Research Article

A Note on Unbounded Hyponormal Composition Operators in L^2 -Spaces

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We construct an unbounded hyponormal composition operator C_ϕ in L^2 -space such that the domains of C_ϕ^2 and C_{ϕ^2} are trivial.

1. Introduction

The class of unbounded composition operators in L^2 -spaces seems to be an interesting object of research. It contains some classical operators (e.g., the shift operator on weighted ℓ^2 space), and, what is more important, it also contains operators with properties not known before, sometimes very surprising (e.g., a nonsubnormal operator which generates Stieltjes moment sequences, see [1]). The theory of it is in early stage of development, and the literature is minimal (cf. [2–5]).

In the present paper we provide an example of a hyponormal composition operator acting in $\ell^2(\mu)$ whose square has trivial domain. Existence of such example is relevant to the problem of characterizing subnormality of unbounded composition operators in L^2 -spaces. In a recent paper [4], it was shown that the celebrated Lambert's characterizations of subnormality for bounded composition operators (cf. [6]) are no longer valid in the unbounded case (see also [1]). The authors also elaborated on the role of density of $\bigcap_{n=1}^{\infty} D(C_\phi^n)$ in the underlying L^2 -space. It is important because of all known general criteria for subnormality of unbounded operators which do not require the density of $\bigcap_{n=1}^{\infty} D(C_\phi^n)$; some auxiliary objects are assumed to exist (cf. [7–10]). However, verifying these additional assumptions seems to be difficult in the context of composition operators. In view of this, the question of existence of a subnormal composition operator C_ϕ such that $\bigcap_{n=1}^{\infty} D(C_\phi^n)$ is not

dense in the underlying L^2 -space arises naturally and appears to be very intriguing (no such example is known). It is well known that subnormal operators are hyponormal. Therefore, our example sheds some light on the problem.

The first example of a nonsymmetric hyponormal operator such that the domain of its square is trivial was provided within the class of weighted shifts on directed trees (cf. [11]). As it turned out very recently (cf. [1, Lemma 4.3.1]), some weighted shifts on directed trees are unitarily equivalent to composition operators in ℓ^2 -spaces. Our construction is direct and independent of any connection between both the classes. It should be also mentioned that our example, which is build over a discrete measure space, can be adapted so as to obtain an example over a nonatomic measure space.

2. Preliminaries

We denote by $\mathbb{Z}$ and $\mathbb{N}$ the sets of integers and positive integers, respectfully. By $\mathbb{Z}_+$ we understand the set $\mathbb{N} \cup \{0\}$. If X is a set, then χ_X stands for the characteristic function of X ; the power set of X will be denoted by 2^X .

If $\mathcal{H}$ is a (complex) Hilbert space and A is a (linear) operator acting in $\mathcal{H}$, then $D(A)$ stands for the domain of A and A^* stands for the adjoint operator of A (if it exists). A densely defined operator A in $\mathcal{H}$ is said to be *hyponormal* if $D(A) \subseteq D(A^*)$ and $\|A^*f\| \leq \|Af\|$ for all $f \in D(A)$. Interested reader is referred to [12–14] for some pieces of the theory of unbounded hyponormal operators.

We recall now some information on composition operators in L^2 -spaces. We restrict ourself to the case of composition operator induced by a transformation of a countable set because our example is built in this way (general setting is presented in [4, 15, 16]). Let X be a countably infinite set, let μ be a measure on 2^X such that $\mu(\{x\}) > 0$ for every $x \in X$, and let $\phi : X \rightarrow X$. By $\ell^2(\mu)$ we denote the Hilbert space of all complex valued functions on X which are square-summable with respect to the measure μ . The following formula

$$\begin{aligned} D(C_\phi) &= \{f \in \ell^2(\mu) : f \circ \phi \in \ell^2(\mu)\}, \\ C_\phi f &= f \circ \phi, \quad f \in D(C_\phi), \end{aligned} \tag{2.1}$$

defines the so-called *composition operator*

$$C_\phi : \ell^2(\mu) \supseteq D(C_\phi) \longrightarrow \ell^2(\mu). \tag{2.2}$$

It is well known that C_ϕ is a closed linear operator in $\ell^2(\mu)$. Typically, properties of C_ϕ are written in terms of the canonical Radon-Nikodym derivative h_ϕ attached to C_ϕ (cf. [2–4, 6, 17–22]). In our setting, h_ϕ is given by

$$h_\phi(x) = \frac{\mu(\phi^{-1}(\{x\}))}{\mu(\{x\})}, \quad x \in X. \tag{2.3}$$

The operator of conditional expectation is also frequently employed when investigating composition operators. Let us recall that for a σ -algebra $\mathcal{A} \subseteq 2^X$ generated by a partition

$\{\sigma_i : i \in \mathcal{I}\}$ of X and for a nonnegative function f on X , the conditional expectation of f with respect to $\mathcal{A}$ can be defined as follows:

$$E(f \mid \mathcal{A}) = \sum_{i \in \mathcal{I}} \frac{1}{\mu(\sigma_i)} \left(\int_{\sigma_i} f d\mu \right) \cdot \chi_{\sigma_i}. \quad (2.4)$$

We refer the reader to [23] for background on conditional expectation.

The following lemma delivers few basic properties of composition operators in $\ell^2(\mu)$ which we use to obtain the example. The proofs are straightforward; nonetheless we include them for completeness. One more piece of notation will be useful: $\mathcal{E}$ stands for the set $\{\chi_{\{x\}} : x \in X\}$.

Lemma 2.1. *The following conditions are equivalent:*

- (i) C_ϕ is densely defined.
- (ii) $\mathcal{E} \subseteq D(C_\phi)$.
- (iii) $\mu(\phi^{-1}(\{x\})) < \infty$ for all $x \in X$.

Moreover, if C_ϕ is densely defined, then the following assertion holds:

- (iv) C_ϕ is hyponormal if and only if

$$\frac{1}{\mu(\{x\})} \sum_{y \in \phi^{-1}(\{x\})} \frac{\mu(\{y\})^2}{\mu(\phi^{-1}(\{y\}))} \leq 1, \quad x \in \phi(X). \quad (2.5)$$

Proof. By definition, $\mathcal{E} \subseteq D(C_\phi)$ if and only if $\chi_{\{x\}} \circ \phi \in \ell^2(\mu)$ for all $x \in X$. Since

$$\int |\chi_{\{x\}} \circ \phi|^2 d\mu = \int |\chi_{\phi^{-1}(\{x\})}|^2 d\mu = \mu(\phi^{-1}(\{x\})), \quad x \in X, \quad (2.6)$$

we see that conditions (ii) and (iii) are equivalent.

That (ii) implies (i) is obvious. The converse can be proved as follows. Let $f \in D(C_\phi)$. Then $f = \sum_{x \in X_f} f(x) \cdot \chi_{\{x\}}$, where $X_f := \{x \in X : f(x) \neq 0\}$. If $x_0 \in X_f$, then $\chi_{\{x_0\}} \in D(C_\phi)$ because of the following

$$|f(x_0)|^2 \int \chi_{\{x_0\}} \circ \phi d\mu = \int |f \circ \phi|^2 \cdot \chi_{\phi^{-1}(\{x_0\})} d\mu \leq \int |f \circ \phi|^2 d\mu < \infty. \quad (2.7)$$

If $\overline{D(C_\phi)} = \ell^2(\mu)$, then clearly $X = \bigcup_{f \in D(C_\phi)} X_f$. Therefore, $\chi_{\{x\}} \in D(C_\phi)$ for every $x \in X$, which gives (ii).

For the proof of the “moreover” statement we recall a general characterization of hyponormality of a densely defined composition operator induced by a nonsingular

transformation of a measure space (X, Σ, μ) (see [2, Corollary 6.7]): C_ϕ is hyponormal if and only if

$$h_\phi \circ \phi \cdot E\left(\frac{1}{h_\phi}\right) \leq 1; \quad (2.8)$$

the conditional expectation above is calculated with respect to the completion of the σ -subalgebra $\phi^{-1}(\Sigma)$ of Σ . In our situation the σ -subalgebra is generated by the partition $\{\phi^{-1}(\{x\}) : x \in \phi(X)\}$. The characterization above combined with (2.3) and (2.4) yields (iv). $\square$

By inspecting the part “(i) implies (ii)” of the proof above we can get the following description of when the domain of a composition operator in $\ell^2(\mu)$ is trivial.

Corollary 2.2. $D(C_\phi) = \{0\}$ if and only if $\mathcal{X} \cap D(C_\phi) = \emptyset$.

3. The Example

The main result of this paper is the existence of a hyponormal composition operator whose square has trivial domain.

Theorem 3.1. *There exists a countably infinite set X , a positive measure μ on 2^X and a map $\phi : X \rightarrow X$ such that the composition operator C_ϕ acting in $\ell^2(\mu)$ is hyponormal and $D(C_{\phi^2}) = D(C_\phi^2) = \{0\}$.*

Proof. Set $X = \mathbb{N} \times \{1, 2\}$. Let $\{X(n, k) : n \in \mathbb{N}, k = 1, 2\}$ be a family of subsets of X defined according to the following scheme:

$$\begin{aligned} X(1, k) &= \{(3, k), (7, k), (12, k), \dots\}, \\ X(2, k) &= \{(1, k), (4, k), (8, k), (13, k), \dots\}, \\ X(3, k) &= \{(2, k), (5, k), (9, k), (14, k), \dots\}, \\ X(4, k) &= \{(6, k), (10, k), (15, k), (21, k), \dots\}, \\ X(5, k) &= \{(11, k), (16, k), (22, k), \dots\}. \\ &\vdots \end{aligned} \quad (3.1)$$

Then the sets $X(n, k)$ are pairwise disjoint, infinite, and $X = \bigcup_{n \in \mathbb{N}, k=1,2} X(n, k)$.

Now, we define an atomic measure μ on X . For this we take any sequence $\{a(n)\}_{n=1}^\infty$ of positive real numbers such that

$$2^{11} \leq 2^{k+2}(a(k))^2 \leq a(k+1), \quad k \in \mathbb{N}. \quad (3.2)$$

Clearly, $\{a(n)\}_{n=1}^{\infty}$ is monotonically increasing and $a(k) \geq 2^{k+3}$ for all $k \in \mathbb{N}$. Suppose $x \in X(n, 1)$ for some $n \in \mathbb{N}$. Since $X(n, 1) = \{(k_0, 1), (k_1, 1), (k_2, 1), \dots\}$, where $k_0 < k_1 < k_2 < \dots$ are nonnegative integers, there exists $i \in \mathbb{Z}_+$ such that $x = (k_i, 1)$. Then we put

$$\mu(\{x\}) = \frac{a(n)}{2^i}. \quad (3.3)$$

On the other hand, if $x \in \mathbb{N} \times \{2\}$, then we set

$$\mu(\{x\}) = 1. \quad (3.4)$$

Since $X = \bigcup_{n \in \mathbb{N}, k=1,2} X(n, k)$, the measure μ is defined on the whole of 2^X . Observe that $\mu(X(n, 1)) = 2a(n)$ for every $n \in \mathbb{N}$. In order to define a transformation ϕ of X , we note that $X = \{x_n : n \in \mathbb{N}\}$, where

$$x_n := \begin{cases} (1, 2), & \text{if } n = 1, \\ (l + 1, 2), & \text{if } n = 2l, l \in \mathbb{N}, \quad n \in \mathbb{N}, \\ (l, 1), & \text{if } n = 2l + 1, l \in \mathbb{N}. \end{cases} \quad (3.5)$$

It is easy to see that $x_m \notin X(m, 1)$ for $m \in \mathbb{N}$. Moreover, if $x_i \in X(m, 1)$ for some $i, m \in \mathbb{N}$, then $i > m$. Now, if $x \in X(n, 1)$ for some $n \in \mathbb{N}$, then we set

$$\phi(x) = x_n, \quad (3.6)$$

if $x = (m, 2)$ for some $m \in \mathbb{N}$, then we put

$$\phi(x) = (m, 1). \quad (3.7)$$

Again, equality $X = \bigcup_{n \in \mathbb{N}, k=1,2} X(n, k)$ yields that the transformation ϕ is defined on the whole of X .

Now, we derive some complementary information on ϕ and μ . Choose $x \in X$. Then $x = (p, k) \in X(n, k)$ for some $p, n \in \mathbb{N}$ and $k \in \{1, 2\}$. Simultaneously, $x = x_i$ for some $i \in \mathbb{N}$. Therefore, we gather that

$$\phi^{-1}(\{x\}) = \begin{cases} X(i, 1), & \text{for } k = 2, \\ X(i, 1) \cup \{(p, 2)\}, & \text{for } k = 1. \end{cases} \quad (3.8)$$

Furthermore, if $y \in \phi^{-1}(\{x\})$, then either $y = (p, 2)$ or $y = (r, 1)$ for some $r \in \mathbb{N}$. In both cases $y = x_m$ for some $m \in \mathbb{N}$. This yields

$$\phi^{-1}(\{y\}) = \begin{cases} X(m, 1) \cup \{(r, 2)\}, & \text{for } k = 2, \\ X(m, 1), & \text{for } k = 1, \quad y = (p, 2), \\ X(m, 1) \cup \{(r, 2)\}, & \text{for } k = 1, \quad y \neq (p, 2). \end{cases} \quad (3.9)$$

Suppose now that $x \in X(n, 1)$. Then, as above, $x = (p, 1) = \mathbf{x}_i$ for $p, i \in \mathbb{N}$. Note that

$$\mu(\{x\}) \geq \frac{a(n)}{2^i} \geq \frac{1}{2^i}. \quad (3.10)$$

Also, if $y = \mathbf{x}_l \in \phi^{-1}(\{x\}) \setminus \{(p, 2)\}$, then $l > i$. This, (3.8), (3.9), definition of μ , and (3.2), implies that

$$\begin{aligned} \sum_{y \in \phi^{-1}(\{x\})} \frac{\mu(\{y\})^2}{\mu(\phi^{-1}(\{y\}))} &= \frac{\mu(\{(p, 2)\})^2}{\mu(\phi^{-1}(\{(p, 2)\}))} + \sum_{y \in X(i, 1)} \frac{\mu(\{y\})^2}{\mu(\phi^{-1}(\{y\}))} \\ &\leq \frac{1}{\mu(X(m', 1))} + \sum_{y \in X(i, 1)} \frac{\mu(\{y\})^2}{\mu(X(i+1, 1))} \\ &\leq \frac{1}{2^{m'+4}} + \frac{\mu(X(i, 1))^2}{\mu(X(i+1, 1))} \leq \frac{1}{2^{m'+4}} + \frac{1}{2^{i+1}}, \end{aligned} \quad (3.11)$$

where $m' \in \mathbb{N}$ obeys $(p, 2) = \mathbf{x}_{m'}$. For such an m' we either have $i = m' + 2$ (for $p = 1$) or $i = m' + 3$ (for $p > 1$), thus we obtain

$$\sum_{y \in \phi^{-1}(\{x\})} \frac{\mu(\{y\})^2}{\mu(\phi^{-1}(\{y\}))} \leq \frac{1}{2^i}. \quad (3.12)$$

This together with (3.10) implies that for all $x \in \bigcup_{n=1}^{\infty} X(n, 1)$ we have

$$\sum_{y \in \phi^{-1}(\{x\})} \frac{\mu(\{y\})^2}{\mu(\phi^{-1}(\{y\}))} \leq \mu(\{x\}). \quad (3.13)$$

On the other hand, if $x = \mathbf{x}_i \in X(n, 2)$, then $\mu(\{x\}) = 1$. Moreover, for such an x we have

$$\sum_{y \in \phi^{-1}(\{x\})} \frac{\mu(\{y\})^2}{\mu(\phi^{-1}(\{y\}))} = \sum_{y \in X(i, 1)} \frac{\mu(\{y\})^2}{\mu(\phi^{-1}(\{y\}))} \leq \sum_{y \in X(i, 1)} \frac{\mu(\{y\})^2}{\mu(X(i+1, 1))}. \quad (3.14)$$

Therefore, arguing as above we see that the inequality in (3.13) holds for all $x \in \mathbb{N} \times \{2\}$ as well.

It is easy to see that the operator C_ϕ is densely defined. This follows immediately from Lemma 2.1, formula (3.8), and the fact that $\mu(X(i, 1)) < \infty$ for all $i \in \mathbb{N}$. Since (3.13) holds for every $x \in X$, by Lemma 2.1(iv) we get hyponormality of C_ϕ . Now, we observe that both operators C_{ϕ^2} and C_ϕ^2 have trivial domains. This is a consequence of the fact that there is no $x \in X$ such that $\chi_{\{x\}} \in D(C_{\phi^2})$. Indeed, take any $x \in X$. Then $x = \mathbf{x}_n$ for some $n \in \mathbb{N}$ and hence $X(n, 2) \subseteq (\phi^2)^{-1}(\{x\})$. By definition, we have $\chi_{\{x\}} \notin D(C_{\phi^2})$. Therefore, by Corollary 2.2, the domain of C_{ϕ^2} is trivial. Since $D(C_\phi^2) \subseteq D(C_{\phi^2})$ (cf. [4, Theorem 5.1]) we get the claim. $\square$

4. Final Remarks

(1) It is known that there are closed symmetric operators whose squares possess trivial domains (cf. [24, 25]). Since each symmetric operator is automatically subnormal, one could expect to find a subnormal composition operator with trivial square among symmetric ones. However, this is not the case of the space $\ell^2(\mu)$. The reason for this is that for every symmetric composition operator C_ϕ acting in $\ell^2(\mu)$ the space $\bigcap_{n=1}^{\infty} D(C_\phi^n)$ is dense in $\ell^2(\mu)$ (cf. [5]).

(2) By using [3, Theorem 2.7] one can adapt our example, so as to obtain a composition operator acting in L^2 -space over a nonatomic measure space (in fact, the measure could be equivalent to the Lebesgue measure on the unit interval), being hyponormal and having trivial square.

(3) A considerable amount of attention has been given to the study of p -hyponormal operators, also in the context of composition operators (cf. [21, 26–29]). Recall that a bounded operator T is said to be p -hyponormal, $p \in (0, \infty)$, if $(TT^*)^p \leq (T^*T)^p$. A complete characterization of p -hyponormal composition operators on L^2 -spaces was given in [21]. It resembles the characterization of hyponormality and states that C_ϕ is p -hyponormal if and only if $h_\phi > 0$ a.e. $[\mu]$ and

$$h_\phi^p \circ \phi \cdot E\left(\frac{1}{h_\phi^p}\right) \leq 1 \quad \text{a.e. } [\mu]. \quad (4.1)$$

To the best of our knowledge, the notion has not been introduced to the theory of unbounded operators yet. One can use condition (4.1) as a definition of “unbounded” p -hyponormality, of course only for composition operators. Our construction can be adjusted so as to obtain a composition operator satisfying (4.1) for any $p \geq 1$ (it suffices to modify (3.2) slightly). It is also worth pointing out that in our setting the “ p -hyponormality” condition takes the following form:

$$\frac{\mu(\phi^{-1}(\{x\}))^{p-1}}{\mu(\{x\})^p} \sum_{y \in \phi^{-1}(\{x\})} \frac{\mu(\{y\})^{p+1}}{\mu(\phi^{-1}(\{y\}))^p} \leq 1, \quad x \in \phi(X). \quad (4.2)$$

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