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Research Article

A Note on Some Uniform Algebra Generated by Smooth Functions in the Plane

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We determine, via classroom proofs, the maximal ideal space, the Bass stable rank as well as the topological and dense stable rank of the uniform closure of all complex-valued functions continuously differentiable on neighborhoods of a compact planar set K and holomorphic in the interior K° of K. In this spirit, we also give elementary approaches to the calculation of these stable ranks for some classical function algebras on K.

1. Introduction

Let K be a compact subset of the complex plane $\mathbb{C}$. We denote by $A_1(K)$ the uniform closure on K of all complex-valued functions continuously differentiable on neighborhoods of K and holomorphic in the interior K° of K. As usual, let $P(K)$ be the uniform closure on K of the set of polynomials $\sum_{j=0}^{n} a_j z^j$, $R(K)$ the uniform closure on K of those rational functions that have no poles in K, and $A(K)$ the uniform algebra of all continuous functions on K that are holomorphic in K°. Finally, $C(K)$ is the algebra of all complex-valued continuous functions on K. We obviously have that $P(K) \subseteq R(K) \subseteq A_1(K) \subseteq A(K) \subseteq C(K)$. Whereas $P(K)$, $R(K)$, $A(K)$, and $C(K)$ are classical objects that are well understood (see, e.g., the books by Browder [1], and Gamelin [9]), we do not know a paper or textbook dealing with this intermediate algebra $A_1(K)$.

In the present paper we are mainly interested in solving the Bézout equation $\sum_{j=1}^{n} x_j f_j = 1$ in these algebras, with particular emphasis on $A_1(K)$. By giving classroom

proofs of our results, we hope that we will make the notions of stable ranks appearing below accessible to a larger group of analysts, especially students willing to work in function theory.

So let A be any of the algebras above. The leading questions will be the following.

(1) Suppose that (f, g) is a pair of functions in A such that f and g have no common zeros on K. Does there exist a solution $(x, y) \in A^2$ of the Bézout equation $xf + yg = 1$ such that x is invertible?

(2) Let $(f, g) \in A^2$ be arbitrary. Is it possible to uniformly approximate (f, g) by a pair (u, v) of functions in A^2 such that u and v have no common zeros on K?

These questions originally stem from algebraic K-theory. In the abstract setting of commutative rings or Banach algebras, they run under the heading "Bass stable rank" and "topological stable rank," and go back to Bass and Rieffel.

The intention of our paper is now twofold. We first present an elementary approach to the calculation of the stable ranks for the classical algebras $P(K), R(K)$, and $C(K)$ without using sophisticated methods or notions from algebraic topology and without using deep results as the Arens-Taylor-Novodvorsky theory (see the introduction to [5], for details of this theory). These results were known.

Then, we apply our techniques to the new algebra $A_1(K)$. So, in Section 4 we determine the maximal ideal space of $A_1(K)$, and the $A_1(K)$-convex sets. These results will be used in Section 5 to determine the Bass, topological, and dense stable rank of $A_1(K)$, which represent the main results of our paper.

2. The Central Definitions

Here we recall those definitions necessary to understand this paper. Let X be a compact Hausdorff space, and let $C(X)$ be the algebra of all complex-valued continuous functions on X endowed with the supremum norm. A uniform algebra A on X is a uniformly closed subalgebra of $C(X)$ separating the points of X and having as unit the constant function 1. Its maximal ideal space, or spectrum, $M(A)$, is the set of all nonzero, multiplicative linear functionals on A. As usual we will identify a function $f \in A$ with its Gelfand transform $\widehat{f}$ defined on $M(A)$ by $\widehat{f}(m) = m(f)$.

In the sequel, let $\|f\|_E := \sup_{z\in E}|f(z)|$. A closed subset E of the spectrum $M(A)$ of A is called *A-convex*, if E coincides with its A-convex hull:

$$\widehat{E} = \left\{ m \in M(A) : \left|f(m)\right| \leq \max_E \left|f\right| \ \forall f \in A \right\}. \tag{2.1}$$

It is well known that $\widehat{E}$ can be identified with the spectrum of the algebra $\overline{A|_E}$ (see [9, page 39]). Let us also note that $M(P(K)) = \widehat{K}$ [9, page 27]. In this setting

$$\widehat{K} = \left\{ z \in \mathbb{C} : \left|f(z)\right| \leq \max_K \left|f\right| \ \forall f \in P(K) \right\} \tag{2.2}$$

and $\widehat{K}$ is called the *polynomial convex hull of* K. Note that $\widehat{K}$ is the union of K with all the bounded components of $\mathbb{C} \setminus K$, which we will call holes.

An n-tuple $(f_1, \ldots, f_n) \in A^n$ is said to be *invertible* (or unimodular) if there exists $(x_1, \ldots, x_n) \in A^n$ such that $\sum_{j=1}^{n} x_j f_j = 1$. The set of all invertible n-tuples is denoted by $U_n(A)$. An $(n+1)$-tuple $(f_1, \ldots, f_n, g) \in U_{n+1}(A)$ is called *reducible* if there exists $(a_1, \ldots, a_n) \in A^n$ such that $(f_1 + a_1 g, \ldots, f_n + a_n g) \in U_n(A)$.

The *Bass stable rank* of A, denoted by $\operatorname{bsr}(A)$, is the smallest integer n such that every element in $U_{n+1}(A)$ is reducible. If no such n exists, then $\operatorname{bsr}(A) = \infty$.

The *topological stable rank*, $\operatorname{tsr}(A)$, of A is the least integer n for which $U_n(A)$ is dense in A^n, or infinite if no such n exists. This notion goes back to Rieffel [19].

The following concept was introduced by Corach and Suárez [7, page 542]. A version from [18] reads as follows.

The *dense stable rank*, $\operatorname{dsr}(A)$, of A is the least integer n such that for every A-convex set E in the character space of A the Gelfand transform of any n-tuple $(f_1, \ldots, f_n) \in A^n$ satisfying $\sum_{j=1}^{n} |\hat{f}_j| \geq \delta > 0$ on E can be uniformly approximated on E by the Gelfand transform of n-tuples $(a_1, \ldots, a_n)$ that are invertible in A.

It is well known that $\operatorname{bsr}(A) \leq \operatorname{dsr}(A) \leq \operatorname{tsr}(A)$ (see [7, 17]).

Stable ranks of various real or complex function algebras have mainly been determined by Corach and Suárez [4, 6, 7], Rupp [20–22], Rupp and Sasane [24, 25], Mikkola and Sasane [14], and Mortini and Wick [16, 17].

3. The Stable Ranks of $P(K)$, $R(K)$, and $C(K)$: A Classroom Approach

As a major tool, we use the following lemma, due to Corach and Suárez [4, 6].

Lemma 3.1 (see [4, page 636] and [6, page 608]). *Let A be a commutative unital Banach algebra. Then, for $g \in A$, the set*

$$R_n(g) = \{(f_1, \ldots, f_n) \in A^n : (f_1, \ldots, f_n, g) \text{ is reducible}\} \tag{3.1}$$

is open-closed inside

$$I_n(g) = \{(f_1, \ldots, f_n) \in A^n : (f_1, \ldots, f_n, g) \in U_{n+1}(A)\}. \tag{3.2}$$

In particular, for $n = 1$, if $\phi : [0,1] \to I_1(g)$ is a continuous curve and $(\phi(0), g)$ is reducible, then $(\phi(1), g)$ is reducible.

In the case where $n = 1$ a pretty proof goes as follows (see [17]).

Let $(x, y) \in A^2$ be so that $1 = xf + yg$. Suppose that $f_n \to f$ and (f_n, g) is reducible, say $f_n + u_n g \in A^{-1}$. Write

$$f_n + u_n g = f(xf_n + yg) + (y(f_n - f) + u_n)g. \tag{3.3}$$

Since $\|f_n - f\|_A \to 0$, we may choose n_0 so big that for all $n \geq n_0$ the elements $xf_n + yg$ are invertible in A. Hence $f + h_n g$ is invertible for some h_n. Thus, $R_1(g)$ is closed within $I_1(g)$. Now if $u = f + hg$ is invertible, then every small perturbation of u is invertible, too. Thus, $R_1(g)$ is open.

3.1. The Algebras $R(K)$ and $P(K)$

Corach and Suárez showed in [4] that $\mathrm{bsr}(P(K)) = \mathrm{bsr}(R(K)) = 1$. In our opinion, their proof on page 638 contains a gap, since, in general, the boundary of a component of K is not accessible via a path staying outside K. For example, let K be the union of the closed disk $\overline{\mathbb{D}}$ and a spiral outside $\overline{\mathbb{D}}$ that clusters at $\mathbb{T} = \partial\mathbb{D}$.

So, for the reader's convenience, we present a short proof here (that is very close to the original one, though). We begin with a Lemma, used in [4], without a proof. We think that this needs a proof though, since a priori it is not clear that a domain admits accessible boundary points in the sense described below.

Lemma 3.2. *Let $K \subseteq \mathbb{C}$ be compact, $a \in K^\circ$. If $g \in A(K)$ does not vanish at a, then there exist $b \in \partial K$ and a piecewise C^1-path outside the zero set of g joining a and b within $K^\circ \cup \{b\}$.*

Proof. Let G be the connected component of a in K°. Since $g(a) \neq 0$, the maximum principle implies that g does not vanish identically on the boundary of G, a set that is contained in K. Let $w \in \partial G$ with $g(w) \neq 0$. Choose a disk U centered at w so that $g \neq 0$ on $U \cap K$. Let $w' \in U \cap G$. Since, as an open set, G is pathconnected, there is a path γ_1 in G joining w' with a. Since at most finitely many zeros of g could belong to such a path, we can easily avoid these zeros by perturbing a little bit the original path γ_1. Next, on the segment joining w' with w, there is a first boundary point of G, say b. The combined path now joins a with b and stays outside the zero set of g. □

Theorem 3.3 (Corach-Suárez). *Let $K \subseteq \mathbb{C}$ be compact. Then,*

(1) $\mathrm{bsr}(R(K)) = 1$;

(2) $\mathrm{tsr}(R(K)) = 1$ *if and only if* $K^\circ = \emptyset$;

(3) $\mathrm{tsr}(R(K)) = 2$ *if and only if* $K^\circ \neq \emptyset$.

Proof. (1) Let (f, g) be an invertible pair in $R(K)$, and let (r_n) be a sequence of rational functions with poles outside K uniformly approximating f on K. By Lemma 3.1, in order to show the reducibility of (f, g), it suffices to show the reducibility of (r_n, g) for n large. Now $r_n = p/q$ for some polynomials p and q. We may assume that p and q have no common zeros. Obviously, q is invertible in $R(K)$. Also, since the reducibility of (f_1, g) and (f_2, g) implies that of $(f_1 f_2, g)$, it suffices to show that invertible pairs of the form $(z - a, g)$ are reducible. We have to deal with three cases.

Case 1. Let $a \in \mathbb{C} \setminus K$. Then, $z - a$ is invertible in $R(K)$ and so $(z - a, g)$ is reducible.

Case 2. Let $a \in \partial K$ satisfy $g(a) \neq 0$. Choose a sequence, (a_n), of points outside K converging to a. Then, the pairs $(z - a_n, g)$ are reducible and hence, by Lemma 3.1, $(z - a, g)$, too.

Case 3. Now let $a \in K^0$ and $g(a) \neq 0$. Choose according to Lemma 3.2 a curve γ in K, joining a with a boundary point b of K such that γ stays outside the zero set of g. The curve $\phi : [0, 1] \to I_1(g)$, given by $\phi(t) = z - \gamma(t)$, is a continuous curve, and $(\phi(1), g) = (z - b, g)$ is reducible. Thus, by Lemma 3.1, $(\phi(0), g) = (z - a, g)$ is reducible.

(2), (3) We first note that $\operatorname{tsr}(R(K)) \leq 2$. In fact, let (f, g) be a pair of functions in $R(K)$. Then, for any $\varepsilon > 0$, there exist two rational functions r_f, r_g with poles off K such that

$$\|f - r_f\|_K + \|g - r_g\|_K < \varepsilon. \tag{3.4}$$

By slightly perturbing, if necessary, the zeros of r_f, we may assume that r_f and r_g have no zeros in common. Thus, (r_f, r_g) is an invertible pair and so $\operatorname{tsr}(R(K)) \leq 2$.

Now if K has no interior points, then any $f \in R(K)$ can be uniformly approximated on K by rational functions without poles and zeros on K. Hence, $\operatorname{tsr}(R(K)) = 1$ if $K^\circ = \emptyset$. If $K^\circ \neq \emptyset$, then, by Rouché's theorem, the function $z - a$ with $a \in K^\circ$ cannot be uniformly approximated by holomorphic functions invertible in a neighborhood of a. Thus, $\operatorname{tsr}(R(K)) \geq 2$. Keeping in mind that $\operatorname{tsr}(R(K)) \leq 2$, we deduce that $\operatorname{tsr}(R(K)) = 2$. □

A similar result holds of course for the algebras $P(K)$, too. Since we were unable to find the assertions on the topological stable rank in the literature, we provide them for the reader's convenience. Recall that for compact sets in $\mathbb{C}^n$ the Bass stable rank for $P(K)$ was determined by Corach and Suárez in [7].

Theorem 3.4. *Let $K \subseteq \mathbb{C}$ be a compact set. Then,*

(1) $\operatorname{bsr}(P(K)) = 1$;

(2) $\operatorname{tsr}(P(K)) = 1$ *if and only if* $K^\circ = \emptyset$ *and K has no holes;*

(3) $\operatorname{tsr}(P(K)) = 2$ *if and only if* $K^\circ \neq \emptyset$ *or K has holes.*

Proof. All three assertions follow as above, by noticing that $M(P(K))$ coincides with the polynomial convex hull, $\widehat{K}$, of K (i.e., the union of K with all its holes) and that $R(\widehat{K}) = P(\widehat{K}) \underset{\|\cdot\|}{\cong} P(K)$. □

Our original intention for the present paper was to give such an elementary proof for the algebra $A(K)$. Our method would have been to approximate $f \in A(K)$ by functions in $A(K)$ that have no zeros on ∂K. A "proof" of that claim that appeared in [13, page 268, Theorem 9] does not seem to be correct, though. In fact, even for functions f in the disk algebra, one can have that 0 is an interior point of $f(\partial\mathbb{D})$ [26], the image of the unit circle under f. Hence, contrary to the claim in that paper, there does not exist η sufficiently small with $\eta \notin f(\partial\mathbb{D})$ (note also the misprint in that paper, where the function g in the assertion $\eta \notin g(\partial\mathbb{D})$ should have been f). Thus, we were led to consider functions continuously differentiable in a neighborhood of K, where such a Peano-curve phenomenon does not occur.

We will need the following classical result based on Sard's Lemma [12, page 81] that tells us that the image of the set of critical points of a C^1-map f in $\mathbb{R}^n$ has Lebesgue measure zero. As it is less time consuming to present a proof here than to browse through monographs not readily available, we add these few lines.

Theorem 3.5. *Let $U \subseteq \mathbb{R}^n$ be open and $f : \mathbb{R}^n \to \mathbb{R}^n$ a C^1-map on U. Suppose that $K \subseteq U$ is compact and nowhere dense. Then, $f(K)$ is nowhere dense, too.*

Proof. Let w be in the target space. We have to show that w is not an interior point of $f(K)$. If $w \notin f(K)$, then we are done. So let $w \in f(K)$. Consider the image $f(C)$ of the set C of critical

points of f within a fixed compact neighborhood W of K. Then, $f(C)$ is a compact set having Lebesgue measure zero by Sard's Lemma. Thus, there exist closed disks D_n with centers w_n and radii ε_n such that $w_n \to w, \varepsilon_n \to 0$, and $D_n \cap f(C) = \emptyset$. It suffices to show that each of these disks contains points that do not belong to $f(K)$. Let $E_n = K \cap f^{-1}(D_n)$. Note that each point in E_n is a regular point for f. Thus, we may cover E_n by a finite number of closed disks $V \subseteq W$ such that f is a diffeomorphism of V onto $f(V)$. By our hypothesis, $E_n \cap V$ is nowhere dense in V. Thus, $f(E_n \cap V)$ is nowhere dense in D_n. Therefore, $f(E_n) = \bigcup_V f(E_n \cap V)$ is a finite union of nowhere dense, compact sets. Hence $f(E_n)$ is nowhere dense, too. Note that $f(E_n) \subseteq D_n$ and so $f(E_n) = f(K) \cap D_n$. Thus, $f(K) \cap D_n$ is nowhere dense. Since the disks D_n are eventually arbitrarily close to w, we conclude that w is not an interior point of $f(K)$. □

3.2. The Algebra $C(K)$

For arbitrary compact Hausdorff spaces X, Vasershtein [27] and Rieffel [19] gave the following formula for the Bass and topological stable ranks of $C(X, \mathbb{C})$ and $C(X, \mathbb{R})$ (see also [15], for a self-contained, easy proof).

Theorem 3.6. *Let X be a compact Hausdorff space. Then,*

$$\begin{gathered} \operatorname{tsr}(C(X,\mathbb{C})) = \operatorname{bsr}(C(X,\mathbb{C})) = \left[\frac{\dim X}{2}\right] + 1, \\ \operatorname{tsr}(C(X,\mathbb{R})) = \operatorname{bsr}(C(X,\mathbb{R})) = \dim X + 1. \end{gathered} \tag{3.5}$$

Using the additional fact from dimension theory that the covering dimension (or Čech-Lebesgue dimension) of $K \subseteq \mathbb{C}$ is less than or equal to one if and only if K has no interior points, and two otherwise, it follows that the Bass stable rank of $C(K)$ is one whenever $K^\circ = \emptyset$ and two if $K^\circ \neq \emptyset$. The proofs of these facts on the covering dimension are rather technical. We would like to present two elementary approaches, independent of dimension theory. The first one determines directly the Bass stable rank of $C(K)$. The second approach determines the topological stable rank and then deduces the Bass stable rank. But first we present an elementary proof of Rieffel's result [19] that the Bass and topological stable ranks of $C(X)$ coincide.

Theorem 3.7. *Let X be a compact Hausdorff space. Then,* $\operatorname{bsr}(C(X)) = \operatorname{tsr}(C(X))$.

Proof. Let $N = \operatorname{bsr}(C(X)) < \infty$, and let $F := (f_1, \ldots, f_N)$ be an N-tuple in $C(X)$. If F is invertible, we are done. So we may assume that the f_j have at least one zero in common. Consider the sets

$$E_n = \left\{ x \in X : \sum_{j=1}^{N} |f_j(x)|^2 \geq \left(\frac{1}{n}\right)^2 \right\}. \tag{3.6}$$

Choose by Urysohn's Lemma a function $g_n \in C(X)$ with $0 \leq g_n \leq 1$ such that g_n vanishes identically on E_n and is constant one on $\bigcap_{j=1}^{N} Z(f_j)$. Then, the $(N+1)$-tuple $(f_1, \ldots, f_N, g_n)$ is

invertible in $C(X)$. Since $\operatorname{bsr}(C(X)) = N$, this tuple is reducible. Hence, there exist $h_j \in C(X)$ so that

$$F_n := (f_1 + h_1 g_n, \ldots, f_N + h_N g_n) \tag{3.7}$$

is invertible in $C(X)$. Now on X

$$\left| F - \left(|F| + \frac{1}{n} \right) \frac{F_n}{|F_n|} \right| \leq \frac{3}{n}, \tag{3.8}$$

where $|Q| = \sqrt{\sum_{j=1}^{N} |c_j|^2}$ whenever $Q = (c_1, \ldots, c_N)$ (this can easily be seen by considering the cases $x \in E_n$, $x \notin E_n$, and $|F|(x) = 0$). Thus, $\operatorname{tsr}(C(X)) \leq \operatorname{bsr}(C(X))$.

The reverse inequality holds for any Banach algebra A, (see [19]). In fact, let $m = \operatorname{tsr}(A) < \infty$, and suppose that $(f_1, \ldots, f_m, g) \in U_{m+1}(A)$. Choose $(x_1, \ldots, x_m, x) \in A^{m+1}$ so that $\sum_{j=1}^{m} x_j f_j + xg = 1$. Approximating $(x_1, \ldots, x_m)$ by an invertible m-tuple $(y_1, \ldots, y_m)$ yields that $F := \sum_{j=1}^{m} y_j f_j + xg \neq 0$ on $M(A)$. Moreover, there is $(h_1, \ldots, h_m) \in A^m$ with $x = \sum_{j=1}^{m} h_j y_j$. Hence,

$$F = \sum_{j=1}^{m} y_j (f_j + h_j g) \neq 0 \quad \text{on } M(A). \tag{3.9}$$

This shows that $(f_1 + h_1 g, \ldots, f_m + h_m g) \in U_m(A)$ and so $\operatorname{bsr}(A) \leq m = \operatorname{tsr}(A)$.

The cases where $\operatorname{tsr}(A) = \infty$ or $\operatorname{bsr}(A) = \infty$ immediately follow. □

Theorem 3.8 (see [27]). *Let $K \subseteq \mathbb{C}$ be compact and suppose that $K^\circ = \emptyset$. Then, $bsr(C(K)) = 1$.*

Proof. Let (f, g) be an invertible pair in $C(K)$. Hence, $f \neq 0$ on $Z(g) = \{x \in K : g(x) = 0\}$. By [2, Theorem 4.29], there exists a rational function r without poles and zeros in $Z(g)$ such that on $Z(g)$ we have $f = re^h$ for some continuous function h. Since K has no interior points, we may shift the poles and zeros of r so that the new rational function, s, has no zeros and poles in K and satisfies

$$\|r - s\|_{Z(g)} < \frac{1}{2} \min_{Z(g)} |r|. \tag{3.10}$$

Thus, $\|s/r - 1\|_{Z(g)} < 1$ and, hence, by a standard reasoning with series in Banach algebras, $s = e^q r$ on $Z(g)$. So

$$f = s\left(\frac{r}{s} e^h \right) = se^{h-q} \quad \text{on } Z(g). \tag{3.11}$$

We may assume that q and h are continuously extended to $\mathbb{C}$. Thus, $F = se^{h-q}$ is a continuous zero-free extension of $f|_{Z(g)}$ to K.

Next we use a simple version of a technique in [23]. Let the closed neighborhood U of $Z(g)$ be chosen so that

$$\|F - f\|_{K\cap U} < \frac{1}{2}\min_{K} |F|. \tag{3.12}$$

Hence, again, $\|f/F - 1\|_{K\cap U} < 1$ and so, on $K \cap U$, $f = e^H F$ for some function $H \in C(K)$.

Now let

$$h = \begin{cases} \dfrac{1}{g}(F - e^{-H} f) & \text{on } K \setminus U, \\ 0 & \text{on } K \cap U. \end{cases} \tag{3.13}$$

Then, $h \in C(K)$ and $f + e^H hg = e^H F$ on K. Therefore, the pair (f, g) is reducible. □

We will now determine with our classroom proofs the stable ranks of $C(K)$ for arbitrary compacta in $\mathbb{C}$.

Theorem 3.9. *Let $K \subseteq \mathbb{C}$ be a compact set. Then,*

(1) $\operatorname{bsr}(C(K)) = \operatorname{tsr}(C(K)) = 1$ *if and only if* $K^\circ = \emptyset$;

(2) $\operatorname{bsr}(C(K)) = \operatorname{tsr}(C(K)) = 2$ *if and only if* $K^\circ \neq \emptyset$.

Proof. In view of Theorem 3.7, it suffices to show the assertions for the topological stable rank.

(1) Suppose that $K^\circ = \emptyset$, and let $f \in C(K)$. By Weierstrass' approximation theorem, there exists for each $\varepsilon > 0$ a polynomial p in the real variables x and y such that $\|f - p\|_K < \varepsilon$. Since $K^\circ = \emptyset$, we may use Theorem 3.5 to conclude that $p(K)$ is nowhere dense in $\mathbb{C}$. In particular, there exists $\eta \in \mathbb{C}$ with arbitrarily small modulus, say $|\eta| < \varepsilon$, such that $p - \eta$ has no zeros on K. Thus,

$$\|f - (p - \eta)\|_K \leq \varepsilon + |\eta| < 2\varepsilon. \tag{3.14}$$

Hence, f has been approximated by invertible functions on K and so $\operatorname{tsr}(C(K)) = 1$.

Now if $K^\circ \neq \emptyset$, then, after a suitable affine transformation, we may assume that $\overline{\mathbb{D}} \subseteq K^\circ$. But the identity function z cannot be uniformly approximated on $\overline{\mathbb{D}}$ by invertible functions u_n in $C(K)$. Otherwise $|z - u_n(z)| < \varepsilon$ for all $z \in \overline{\mathbb{D}}$; hence, by Brouwer's fixed-point theorem, [2, page 108], $z - u_n(z)$ has a fixed point in $\overline{\mathbb{D}}$ and so u_n has a zero, a contradiction. Thus, $\operatorname{tsr}(C(K)) = 1$ implies that $K^\circ = \emptyset$.

(2) We show that $\operatorname{tsr}(C(K)) \leq 2$. Having done this, (2) is the logical negation of (1).

So let $(f, g) \in C(K)^2$. As above, by Weierstrass' approximation theorem, there exist for each $\varepsilon > 0$ two polynomials p and q in the real variables x and y such that $\|f - p\|_K + \|g - q\|_K < \varepsilon$. Now we consider the set $E := K \times \{0\}$ as a subset in $\mathbb{C}^2 = \mathbb{R}^4$ and look upon the map (p, q) as a map from $\mathbb{R}^4$ to $\mathbb{R}^4$. Since E has no interior points, we may conclude with Theorem 3.5 as in (1) that there exist $(\eta_1, \eta_2) \in \mathbb{C}^2$ with $|\eta_1| + |\eta_2| < \varepsilon$ such that $(p - \eta_1, q - \eta_2)$ is an invertible pair in $C(E)$. Hence, (f, g) has been approximated by invertible pairs in $C(K)$. □

4. The Algebra $A_1(K)$: Its Spectrum, Its $A_1(K)$-Convex Sets

Suppose that $K \subseteq \mathbb{C}$ is given. Does the algebra $A_1(K)$ resemble $A(K)$ or $R(K)$? For which K the algebra $A_1(K)$ is strictly contained in $A(K)$? We refer the reader to the books by Gaier [8] and Gamelin [9] for a thorough study of the algebras $R(K)$ and $A(K)$. In particular, they give descriptions of those compacta for which both coincide (Vitushkin's theorem).

Concerning our algebra $A_1(K)$, since every continuous function can be uniformly approximated on compact planar sets by polynomials in the real variables x and y, it would be natural to guess that $A_1(K)$ coincides with $A(K)$. For instance, if $K^\circ = \emptyset$, then, by the Stone-Weierstrass theorem just mentioned, $A_1(K) = C(K)$ (note that $z, \overline{z} \in A_1(K)$). It will follow from our representation of $A_1(K)$ (Theorem 4.2) that this is not true in general. Indeed, $A_1(K)$ will be "closer" to $R(K)$ than to $A(K)$, and this is due to the following theorem that will be the main tool for our study of $A_1(K)$.

Theorem 4.1 (see [1, page 160] or [9, page 26]). *Let $f \in C^1(U)$ for some neighborhood U of K, and suppose that $\overline{\partial} f = 0$ on K. Then, $f|_K \in R(K)$.*

We start now with a representation of $A_1(K)$ in terms of the algebras $R(S_1)$ and $C(S_2)$ for some specified compact sets S_1 and S_2. Let $H(\Omega)$ be the set of all holomorphic functions in an open set $\Omega \subseteq \mathbb{C}$.

Theorem 4.2. *If $K \subseteq \mathbb{C}$ is compact, then $A_1(K) = C(K) \cap R(\overline{K^\circ})$.*

Proof. (1) Let $f \in C^1(U)$ for a neighborhood U of K, and suppose that f is holomorphic in K°. Then, $\overline{\partial} f = 0$ on $\overline{K^\circ}$, and so, by Theorem 4.1, $f \in R(\overline{K^\circ})$. Since $C(K) \cap R(\overline{K^\circ})$ is uniformly closed, we obtain that $A_1(K) \subseteq C(K) \cap R(\overline{K^\circ})$.

(2) To show the reverse inclusion, let $f \in C(K) \cap R(\overline{K^\circ})$. We may think of f as being continuous on the whole plane (Tietze). Let $\varepsilon > 0$. Choose a rational function r without poles in $\overline{K^\circ}$ such that $\|r - f\|_{\overline{K^\circ}} < \varepsilon$. Note that possible poles of r belong to ∂K or to $\mathbb{C} \setminus K$. By shifting the boundary poles, if necessary, we may assume that r has no poles on K. Let V be a neighborhood of $\overline{K^\circ}$, so that $\|r - f\|_V < \varepsilon$, too. By the Stone-Weierstrass theorem, let $p = p(x, y)$ be a polynomial of the two real variables x and y such that $\|p - f\|_K < \varepsilon$. Now choose $q \in C^\infty(\mathbb{C})$ such that $q = 1$ on $\overline{K^\circ}$, $q = 0$ on $K \setminus V$, and $0 \le q \le 1$. Finally let

$$g = rq + (1 - q)p. \tag{4.1}$$

Then, $g \in C^1(U)$ for some open set U with $K \subseteq U$. Also, $g = r$ on $\overline{K^\circ}$; hence, g is holomorphic in K°, and so $g \in A_1(K)$. Moreover, on $K \setminus V$, $|g - f| = |p - f| < \varepsilon$ and on $K \cap V$ we have

$$\begin{aligned} |g - f| &= |q(r - p) + (p - f)| \le |r - p| + |p - f| \\ &\le (|r - f| + |f - p|) + |p - f| \le \varepsilon + 2\varepsilon = 3\varepsilon. \end{aligned} \tag{4.2}$$

Thus, we have uniformly approximated each $f \in C(K) \cap R(\overline{K^\circ})$ by functions in $A_1(K)$. Thus, $C(K) \cap R(\overline{K^\circ}) \subseteq A_1(K)$. □

The proof shows that a set of generators of $A_1(K)$ is given by

$$\mathcal{G} = \{f \in C(K) : f|_{\overline{K^\circ}} \text{ rational without poles in } K\}. \tag{4.3}$$

Corollary 4.3. *The following assertions hold.*

(a) *If* $K = \overline{K^\circ}$, *then* $A_1(K) = R(K)$, *and if* $K^\circ = \emptyset$, *then* $A_1(K) = C(K)$.

(b) *There exist compact sets* $K \subseteq \mathbb{C}$ *with* $K^\circ = \emptyset$ *for which* $R(K) \subsetneqq A_1(K)$.

(c) *There exist compact sets* $K \subseteq \mathbb{C}$ *with* $R(K) \subsetneqq A_1(K) \subsetneqq A(K) \subsetneqq C(K)$.

(d) *There exist compact sets* $K \subseteq \mathbb{C}$ *with* $R(K) \subsetneqq A_1(K) = A(K) \subsetneqq C(K)$.

Proof. (a) This immediately follows from Theorem 4.2.

(b) A Swiss cheese S with $S^\circ = \emptyset$ for which $R(S) \neq C(S)$ appears in [8, page 103]. Hence, $R(S) \subsetneqq A_1(S) = C(S)$.

(c) Let S be the Swiss cheese above (with $S^\circ = \emptyset$), and let S' be a Swiss cheese satisfying $\overline{(S')^\circ} = S'$, $R(S') \neq A(S')$ (see [8, page 104]). We may assume that $S \cap S' = \emptyset$. Then, $K = S \cup S'$ satisfies $R(K) \subsetneqq A_1(K) \subsetneqq A(K) \subsetneqq C(K)$.

(d) Let S be as above, and choose any S'' with $(S'')^\circ \neq \emptyset$ such that $A(S'') = R(S'')$. If $S \cap S'' = \emptyset$, then $K = S \cup S''$ satisfies $R(K) \subsetneqq A_1(K) = A(K) \subsetneqq C(K)$. □

Theorem 4.4. *The maximal ideal space of* $A_1(K)$ *can be identified with* K *via point evaluations. Also, the Shilov boundary of* $A_1(K)$ *coincides with* ∂K, *the topological boundary of* K.

Proof. We use Theorem 4.2 and the fact that all functions defined on K, respectively, $\overline{K^\circ}$, and appearing here admit continuous extensions to the whole plane. Let $(f_1, \ldots, f_n)$ satisfy $\sum_{j=1}^n |f_j| \geq \delta > 0$ on K. Since $M(R(E)) = E$ for every compact set $E \subseteq \mathbb{C}$ (see [9, page 27]), there exists $(r_1, \ldots, r_n) \in R(\overline{K^\circ})^n$ such that

$$\sum_{j=1}^n r_j f_j = 1 \text{ on } \overline{K^\circ}. \tag{4.4}$$

Since $M(C(K)) = K$, there is $(q_1, \ldots, q_n) \in C(K)^n$ such that

$$\sum_{j=1}^n q_j f_j = 1 \text{ on } K. \tag{4.5}$$

Let V be a neighborhood of $\overline{K^\circ}$ so that $|\sum_{j=1}^n r_j f_j - 1| < 1/2$ on V. Choose a function $q \in C_\circ^\infty(\mathbb{C})$, $\|q\|_\infty \leq 1$, with $q = 1$ on $\overline{K^\circ}$ and $q = 0$ on $K \setminus V$. Let $s_j = r_j q + (1-q) q_j$. Then, $s_j \in C(K) \cap R(\overline{K^\circ}) = A_1(K)$. Moreover, $\sum_{j=1}^n s_j f_j \neq 0$ on K. In fact, on $K \setminus V$ we have

$$\sum_{j=1}^n s_j f_j = \left(\sum_{j=1}^n r_j f_j\right) q + (1-q) \sum_{j=1}^n q_j f_j = \sum_{j=1}^n q_j f_j = 1 \tag{4.6}$$

and on $K \cap V$ we have

$$\begin{aligned} \left|\sum_{j=1}^n s_j f_j\right| &= \left|\left(\sum_{j=1}^n r_j f_j\right) q + (1-q) \cdot 1\right| = \left|q\left(\sum_{j=1}^n r_j f_j - 1\right) + 1\right| \\ &\geq 1 - \left|\sum_{j=1}^n r_j f_j - 1\right| \geq \frac{1}{2}. \end{aligned} \tag{4.7}$$

Since zero-free functions in $A_1(K)$ are obviously invertible, we obtain that $\sum_{j=1}^{n} s_j f_j$ is invertible, too. Hence, the ideal I generated by $(f_1, \ldots, f_n)$ coincides with $A_1(K)$ and so $M(A_1(K)) = K$.

That the Shilov boundary of $A_1(K)$ is ∂K works in the same manner as for $R(K)$ (see [9, page 27]). □

Next we determine the A-convex subsets of $M(A)$ for a large class of function algebras. It applies in particular to $R(K)$, $A_1(K)$, and $A(K)$. We do not claim originality of this result, but we could not find a reference.

Recall that a hole H of a compact set $K \subseteq \mathbb{C}$ is a bounded component of $\mathbb{C} \setminus K$.

Theorem 4.5. *Let $K \subseteq \mathbb{C}$ be compact, and let A be a uniform algebra with $P(K) \subseteq A \subseteq A(K)$ and such that $M(A) = K$ (via point evaluation). Suppose that S is a closed subset of K. Then S is A-convex if and only if each hole of S contains a hole of K.*

Proof. Let $S \subseteq K$ be A-convex. Suppose that there exists a hole H of S with $H \subseteq K^\circ$. Note that $\partial H \subseteq S$. By virtue of the maximum principle for holomorphic functions, we see that for any $z \in H$ and $f \in A$

$$|f(z)| \leq \max_{\partial H} |f| \leq \max_{S} |f|. \tag{4.8}$$

Thus, H is contained in the A-convex hull of S, which coincides, by our hypothesis, with S; a contradiction. Hence, if S is A-convex, then each hole of S contains a hole of K.

To prove the converse, let S satisfy the condition above. We first note that our hypothesis on the spectrum actually implies that $R(K) \subseteq A$. In fact, for $a \in \mathbb{C} \setminus K$, the function $z - a$ has no zeros on the spectrum K of the algebra, so is invertible. Thus, each rational function with poles outside K belongs to A. So actually $R(K) \subseteq A$.

Choose any point $a \in K \setminus S$, and let $f \equiv 0$ in a neighborhood of S and $f \equiv 1$ in a neighborhood of a. By Runge's theorem, there exists a rational function r with poles off $S \cup \{a\}$ such that $|f - r| < 1/4$ on $S \cup \{a\}$. Since each hole of S contains a hole of K and the unbounded component of $\mathbb{C} \setminus S$ contains the unbounded component of $\mathbb{C} \setminus K$, we may assume, by Runge again, that the poles of r are outside K. So $r \in R(K)$ and

$$|r(a)| \geq \frac{3}{4} > \frac{1}{4} \geq \max_{S} |r|. \tag{4.9}$$

Thus, a does not belong to the A-convex hull of S. Since $a \in K \setminus S$ was arbitrary, S is A-convex. □

Theorem 4.6. *Let $K \subseteq \mathbb{C}$ be compact and S an $A_1(K)$-convex subset of K. Then,*

$$\overline{(A_1(K))|_S} = R\left(S \cap \overline{K^\circ}\right) \cap C(S \cap \partial K). \tag{4.10}$$

Proof. Let $f \in A_1(K)$. To show the inclusion $\subseteq$, we may assume, by Theorem 4.2, that $r = f|_{\overline{K^\circ}}$ is a rational function without poles in K. Thus, $r|_{S \cap \overline{K^\circ}} \in R(S \cap \overline{K^\circ})$. So

$$f|_S \in R\left(S \cap \overline{K^\circ}\right) \cap C(S \cap \partial K) =: R. \tag{4.11}$$

To show the reverse inclusion, let $f \in R$. Note that $f \in C(S)$. We may extend f continuously to $\mathbb{C}$. Let r be a rational function without poles in $S \cap \overline{K^\circ}$ so that $|r - f| < \varepsilon$ on $S \cap \overline{K^\circ}$. By moving possible poles on ∂K a little bit, we may assume that r has no poles in S. Note that $S = (S \cap \overline{K^\circ}) \cup (S \cap \partial K)$. Since S is $A_1(K)$-convex, by Theorem 4.5, each hole of S contains a hole of K. Also, the unbounded component of $\mathbb{C} \setminus S$ contains the unbounded component of $\mathbb{C} \setminus K$. Thus, we may assume, using Runge's theorem, that r has no poles in K. Let V be a neighborhood of $S \cap \overline{K^\circ}$ so that $|r - f| < \varepsilon$ on V.

Also, let h be a polynomial of the real variables x and y with $|f - h| < \varepsilon$ on K. In particular, $|f - h| < \varepsilon$ on $S \cap \partial K$. Choose $q \in C^\infty(\mathbb{C})$, $0 \le q \le 1$, so that $q = 1$ on $\overline{K^\circ}$ and $q = 0$ on $K \setminus V$. Then, $F := rq + (1 - q)h$ is in $C^1(U)$ for some neighborhood U of K. Moreover, F is holomorphic in K°. Thus, $F \in A_1(K)$. Now, on S, we have that $|F - f| < 3\varepsilon$ (see the estimate in Theorem 4.2). Thus, $f \in \overline{(A_1(K))|_S}$. □

Remark 4.7. Let us mention the following interesting and related result by Izzo [11, Theorem 2.2 and Corollary 2.5]. Let E be a closed subset of K. Then, $A(K)|_E$ is dense in $C(E)$ if and only if E has no interior points and each component of $\mathbb{C} \setminus E$ contains a component of $\mathbb{C} \setminus K$. Note that the second condition is, in view of Theorem 4.5, equivalent to E being $A(K)$-convex.

5. The Stable Ranks of $A_1(K)$

Let us call an algebra A of holomorphic functions on a planar open set Ω stable if $(f(z) - f(a))/(z - a) \in A$ whenever $f \in A$ and $a \in \Omega$. It is clear that the algebras $P(K), R(K), A_1(K)$, and $A(K)$ are all stable, where for Ω we take K°.

Lemma 5.1. *For all compact subsets $K \subseteq \mathbb{C}$, one has* $\operatorname{tsr}(A_1(K)) \le 2$.

Proof. Let (f, g) be a pair of functions in $A_1(K)$. By definition, for $\varepsilon > 0$, there is a neighborhood U of K and functions F and G in $C^1(U) \cap H(K^\circ)$ so that $\|F - f\|_K + \|G - g\|_K < \varepsilon$. Since ∂K has no interior points, we conclude from Theorem 3.5 that the images $F(\partial K)$ and $G(\partial K)$ have no interior points. In particular, the value zero is not an interior point of $F(\partial K)$ and $G(\partial K)$. Thus, we may choose η_1 and η_2 in $\mathbb{C}$ with $|\eta_j| < \varepsilon/2$ so that $\eta_1 \notin F(\partial K)$ and $\eta_2 \notin G(\partial K)$. Therefore, $F - \eta_1$ and $G - \eta_2$ have no zeros on ∂K. Since both functions are holomorphic in K°, they have only finitely many zeros at all. Hence, there exist (holomorphic) polynomials p and q and functions u and v zero-free on K such that $F - \eta_1 = pu$ and $G - \eta_2 = qv$. Since $A_1(K)$ is a stable algebra, u and v actually belong to $A_1(K)$. By moving, if necessary, the zeros of p a little bit, we may assume that p and q have no common zeros. By construction we have

$$\|F - pu\|_K + \|G - qv\|_K \le |\eta_1| + |\eta_2| \le \varepsilon, \tag{5.1}$$

and so

$$\|f - pu\|_K + \|g - qv\|_K \le 2\varepsilon. \tag{5.2}$$

Since the functions pu and qv have no common zeros on K, they form, by Theorem 4.4, an invertible pair in $A_1(K)$. □

We are now able to prove the main results of this paper.

Theorem 5.2. *Let $K \subseteq \mathbb{C}$ be a compact set. Then,*

(1) $\operatorname{bsr}(A_1(K)) = 1$;

(2) $\operatorname{tsr}(A_1(K)) = 1$ *if and only if* $K^\circ = \emptyset$;

(3) $\operatorname{tsr}(A_1(K)) = 2$ *if and only if* $K^\circ \neq \emptyset$.

Proof. (1) follows from [21, Corollary 1.3] (that was based on the Arens-Taylor-Novodvorsky theory) and Theorem 4.4. It can also be seen in the following way, using the simpler results that $\operatorname{bsr}(R(E)) = 1$ for every compact set $E \subseteq \mathbb{C}$ and $\operatorname{bsr}(C(S)) = 1$ whenever S is a compact set in $\mathbb{C}$ with empty interior (see Theorem 3.8). So let (f, g) be an invertible pair in $A_1(K)$. By Theorems 4.2 and 3.3, there exists $r \in R(\overline{K^\circ})$ so that

$$f + rg \neq 0 \quad \text{on } \overline{K^\circ}. \tag{5.3}$$

We may assume that r is a rational function without poles in $\overline{K^\circ}$. By moving the poles a little bit, if necessary, we may also assume that r has no poles on K.

Let $q \in C^\infty(\mathbb{C})$ be taken so that $q = 0$ on $\overline{K^\circ}$ and that

$$Z(q) \cap Z(f + rg) = \emptyset, \tag{5.4}$$

where $Z(h) = \{z \in K : h(z) = 0\}$. Then, $(f + rg, qg)$ is an invertible pair in $C(K)$. Since $\operatorname{bsr}(C(\partial K)) = 1$, there is $h \in C(K)$ so that

$$H := (f + rg) + h(qg) \neq 0 \quad \text{on } \partial K. \tag{5.5}$$

Now by Theorem 4.2, $hq \in A_1(K)$ and $r + hq \in A_1(K)$ (note that q vanishes identically on $\overline{K^\circ}$). Thus, $H \in A_1(K)$. Moreover, by (5.3) and (5.5), we obtain that $f + (r + hq)g \neq 0$ on K. Since, by Theorem 4.4, $M(A_1(K)) = K$, we conclude that $\operatorname{bsr}(A_1(K)) = 1$.

(2), (3) By Lemma 5.1 we have that $\operatorname{tsr}(A_1(K)) \leq 2$. Thus, it suffices to show (2). So assume that $K^\circ = \emptyset$. Then, by Corollary 4.3, $A_1(K) = C(K)$. Hence, by Theorem 3.9, $\operatorname{tsr}(A_1(K)) = \operatorname{tsr}(C(K)) = 1$. If $a \in K^\circ \neq \emptyset$, then, by Rouché's theorem, the function $z - a$ cannot be uniformly approximated by functions holomorphic and invertible in a neighborhood of a. Hence, $\operatorname{tsr}(A_1(K)) \geq 2$. □

Our final theorem determines the dense stable rank of a large class of function algebras whose spectrum is, via point evaluation, a compact set in $\mathbb{C}$. In particular it applies to $R(K), A_1(K), A(K)$, and $P(K)$. For compact sets in $\mathbb{C}^n$ the dense stable rank for $P(K)$ has been determined by Corach and Suárez in [7].

Theorem 5.3. *Let $K \subseteq \mathbb{C}$ be a compact set. Suppose that A is a uniform algebra with $P(K) \subseteq A \subseteq A(K)$ and that $M(A) = K$. Then, $dsr(A) = 1$.*

Proof. As in the proof of Theorem 4.5, we observe that the hypothesis implies that $R(K) \subseteq A$. Let E be an A-convex set, and suppose that $f \in A$ does not vanish on E. Since E is A-convex, by Theorem 4.5, each hole of E contains a hole of K. Moreover, the unbounded component of $\mathbb{C} \setminus E$ contains the unbounded component of $\mathbb{C} \setminus K$. Hence, by [2, Corollary 4.30], there exists

a rational function r with zeros and poles off K and some continuous function $h \in C(E)$ so that $f|_E = re^h$. Now r is invertible in A and so the function $(f/r)|_E$ belongs to $A|_E$ and admits a continuous logarithm on E. Note that $E = M(\overline{A|_E})$, because E is A-convex. Thus, by [9, Corollary 6.2, page 88], $(f/r)|_E$ actually admits a logarithm in $\overline{A|_E}$; that is, $f|_E = re^g$ for some $g \in \overline{A|_E}$. Uniformly approximating g on E by functions h_n in A now shows that, on E, f is the uniform limit of functions of the form re^{h_n} that are invertible in A. Hence, $\mathrm{dsr}(A) = 1$. □

We remark that Theorem 5.3 yields another proof that $\mathrm{bsr}(A) = 1$ for these algebras, since the Bass stable rank is always dominated by the dense stable rank (see [7, 17]). In particular, we therefore have a rather short proof for $\mathrm{bsr}(A(K)) = 1$ (see [5, 25]). We do not know, though, whether $\mathrm{tsr}(A(K)) \leq 2$ for every compact set $K \subseteq \mathbb{C}$.

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