(w-v)\varphi_{zw}((z-u)(w-v))]x(uv)dv du \\ = y(zw) \end{aligned} \quad (2.34)$$

has a unique solution for any right-hand side $y \in E_{zw}$.

Corollary 2.8. *The unique maximal ideal of the Banach algebra $(E_{zw}, \otimes)$ is $\{f \in E_{zw} : f(0) = 0\}$; that is, the maximal ideal space of $(E_{zw}, \otimes)$ consists of one homomorphism, namely evaluation at the origin $h(f) = f(0)$.*

3. Cyclic Vectors of W_{zw}

Let us consider the restricted operator $W_{zw} = W|_{H_{zw}^p}$. In this section we will describe the set of all cyclic vectors of this operator. The main result of this section is the following.

Theorem 3.1. *Let $f \in H_{zw}^p$. Then*

$$\text{span}\{W_{zw}^n f : n = 0, 1, 2, \dots\} = H_{zw}^p, \quad (3.1)$$

if and only if $f|_{zw=0} \neq 0$.

Proof. It is easy to verify that

$$W_{zw}^k g(zw) = \frac{(zw)^k}{(k!)^2} \otimes g(zw) \quad (k \geq 0), \quad (3.2)$$

for all $g \in H_{zw}^p$. Let us define the integrodifferential operator (or, briefly, the Duhamel operator) $\mathfrak{D}_f$ defined by $(\mathfrak{D}_f g)(zw) = (f \otimes g)(zw)$, $g \in H_{zw}^p$. By the known result of Merryfield and Watson (see [3, Corollary 2.6]), $H^p(\mathbb{D}^2)$, $p \geq 1$, is the Banach algebra with respect to the Duhamel product $\otimes$ defined by (1.1). Therefore, it is easy to see that $(H_{zw}^p, \otimes)$ is also Banach algebra, and hence, $\mathfrak{D}_f$ is a bounded operator on H_{zw}^p . Then it follows from (3.2) that

$$\begin{aligned} \text{span}\{W_{zw}^k f : k \geq 0\} &= \text{span}\left\{\frac{(zw)^k}{(k!)^2} \otimes f(zw) : k \geq 0\right\} \\ &= \text{span}\left\{\mathfrak{D}_f\left(\frac{(zw)^k}{(k!)^2}\right) : k \geq 0\right\} \\ &= \text{clos } \mathfrak{D}_f \text{span}\{(zw)^k : k \geq 0\} \\ &= \text{clos } \mathfrak{D}_f H_{zw}^p. \end{aligned} \quad (3.3)$$

Thus, f is a cyclic vector for W_{zw} if and only if $\mathfrak{D}_f$ has a dense range. Let us show that the latter is equivalent to the condition $f(0) \neq 0$. Clearly, if $\mathfrak{D}_f$ has a dense range then $f(0) \neq 0$. Conversely, let $f(0) \neq 0$. We will prove actually more strong result that $\mathfrak{D}_f$ is invertible in H_{zw}^p . Really, let us rewrite the operator $\mathfrak{D}_f$ in the form $\mathfrak{D}_f = f(0)I + \mathcal{K}_{\partial^2 f / \partial z \partial w}$, where I is the identity operator in H_{zw}^p and

$$\begin{aligned} \mathcal{K}_{\partial^2 f / \partial z \partial w} g(zw) &= \int_0^z \int_0^w [f_z((z-u)(w-v)) \\ &\quad + (z-u)(w-v)f_{zw}((z-u)(w-v))] g(uv) dv du. \end{aligned} \quad (3.4)$$

Since $\partial^2 f / \partial z \partial w$ is a continuous function, it is easy to see that $\mathcal{K}_{\partial^2 f / \partial z \partial w}$ is a compact operator (even Volterra operator) on H_{zw}^p . Now, as in the proof of Theorem 2.5, it follows from Titchmarsh Convolution Theorem that $\ker \mathfrak{D}_f = \{0\}$. Then, again by the Fredholm theorem we assert that $\mathfrak{D}_f$ is invertible, which completes the proof. $\square$

In conclusion, note that the study of the double integration operator W in the Lebesgue space $L^2([0, 1] \times [0, 1])$ was originated by Donoghue, Jr., in [18]. He showed that the operator W is not unicellular. Atzmon and Manos [19] proved that the multiplicity of spectrum $\mu(W)$ of the operator W is equal to $+\infty$ (we recall that the multiplicity of spectrum of the Banach space operator $A \in \mathcal{L}(X)$ is defined by $\mu(A) := \min\{\text{card } E : \text{span}\{A^n E : n \geq 0\} = X\}$). Some related results for W are also contained in the paper [15] by Karaev.

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