Research Article

BMO-Boundedness of Maximal Operators and g -Functions Associated with Laguerre Expansions

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Received 29 February 2012; Revised 21 May 2012; Accepted 21 June 2012

Academic Editor: Aurelian Gheondea

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Let $\{\varphi_n^\alpha\}_{n \in \mathbb{N}}$ be the Laguerre functions of Hermite type with index α . These are eigenfunctions of the Laguerre differential operator $L_\alpha = 1/2(-d^2/dy^2 + y^2 + (1/y^2)(\alpha^2 - 1/4))$. In this paper, we investigate the boundedness of the Hardy-Littlewood maximal function, the heat maximal function, and the Littlewood-Paley g -function associated with L_α in the localized BMO space BMO_{L_α} , which is the dual space of the Hardy space $H_{L_\alpha}^1$.

1. Introduction

Let $n \in \mathbb{N}$, $\alpha > -1$. The Laguerre function of Hermite type φ_α on $(0, \infty)$ is defined as

$$\varphi_n^\alpha(y) = \left(\frac{\Gamma(n+1)}{\Gamma(n+1+\alpha)} \right)^{1/2} e^{-y^2/2} y^\alpha L_n^\alpha(y^2) (2y)^{1/2}, \quad y \in (0, \infty), \quad (1.1)$$

where $L_n^\alpha(x)$ denotes the Laguerre polynomial of degree n and order α , see [1]. It is well known that for every $\alpha > -1$ the system $\{\varphi_n^\alpha\}_{n=0}^\infty$ forms an orthonormal basis of $L^2(0, \infty)$. Moreover, these functions are eigenfunctions of the Laguerre differential operator

$$L_\alpha = \frac{1}{2} \left(-\frac{d^2}{dy^2} + y^2 + \frac{1}{y^2} \left(\alpha^2 - \frac{1}{4} \right) \right) \quad (1.2)$$

satisfying $L_\alpha \varphi_n^\alpha = (2n + \alpha + 1) \varphi_n^\alpha$. The operator L_α can be extended to a positive self-adjoint operator on $L^2(0, \infty)$ by giving a suitable domain of definition, see [2]; we also denote

the extension by L_α . Let $\{T_t^\alpha\}_{t \geq 0}$ be the heat-diffusion semigroup generated by $-L_\alpha$. More precisely, for $f \in L^2(0, \infty)$, we define

$$T_t^\alpha f(x) = \int_0^\infty W_t^\alpha(x, y) f(y) dy, \quad (1.3)$$

where

$$W_t^\alpha(x, y) = \left(\frac{2e^{-t}}{1 - e^{-2t}} \right)^{1/2} \left(\frac{2xye^{-t}}{1 - e^{-2t}} \right)^{1/2} I_\alpha \left(\frac{2xye^{-t}}{1 - e^{-2t}} \right) \exp \left(-\frac{1}{2} \frac{1 + e^{-2t}}{1 - e^{-2t}} (x^2 + y^2) \right). \quad (1.4)$$

I_α is the modified Bessel function of the first kind and order α .

In [3], we introduced and developed a localized BMO space BMO_{L_α} associated with the operator L_α , which is the dual space of the Hardy space $H_{L_\alpha}^1$ introduced by Dziubański [4]. More precisely, let

$$\rho_{L_\alpha}(x) = \frac{1}{8} \min \left(x, \frac{1}{x} \right), \quad x > 0. \quad (1.5)$$

Definition 1.1. Let $\alpha > -1/2$, $B_s(y)$ be any ball in $(0, \infty)$ with the center y and the radius s and f a locally integrable function on $(0, \infty)$. We say $f \in \text{BMO}_{L_\alpha}$ if there exists a constant $C \geq 0$ independent of s and y such that

$$\begin{aligned} \frac{1}{|B_s(y)|} \int_{B_s(y)} |f - f_{B_s(y)}| &\leq C, \quad \text{if } s < \rho_{L_\alpha}(y), \\ \frac{1}{|B_s(y)|} \int_{B_s(y)} |f| &\leq C, \quad \text{if } s \geq \rho_{L_\alpha}(y). \end{aligned} \quad (1.6)$$

Here, $f_{B_s(y)} = (1/|B_s(y)|) \int_{B_s(y)} f dx$. We let $\|f\|_{\text{BMO}_{L_\alpha}}$ denote the smallest C in the two inequalities above.

It is readily seen that BMO_{L_α} is a Banach space with norm $\|\cdot\|_{\text{BMO}_{L_\alpha}}$.

In this paper, we obtain the boundedness on BMO_{L_α} of several operators including the Hardy-Littlewood maximal operator defined on $(0, \infty)$, the heat maximal function, and the Littlewood-Paley g -function associated with T_t^α .

These results were investigated by Dziubański et al. in [5] for Schrödinger operators on $\mathbb{R}^d$ with $d \geq 3$ and with potentials satisfying a reverse Hölder's inequality. Recently, a theory of localized BMO spaces on RD-spaces associated with an admissible function ρ was investigated in [6]; the authors also established the similar results above for their BMO spaces. The admissible function ρ in [6] is required to satisfy

$$\frac{1}{\rho(x)} \leq C_0 \frac{1}{\rho(y)} \left(1 + \frac{d(x, y)}{\rho(y)} \right)^{k_0}. \quad (1.7)$$

Obviously, our ρ_{L_α} in (1.5) does not satisfy this condition. Indeed, let x tend to zero and $y = 1$; then the left side becomes greater than the right.

It is notable the generalized square functions associated to Schrödinger operators are studied in [7]. The authors of [7] gave several of equivalent conditions for BMO-boundedness of square functions.

In this paper, in order to obtain some key estimates, we will employ the differences in integral kernels (the heat kernel, the g -function kernel) associated with the Hermite operator and the Laguerre operator, respectively (see [8, 9]).

The paper is organized as follows. In the next section we present some preliminary lemmas and collect some useful estimates of the kernels associated with the heat semigroups and the g -functions. In Section 3, we establish the boundedness of two maximal operators (the Hardy-Littlewood maximal operator and the heat maximal function) from BMO_{L_α} to BMO_{L_α} . In Section 4, we obtain the boundedness on BMO_{L_α} of the Littlewood-Paley g -function associated with the heat semigroup for L_α . We make some conventions. Throughout this paper by C we always denote a positive constant that may vary at each occurrence; $B_r(y_0)$ stands for $\{y > 0, |y - y_0| \leq r\}$; $A \sim B$ means $(1/C)A \leq B \leq CA$, and the notation $X \lesssim Y$ is used to indicate that $X \leq CY$ with an independent positive constant C .

2. Preliminaries

Now we give the following covering lemma for $(0, \infty)$ which will be used frequently below. The proof is trivial and left to the reader.

Lemma 2.1. *Let $x_0 = 1$, $x_j = x_{j-1} + \rho_{L_\alpha}(x_{j-1})$ for $j \geq 1$, and $x_j = x_{j+1} - \rho_{L_\alpha}(x_{j+1})$ for $j < 0$. One defines the family of “critical balls” of $\mathcal{B} = \{B_k\}_{k=-\infty}^\infty$, where $B_k := \{x \in (0, \infty) : |x - x_k| < \rho_{L_\alpha}(x_k)\}$. Then one has*

- (a) $\bigcup_{k=-\infty}^\infty B_k = (0, \infty)$,
- (b) for every $k \in \mathbb{Z}$, $B_k \cap B_j = \emptyset$ provided that $j \notin \{k-1, k, k+1\}$,
- (c) for any $y_0 \in (0, \infty)$, at most three balls in $\mathcal{B}$ have nonempty intersection with $B(y_0, \rho_{L_\alpha}(y_0))$.

Corollary 2.2. *There exists a constant $C > 0$ such that for every $B_R(x) \subseteq (0, \infty)$ with $R > \rho_{L_\alpha}(x)$, one has*

$$|B_R(x)| \leq \sum_{\{B_k \in \mathcal{B} : B_k \cap B_R(x) \neq \emptyset\}} |B_k| \leq C|B_R(x)|. \quad (2.1)$$

Corollary 2.3. *There exists a constant C such that, for $f \in BMO_{L_\alpha}$, one has*

$$\|f\|_{BMO_{L_\alpha}} \leq C \sup_k \left(|f|_{B_k} + \|f\|_{BMO(B_k^*)} \right), \quad (2.2)$$

where, for any ball B , the norm $\|\cdot\|_{BMO(B)}$ is given by

$$\begin{aligned} \|f\|_{BMO(B)} &= \sup_{B_r(x) \subset B} \frac{1}{B_r(x)} \int_{B_r(x)} |f - f_{B_r(x)}| dy \\ &\sim \sup_{B_r(x) \subset B} \inf_{c \in \mathbb{C}} \frac{1}{B_r(x)} \int_{B_r(x)} |f - c| dy. \end{aligned} \quad (2.3)$$

Corollary 2.4 (see [3, Corollary 3]). *Let $B = B_r(y_0) \subset (0, \infty)$. There exists a constant $C > 0$ such that, for all $f \in BMO_{L_\alpha}$, one has*

- (1) *if $r \geq \rho_{L_\alpha}(y_0)/2$, then $((1/|B|) \int_B |f(x)|^2 dx)^{1/2} \leq C \|f\|_{BMO_{L_\alpha}}$,*
- (2) *if $r < \rho_{L_\alpha}(y_0)/2$, then $((1/|B|) \int_B |f(x) - f_B|^2 dx)^{1/2} \leq C \|f\|_{BMO_{L_\alpha}}$.*

We give two elementary lemmas, which will be used frequently in next section. The proofs are trivial, and the reader also refer to Lemmas 9 and 2 in [5].

Lemma 2.5. *Let $h \in BMO(B_k^*)$ and g_1 and g_2 be functions in $L^\infty(0, \infty)$. If f is any measurable function satisfying*

$$h + g_1 \leq f \leq h + g_2, \quad a.e., \quad (2.4)$$

then $f \in BMO(B_k^)$ and $\|f\|_{BMO(B_k^*)} \leq \|h\|_{BMO(B_k^*)} + \max(\|g_1\|_\infty, \|g_2\|_\infty)$.*

Lemma 2.6. *For all $f \in BMO_{L_\alpha}$ and $B = B_r(y_0)$ with $r < \rho_{L_\alpha}(y_0)$. There exists a constant $C > 0$ such that*

$$|f_{B^*}| \leq C \left(1 + \log \frac{\rho_{L_\alpha}(y_0)}{r} \right) \|f\|_{BMO_{L_\alpha}}. \quad (2.5)$$

Let H be the Hermite operator

$$H = \frac{1}{2} \left(-\frac{d^2}{dx^2} + x^2 \right). \quad (2.6)$$

One considers the heat diffusion semigroup $\{W_t\}_{t>0}$ associated with H and defined by, for every $f \in L^2(\mathbb{R})$,

$$W_t f(x) = \int_{\mathbb{R}} W_t(x, y) f(y) dy, \quad x \in \mathbb{R}, \quad (2.7)$$

where for each $x, y \in \mathbb{R}$ and $t > 0$,

$$W_t(x, y) = \left[\frac{e^{-t}}{\pi(1 - e^{-2t})} \right]^{1/2} \exp \left(-\frac{1}{2} \frac{1 + e^{-2t}}{1 - e^{-2t}} (x^2 + y^2) + 2xy \frac{e^{-t}}{1 - e^{-2t}} \right) \quad (2.8)$$

(see [10]).

Proposition 2.7. *Let $\alpha > -1/2$, $W_t(x, y)$ be in (2.8). There exists $C > 0$ such that, for $t > 0$,*

- (a) $W_t^\alpha(x, y) \leq C y^{\alpha+1/2} x^{-\alpha-3/2}$, $0 < y < x/2$,
- (b) $W_t^\alpha(x, y) \leq C x^{\alpha+1/2} y^{-\alpha-3/2}$, $0 < 2x < y$,

- (c) $|W_t^\alpha(x, y) - W_t(x, y)| \leq C(1/y)$, $x/2 < y < 2x$,
 (d) $|W_t^\alpha(x, y)| \leq C(1/\sqrt{t})e^{-|x-y|^2/10t}$.

Parts (a), (b), and (c) are the contents of Lemma 2.11 in [8]. Part (d) is from (2.6) in [4].

Remark 2.8. The ranges $0 < y < x/2$ and $0 < 2x < y$ are not critical; Proposition 2.7 also holds when $0 < y < x/c$ and $0 < cx < y$, where $c > 1$.

Now we consider the estimates of the integral kernel for the g -function, which will be defined in Section 4:

$$Q_t(x, y) = t^2 \frac{\partial W_s^\alpha(x, y)}{\partial s} \Big|_{s=t^2}, \quad (2.9)$$

$$P_t(x, y) = t^2 \frac{\partial W_s(x, y)}{\partial s} \Big|_{s=t^2}. \quad (2.10)$$

Proposition 2.9. *One has,*

- (a) *for every $t, x, y \in (0, \infty)$ such that $e^{-t^2}xy/(1 - e^{-2t^2}) \leq 1$,*

$$|Q_t(x, y)| \leq Ct^2(xy)^{(\alpha+1/2)} e^{-(x^2+y^2)/8t^2} \frac{e^{-(\alpha+1)t^2}}{(1 - e^{-2t^2})^{\alpha+2}}, \quad (2.11)$$

- (b) *for every $t, x, y \in (0, \infty)$ such that $e^{-t^2}xy/(1 - e^{-2t^2}) > 1$,*

$$|Q_t(x, y) - P_t(x, y)| \leq Ct^2 e^{-(x-y)^2/2t^2} \frac{e^{t^2/2}}{xy(1 - e^{-t^2})^{1/2}}. \quad (2.12)$$

Parts (a) and (b) are contained in [9, (3.4) and (3.6)].

Proposition 2.10. *For every $N \geq 1$, there is a constant C_N such that*

- (a) *if $t > 0$, $|P_t(x, y)| \leq C_N e^{-t^2/8} (1/t) e^{-|x-y|^2/10t^2} (1 + t|x|)^{-N}$;*
 (b) *for $|h| \leq t$, $|P_t(x+h, y) - P_t(x, y)| \leq C(|h|/t)(1/t) e^{-|x-y|^2/20t^2}$, C is independent of x, y, t ;*
 (c) $|\int_{-\infty}^{\infty} P_t(x, y) dy| \leq C(t/\rho_{L_\alpha}(x))^2$, C is independent of x and t .

Proof. By using (2.8) we can write, for every $x, y \in \mathbb{R}$ and $s > 0$,

$$\begin{aligned} \frac{\partial}{\partial s} W_s(x, y) &= -\frac{1}{2\sqrt{\pi}} e^{-((x-e^{-s}y)^2 + (y-e^{-s}x)^2)/2(1-e^{-2s})} \frac{e^{-s/2}}{(1 - e^{-2s})^{3/2}} \\ &\times \left\{ 1 + e^{-2s} + 2e^{-s}(y(x - e^{-s}y) + x(y - e^{-s}x)) \right. \\ &\quad \left. - 2e^{-2s} \frac{(x - e^{-s}y)^2 + (y - e^{-s}x)^2}{1 - e^{-2s}} \right\}. \end{aligned} \quad (2.13)$$

By the simple fact

$$(x - e^{-s}y)^2 + (y - e^{-s}x)^2 = 2(x - y)^2 e^{-s} + (x^2 + y^2)(1 - e^{-s})^2, \quad (2.14)$$

a straightforward manipulation leads to

$$\begin{aligned} \left| \frac{\partial}{\partial s} W_s(x, y) \right| &\leq C e^{-((x-y)^2 + (x^2 + y^2)(1 - e^{-s})^2)/8(1 - e^{-2s})} \frac{e^{-s/2}}{(1 - e^{-2s})^{3/2}} \\ &\leq C_N \frac{e^{-s/8}}{(1 - e^{-2s})^{3/2}} e^{-c(|x-y|^2/10s)} (1 + \sqrt{s}|x|)^{-N}, \end{aligned} \quad (2.15)$$

which implies (a).

To prove (b), we also directly compute the x partial derivative:

$$\begin{aligned} \frac{\partial}{\partial x \partial s} W_s(x, y) &= -\frac{1}{2\sqrt{\pi}} e^{-((x - e^{-s}y)^2 + (y - e^{-s}x)^2)/2(1 - e^{-2s})} \frac{e^{-s/2}}{(1 - e^{-2s})^{3/2}} \\ &\quad \times \left\{ 4e^{-s}(y - e^{-s}x) - 4e^{-2s} \frac{(x - y)}{1 + e^{-s}} \right\} \\ &\quad + \frac{1}{2\sqrt{\pi}} e^{-((x - e^{-s}y)^2 + (y - e^{-s}x)^2)/2(1 - e^{-2s})} \frac{e^{-s/2}}{(1 - e^{-2s})^{5/2}} ((x - e^{-s}y) - e^{-s}(y - e^{-s}x)) \\ &\quad \times \left\{ 1 + e^{-2s} + 2e^{-s}(y(x - e^{-s}y) + x(y - e^{-s}x)) \right. \\ &\quad \left. - 2e^{-2s} \frac{(x - e^{-s}y)^2 + (y - e^{-s}x)^2}{1 - e^{-2s}} \right\}. \end{aligned} \quad (2.16)$$

By an elementary manipulation and (2.14), we have

$$\left| \frac{\partial}{\partial x \partial s} W_s(x, y) \right| \leq C \frac{1}{s^2} e^{-(x-y)^2/16s}. \quad (2.17)$$

This together with the mean value theorem and the condition $|h| \leq t$ leads to (b).

Let $\phi_n(y) = \phi(y/n)$; $\phi(y)$ is a smooth function satisfying $\phi(y) = 1$ for $|y| \leq 1$, $\phi(y) = 0$ for $|y| \geq 2$ and $\Delta \phi(y) \leq 1$ for $y \in \mathbb{R}$. From the above, for fixed s and x , a straightforward manipulation shows that

$$\int_{-\infty}^{+\infty} \left| \frac{\partial W_s(x, y)}{\partial s} \right| dy < \infty. \quad (2.18)$$

Hence, we have

$$\begin{aligned}
 \left| \int_{-\infty}^{\infty} \frac{\partial W_s(x, y)}{\partial s} dy \right| &= \left| \lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} \frac{\partial W_s(x, y)}{\partial s} \phi_n(y) dy \right| \\
 &= \left| \lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} W_s(x, y) H \phi_n(y) dy \right| \\
 &\leq C \int_{-\infty}^{\infty} W_s(x, y) y^2 dy.
 \end{aligned} \tag{2.19}$$

Using (2.8) again,

$$\begin{aligned}
 I &= \left| \int_{-\infty}^{\infty} \frac{\partial W_s(x, y)}{\partial s} dy \right| \\
 &\leq C \int_{-\infty}^{\infty} \frac{e^{-s/4}}{\sqrt{s}} \exp\left(-\frac{(x-y)^2 e^{-s} + (x^2 + y^2)(1 - e^{-s})^2}{2(1 - e^{-2s})}\right) ((y-x)^2 + x^2) dy,
 \end{aligned} \tag{2.20}$$

which implies (c). $\square$

Lemma 2.11 (see [3, Theorem 2]). *For all $f \in BMO_{L_\alpha}$ and $B = B_r(y_0) \subseteq (0, \infty)$, there exists a constant $C > 0$ such that*

$$\frac{1}{|B|} \int_0^r \int_B Q_t^2 f(x) \frac{dx dt}{t} \leq C \|f\|_{BMO_{L_\alpha}}^2. \tag{2.21}$$

3. Maximal Operators

First of all, we define the following notions:

$$M_+ f(x) = \sup_{x \in B \subseteq (0, \infty)} \frac{1}{|B|} \int_B |f(y)| dy, \tag{3.1}$$

$$\mathcal{I}_\alpha^* f(x) = \sup_{t>0} |T_t^\alpha f(x)|. \tag{3.2}$$

In this section, we will show $\mathcal{I}_\alpha^*$ and M_+ are bounded on BMO_{L_α} .

Theorem 3.1. *There exists a constant $C > 0$ such that, for all $f \in BMO_{L_\alpha}$, $M_+ f < \infty$, for a.e. $x \in (0, \infty)$, and*

$$\|M_+ f\|_{BMO_{L_\alpha}} \leq C \|f\|_{BMO_{L_\alpha}}. \tag{3.3}$$

Proof. First of all, we show that for a.e. $x \in (0, \infty)$, $M_+ f < \infty$. To do this, we only need to show that for, at almost $x \in B_k \subseteq \mathcal{B}$ in Lemma 2.1, $M_+ f(x) < \infty$. Let us split $f = f_1 + f_2$ with $f_1 = f \chi_{B_k^*}$. Obviously, since f is locally integrable, we have $M_+ f_1 < \infty$ for a.e. $x \in (0, \infty)$. For f_2 , if

$x \in B$ and $B \cap B_k^* = \emptyset$, since $\text{supp } f_2$ is in the complement of B_k^* , we have $(1/|B|) \int_B |f(y)| dy = 0$. Otherwise, by the definition of BMO_{L_α} , $(1/|B|) \int_B |f(y)| dy \leq (4/|B_{4r}(x_k)|) \int_{B_{4r}(x_k)} |f(y)| dy \leq c \|f\|_{\text{BMO}_{L_\alpha}}$.

We turn to the boundedness in BMO_{L_α} . Let M denote the Hardy-Littlewood function on $\mathbb{R}$; it is well known in [11] that M is bounded on $\text{BMO}(\mathbb{R})$. Let f_0 be a function defined on $\mathbb{R}$ which is f on $(0, \infty)$ and 0 on $(-\infty, 0]$. Notice that $M_+ f = M f_0$, for $x \in (0, \infty)$, so

$$\|M_+ f\|_{\text{BMO}(B_k^*)} = \|M f_0\|_{\text{BMO}(B_k^*)} \leq C \|f_0\|_{\text{BMO}}. \quad (3.4)$$

Now, we need to show that $\|f_0\|_{\text{BMO}} \leq C \|f\|_{\text{BMO}_{L_\alpha}}$. Indeed, if $B \subseteq (0, \infty)$, it is obvious that $(1/|B|) \int_B |f_0 - (f_0)_B| dy \leq \|f\|_{\text{BMO}_{L_\alpha}}$. If $B \cap (0, \infty) = \emptyset$, then $(1/|B|) \int_B |f_0 - (f_0)_B| dy = 0$. If $B \cap (0, \infty) \neq \emptyset$ and $B \cap (-\infty, 0) \neq \emptyset$, let $B = B_1 \cup B_2$, here $B_1 = B \cap (-\infty, 0)$ and $B_2 = B \cap (0, \infty)$, then

$$\begin{aligned} \frac{1}{|B|} \int_B |f_0 - (f_0)_B| dy &\leq 2 \frac{1}{|B|} \int_{B_2} |f_0| dy \\ &\leq 2 \|f\|_{\text{BMO}_{L_\alpha}}. \end{aligned} \quad (3.5)$$

On the other hand, we again split $f = f_1 + f_2$ with $f_1 = f \chi_{B_k^*}$, from the argument above, $M_+ f_2(x) \leq c \|f\|_{\text{BMO}_{L_\alpha}}$, for a.e $x \in B_k$. So

$$\begin{aligned} \frac{1}{|B_k|} \int_{B_k} |M_+ f| dy &\leq \frac{1}{|B_k|} \int_{B_k} |M_+ f_1| dy + \frac{1}{|B_k|} \int_{B_k} |M_+ f_2| dy \\ &\lesssim \left(\frac{1}{|B_k|} \int_{B_k} |M_+ f_1|^2 dy \right)^{1/2} + \|f\|_{\text{BMO}_{L_\alpha}} \\ &\lesssim \|f\|_{\text{BMO}_{L_\alpha}}, \end{aligned} \quad (3.6)$$

where in the last inequality we have used Corollary 2.4. □

Theorem 3.2. *Let $\alpha > -1/2$. There exists a constant $C > 0$ such that*

$$\|\mathcal{I}_\alpha^* f\|_{\text{BMO}_{L_\alpha}} \leq C \|f\|_{\text{BMO}_{L_\alpha}}. \quad (3.7)$$

Proof. By the definition of BMO_{L_α} and Corollary 2.3, it suffices to prove the following: for every fixed “critical ball” $B_k \in \mathcal{B}$ (see Lemma 2.1) we have

- (1) $(1/|B_k|) \int_{B_k} |\mathcal{I}_\alpha^* f| dx \leq C \|f\|_{\text{BMO}_{L_\alpha}}$,
- (2) $\|\mathcal{I}_\alpha^* f\|_{\text{BMO}(B_k^*)} \leq C \|f\|_{\text{BMO}_{L_\alpha}}$.

Let us start to prove (1). It is immediate from Theorem 3.1 and (d) of Proposition 2.7; since $\mathcal{I}_\alpha^* f(x) \leq M_+ f(x)$, for $x > 0$, therefore,

$$\frac{1}{|B_k|} \int_{B_k} |M_+ f| dy \leq C \|f\|_{\text{BMO}_{L_\alpha}}. \quad (3.8)$$

It remains to show (2). By Lemma 2.5, we split $\mathcal{O}_\alpha^* f(x)$ into several parts. First, we shall show

$$\left\| \sup_{t > \rho_{L_\alpha}^2(x_k)} |T_t^\alpha f(x)| \right\|_{L^\infty(B_k^*)} \leq C \|f\|_{\text{BMO}_{L_\alpha}}. \quad (3.9)$$

From (d) of Proposition 2.7, we have

$$\begin{aligned} |T_t^\alpha f(x)| &\lesssim \int_0^\infty |f(y)| \frac{1}{t^{1/2}} \left(1 + \frac{|x-y|}{\sqrt{t}}\right)^{-N} dy \\ &\lesssim \sum_{j=0}^\infty \frac{1}{2^{jN}} \frac{1}{t^{1/2}} \int_{\{y>0, |y-x|<2^j\sqrt{t}\}} |f(y)| dy. \end{aligned} \quad (3.10)$$

Notice that, for $j \geq 0$ and $t > \rho_{L_\alpha}^2(x_k)$, we have $2^j \sqrt{t} \geq \rho_{L_\alpha}(x) \sim \rho_{L_\alpha}(x_k)$, for $x \in B_k^*$. Thus

$$\frac{1}{\sqrt{t}} \int_{\{y>0, |y-x|<2^j\sqrt{t}\}} |f(y)| dy \leq C 2^j \|f\|_{\text{BMO}_{L_\alpha}}. \quad (3.11)$$

Therefore,

$$\sup_{t > \rho_{L_\alpha}^2(x_k)} |T_t^\alpha f(x)| \lesssim \sum_{j=0}^\infty \frac{1}{2^{N-1}} \|f\|_{\text{BMO}_{L_\alpha}} \lesssim \|f\|_{\text{BMO}_{L_\alpha}}. \quad (3.12)$$

By Lemma 2.5, it suffices to show that $\sup_{0 < t \leq \rho_{L_\alpha}^2(x_k)} |T_t^\alpha f(x)|$ satisfies (2). Write

$$f = f \chi_{\{x_k/2 \leq y \leq 2x_k\}} + f \chi_{\{y < x_k/2\}} + f \chi_{\{y > 2x_k\}} = f_1 + f_2 + f_3. \quad (3.13)$$

By Proposition 2.7, it easily follows that

$$\begin{aligned} \left\| \sup_{0 < t \leq \rho_{L_\alpha}^2(x_k)} |T_t^\alpha f_2(x)| \right\|_{L^\infty(B_k^*)} &\leq C \|f\|_{\text{BMO}_{L_\alpha}}, \\ \left\| \sup_{0 < t \leq \rho_{L_\alpha}^2(x_k)} |T_t^\alpha f_3(x)| \right\|_{L^\infty(B_k^*)} &\leq C \|f\|_{\text{BMO}_{L_\alpha}}. \end{aligned} \quad (3.14)$$

Indeed, since $x \sim x_k$, for $x \in B_k^*$, by (a) of Proposition 2.7 and Remark 2.8, we have

$$\begin{aligned} |T_t^\alpha f_2(x)| &\lesssim \int_0^{x_k/2} y^{\alpha+1/2} x_k^{-\alpha-3/2} |f(y)| dy \\ &\lesssim \frac{1}{x_k} \int_0^{x_k/2} |f(y)| dy \\ &\lesssim \|f\|_{\text{BMO}_{L_\alpha}}. \end{aligned} \quad (3.15)$$

Similarly,

$$\begin{aligned} |T_t^\alpha f_3(x)| &\lesssim \sum_{n=1}^{\infty} (2^n x_k)^{-\alpha-3/2} (x_k)^{\alpha+1/2} \int_{2^n x_k}^{2^{n+1} x_k} |f(y)| dy \\ &\lesssim \sum_{n=1}^{\infty} (2^n)^{-\alpha-1/2} \|f\|_{\text{BMO}_{L_\alpha}} \\ &\lesssim \|f\|_{\text{BMO}_{L_\alpha}}. \end{aligned} \quad (3.16)$$

Now, we come to treat f_1 . We make further decompositions. Split

$$T_t^\alpha f_1 = (T_t^\alpha f_1 - W_t f_1) + (W_t f_1 - H_t f_1) + H_t f_1, \quad (3.17)$$

where

$$H_t g(x) = \int_0^\infty \left[\frac{e^{-t}}{\pi(1-e^{-2t})} \right]^{1/2} \exp\left(-\frac{1}{2} \frac{1+e^{-2t}}{1-e^{-2t}} (x-y)^2\right) g(y) dy. \quad (3.18)$$

For the first term, by (c) of Proposition 2.7, we have

$$|T_t^\alpha f_1 - W_t f_1| \leq C \frac{1}{x_k} \int_{x_k/2}^{2x_k} |f(y)| dy \leq C \|f\|_{\text{BMO}_{L_\alpha}}. \quad (3.19)$$

By (2.8),

$$\begin{aligned} &W_t f_1(x) - H_t f_1(x) \\ &= \int_0^\infty \left[\frac{e^{-t}}{\pi(1-e^{-2t})} \right]^{1/2} \exp\left(-\frac{1}{2} \frac{1+e^{-2t}}{1-e^{-2t}} (x-y)^2\right) \left[e^{-2xy(1-e^{-t})^2/1-e^{-2t}} - 1 \right] f_1(y) dy. \end{aligned} \quad (3.20)$$

Notice that $|e^{-2xy(1-e^{-t})^2/(1-e^{-2t})} - 1| \leq ct x_k^2$, when $t \leq \rho_{L_\alpha}(x_k)^2$, $x_k/2 \leq y \leq 2x_k$ and $x \in B_k^*$. Therefore, for $x \in B_k^*$ and $t \leq \rho_{L_\alpha}(x_k)^2$, we obtain

$$\begin{aligned}
|W_t f_1(x) - H_t f_1(x)| &\lesssim t x_k^2 \int_0^\infty \frac{1}{\sqrt{t}} e^{-c_0|x-y|^2/t} |f_1(y)| dy \\
&\lesssim t x_k^2 \sum_{1 \leq 2^j \leq \rho_{L_\alpha}(x_k)/\sqrt{t}} 2^{-j(N-1)} \frac{1}{2^j \sqrt{t}} \int_{\{y>0, |y-x|<2^j\sqrt{t}\}} |f(y)| dy \\
&\quad + t x_k^2 \sum_{\rho_{L_\alpha}(x_k)/\sqrt{t} < 2^j} 2^{-j(N-1)} \frac{1}{2^j \sqrt{t}} \int_{\{y>0, |y-x|<2^j\sqrt{t}\}} |f(y)| dy \\
&\lesssim \sum_{1 \leq 2^j \leq \rho_{L_\alpha}(x_k)/\sqrt{t}} 2^{-j(N-1)} t x_k^2 \left(1 + \log\left(\frac{\rho_{L_\alpha}(x_k)}{2^j \sqrt{t}}\right)\right) \|f\|_{\text{BMO}_{L_\alpha}} \\
&\quad + \sum_{j=0}^\infty 2^{-j(N-1)} \|f\|_{\text{BMO}_{L_\alpha}} \\
&\lesssim \sum_{j=0}^\infty 2^{-j(N-1)} \|f\|_{\text{BMO}_{L_\alpha}} \\
&\lesssim \|f\|_{\text{BMO}_{L_\alpha}}.
\end{aligned} \tag{3.21}$$

Finally, by Lemma 2.5 again, we need to show that $\sup_{0 < t \leq \rho_{L_\alpha}^2(x_k)} |H_t f_1(x)|$ satisfies (2). Consider $B = B_r(x_0) \subset B_k^*$ and write

$$\begin{aligned}
f_1 &= (f_1 - f_{B^*}) \chi_{B^*} + [(f_1 - f_{B^*}) \chi_{(B^*)^c \cap (0, \infty)} + f_{B^*} \chi_{(0, \infty)}] \\
&= f_{11} + f_{12}.
\end{aligned} \tag{3.22}$$

By Corollary 2.3, we choose a constant $C_B = \sup_{0 < t \leq \rho_{L_\alpha}(x_k)^2} |H_t f_{12}(x_0)|$,

$$\begin{aligned}
\frac{1}{|B|} \int_B \left| \sup_{0 < t \leq \rho_{L_\alpha}(x_k)^2} |H_t f_1(x)| - C_B \right| dx &\leq \frac{1}{|B|} \int_B \sup_{0 < t \leq \rho_{L_\alpha}(x_k)^2} |H_t f_1(x) - H_t f_{12}(x_0)| dx \\
&\leq \frac{1}{|B|} \int_B \sup_{0 < t \leq \rho_{L_\alpha}(x_k)^2} |H_t f_{11}(x)| dx + \frac{1}{|B|} \int_B \sup_{0 < t \leq \rho_{L_\alpha}(x_k)^2} |H_t f_{12}(x) - H_t f_{12}(x_0)| dx.
\end{aligned} \tag{3.23}$$

For the first integral, by Corollary 2.4 it easily follows that

$$\frac{1}{|B|} \int_B \sup_{0 < t \leq \rho_{L_\alpha}(x_k)^2} |H_t f_{11}(x)| dx \leq \left(\frac{1}{|B|} \int_B |f_{11}|^2 dx \right)^{1/2} \leq C \|f\|_{\text{BMO}_{L_\alpha}}. \tag{3.24}$$

For the second integral,

$$\begin{aligned}
 & |H_t f_1(x) - H_t f_{12}(x_0)| \\
 & \leq \left| \int_{(B^*)^c \cap (0, \infty)} (H_t(x, y) - H_t(x_0, y)) (f_1(y) - f_B) dy \right| + \left| \int_0^\infty (H_t(x, y) - H_t(x_0, y)) f_B dy \right| \\
 & = I_1^t(x) + I_2^t(x).
 \end{aligned} \tag{3.25}$$

By the mean value theorem and the elementary inequality

$$\frac{1}{2^n r} \int_{\{y > 0, |y - x_0| < 2^n r\}} |f(y) - f_B| dy \leq cn \|f\|_{\text{BMO}_{L_\alpha}}, \tag{3.26}$$

we have

$$\begin{aligned}
 I_1^t(x) & \lesssim \int_{(B^*)^c \cap (0, \infty)} \frac{1}{\sqrt{t}} e^{-|y - x_0|^2 / 10t} \frac{|x - x_0|}{\sqrt{t}} |f_1(y) - f_B| dy \\
 & \lesssim \int_{(B^*)^c \cap (0, \infty)} \frac{|x - x_0|}{|y - x_0|^2} |f_1(y) - f_B| dy \\
 & \lesssim r \sum_{n=0}^\infty \frac{1}{(2^n r)^2} \int_{\{y > 0, |y - x_0| < 2^n r\}} (|f(y) - f_B| + |f(y) - f_1(y)|) dy \\
 & \lesssim \sum_{n=0}^\infty 2^{-n} (n+1) \|f\|_{\text{BMO}_{L_\alpha}}.
 \end{aligned} \tag{3.27}$$

On the other hand, by the fact $|f_B| \leq C(1 + \log(\rho_{L_\alpha}(x_0)/r)) \|f\|_{\text{BMO}_{L_\alpha}}$ in Lemma 2.6, we obtain

$$\begin{aligned}
 I_2^t(x) & \lesssim \left| \int_{-\infty}^0 (H_t(x, y) - H_t(x_0, y)) f_B dy \right| \\
 & \lesssim \int_{\{y < 0\}} \frac{|x - x_0|}{|y - x_0|^2} |f_B| dy \\
 & \lesssim \frac{r}{\rho_{L_\alpha}(x_0)} \left(1 + \log \frac{\rho_{L_\alpha}(x_0)}{r} \right) \|f\|_{\text{BMO}_{L_\alpha}} \lesssim \|f\|_{\text{BMO}_{L_\alpha}}.
 \end{aligned} \tag{3.28}$$

Therefore, we obtain

$$\frac{1}{|B|} \int_B \sup_{0 < t \leq \rho_{L_\alpha}(x_k)^2} |H_t f_{12}(x) - H_t f_{12}(x_0)| dx \lesssim \|f\|_{\text{BMO}_{L_\alpha}}, \tag{3.29}$$

which establishes the proof. $\square$

4. g -Function

For all $f \in L^1_{\text{loc}}(0, \infty)$ and $x \in (0, \infty)$, define the Littlewood-Paley g -function by

$$g(f)(x) \equiv \left(\int_0^\infty |Q_t f(x)|^2 \frac{dt}{t} \right)^{1/2}, \quad (4.1)$$

where, $\{Q_t\}_{t>0}$ is a family of operators with the integral kernels

$$Q_t(x, y) = t^2 \frac{\partial W_s^\alpha(x, y)}{\partial s} \Big|_{s=t^2}. \quad (4.2)$$

Theorem 4.1. *Let $\alpha > -1/2$. There exists a constant $C > 0$ such that, for all $f \in BMO_{L_\alpha}$, $g(f) \in BMO_{L_\alpha}$ and $\|g(f)\|_{BMO_{L_\alpha}} \leq C\|f\|_{BMO_{L_\alpha}}$.*

Proof. By Proposition 2.9 and (a) of Proposition 2.10, we have

$$Q_t(x, y) \leq c \frac{1}{t} e^{-c_1|x-y|^2/t^2}. \quad (4.3)$$

For $f \in BMO_{L_\alpha}$, because of this and the integrability of $(1 + |x|)^{-2}f(x)$ (see [12, page 141]),

$$Q_t f(x) = \int_0^\infty Q_t(x, y) f(y) dy \quad (4.4)$$

is well defined absolutely convergent integral for all $(x, t) \in (0, \infty) \times (0, \infty)$. Similar to the proof of Theorem 3.2, we will try to show that, for $B_k \subset \mathcal{B}$ in Lemma 2.1,

- (1) $(1/|B_k|) \int_{B_k} |g(f)(x)| dx \leq C\|f\|_{BMO_{L_\alpha}}$,
- (2) $\|g(f)(x)\|_{BMO(B_k^*)} \leq C\|f\|_{BMO_{L_\alpha}}$.

We split

$$\begin{aligned} [g(f)(x)]^2 &= [g_1(f)(x)]^2 + [g_2(f)(x)]^2 \\ &= \int_0^{20\rho_{L_\alpha}(x_k)} |Q_t f(x)|^2 \frac{dt}{t} + \int_{20\rho_{L_\alpha}(x_k)}^\infty |Q_t f(x)|^2 \frac{dt}{t}. \end{aligned} \quad (4.5)$$

By Lemma 2.11 and Hölder inequality, assertion (1) holds for $g_1(f)(x)$. To finish the proof of (1), it suffices to show that

$$\|g_2(f)\|_{L^\infty(B_k^*)} \leq c\|f\|_{BMO_{L_\alpha}}. \quad (4.6)$$

In the next proof, for the sake of brevity we introduce the additional notations:

$$X_1^t(x) = \left\{ y \in (0, \infty) : \frac{e^{-t^2}xy}{1 - e^{-2t^2}} \leq 1 \right\}, \quad (4.7)$$

$$X_2^t(x) = \left\{ y \in (0, \infty) : \frac{e^{-t^2}xy}{1 - e^{-2t^2}} > 1 \right\}. \quad (4.8)$$

By $X_1^t(x)$ and $X_2^t(x)$, we split $Q_t f(x)$ as

$$\begin{aligned} |Q_t f(x)| &\leq \int_{X_1^t(x)} |f(y)| |Q_t(x, y)| dy + \int_{X_2^t(x)} |f(y)| |Q_t(x, y) - P_t(x, y)| dy \\ &\quad + \int_{X_2^t(x)} |f(y)| |P_t(x, y)| dy \\ &= I_1^t(x) + I_2^t(x) + I_3^t(x). \end{aligned} \quad (4.9)$$

For $I_1^t(x)$ and $x \in B_k^*$, we shall first show the inequality

$$J_1(x) = \int_{20\rho_{L_\alpha}(x_k)}^\infty |I_1^t f(x)|^2 \frac{dt}{t} \leq c \|f\|_{\text{BMO}_{L_\alpha}}^2. \quad (4.10)$$

Using (a) of Proposition 2.9, if $x_k \leq 1$, $\rho_{L_\alpha}(x_k) \sim x_k$, we get

$$|Q_t(x, y)| \leq C t^{2\alpha+3} \left(\frac{xy}{t^2} \right)^{\alpha+1/2} e^{-(x^2+y^2)/8t^2} \frac{e^{-(\alpha+1)t^2}}{(1 - e^{-2t^2})^{\alpha+2}} \leq C \left(\frac{x_k}{t} \right)^{\alpha+1/2} \frac{1}{t} e^{-|x-y|^2/16t^2}. \quad (4.11)$$

If $x_k \geq 1$, $\rho_{L_\alpha}(x_k) \sim 1/x_k$, we have

$$|Q_t(x, y)| \leq C \frac{1}{t} e^{-(x^2+y^2)/8t^2} e^{-((\alpha+1)/2)t^2} \leq C \frac{1}{tx_k} \frac{1}{t} e^{-|x-y|^2/16t^2}. \quad (4.12)$$

The previous two inequalities above imply

$$\begin{aligned} J_1(x) &\lesssim \int_{20\rho_{L_\alpha}(x_k)}^\infty \left| \int_0^\infty \left(\frac{\rho_{L_\alpha}(x_k)}{t} \right)^\sigma \frac{1}{t} e^{-|x-y|^2/16t^2} |f(y)| dy \right|^2 \frac{dt}{t} \\ &\lesssim \int_{20\rho_{L_\alpha}(x_k)}^\infty \left(\frac{\rho_{L_\alpha}(x_k)}{t} \right)^{2\sigma} \left| \sum_{j=0}^\infty 2^{-j(N-1)} \frac{1}{2^j t} \int_{\{y>0, |y-x|<2^j t\}} |f(y)| dy \right|^2 \frac{dt}{t} \lesssim \|f\|_{\text{BMO}_{L_\alpha}}^2. \end{aligned} \quad (4.13)$$

For $I_2^t(x)$ and $x \in B_k^*$, we shall also prove the inequality

$$J_2(x) = \int_{20\rho_{L_\alpha}(x_k)}^\infty |I_2^t f(x)|^2 \frac{dt}{t} \leq c \|f\|_{\text{BMO}_{L_\alpha}}^2. \quad (4.14)$$

We split this integral as

$$\begin{aligned} J_2(x) &= \int_{20\rho_{L_\alpha}(x_k)}^1 |I_2^t f(x)|^2 \frac{dt}{t} + \int_1^\infty |I_2^t f(x)|^2 \frac{dt}{t} \\ &= J_3(x) + J_4(x). \end{aligned} \quad (4.15)$$

To deal with $J_3(x)$, we discuss two cases. In the first case of $x_k \leq 1$, notice that $y > x$, when $x \in B_k^*$, $y \in X_2^t(x_k)$ and $t \geq 20\rho_{L_\alpha}(x_k)$. According to (b) of Proposition 2.9,

$$\begin{aligned} J_3(x) &\lesssim \int_{20\rho_{L_\alpha}(x_k)}^1 \left| \int_{(1-e^{-2t^2})e^{t^2}/x}^\infty \frac{1}{t} e^{-|x-y|^2/4t^2} \frac{xt}{(1-e^{-2t^2})e^{t^2}-x^2} |f(y)| dy \right|^2 \frac{dt}{t} \\ &\lesssim \int_{20\rho_{L_\alpha}(x_k)}^\infty \frac{x_k^2}{t^2} \left| \int_0^\infty \frac{1}{t} e^{-|x-y|^2/4t^2} |f(y)| dy \right|^2 \frac{dt}{t} \\ &\lesssim \|f\|_{\text{BMO}_{L_\alpha}}^2. \end{aligned} \quad (4.16)$$

The last inequality is from the same proof of $J_1(x)$. In the second case of $x_k > 1$, using (b) of Proposition 2.9 again, for $t > 20\rho_{L_\alpha}(x_k)$ we obtain

$$\begin{aligned} |I_2^t(x)| &\lesssim \frac{1}{x_k} \int_{x_k/2}^{2x_k} \frac{t}{x_k} |f(y)| dy + \int_{(0,\infty) \cap (x_k/2, 2x_k)^c} \frac{1}{t} e^{-|x-y|^2/2t^2} |f(y)| dy \\ &\lesssim \frac{t}{x_k} \|f\|_{\text{BMO}_{L_\alpha}} + \frac{t}{x_k} \int_0^\infty \frac{1}{t} e^{-|x-y|^2/4t^2} |f(y)| dy \\ &\lesssim \frac{t}{x_k} \|f\|_{\text{BMO}_{L_\alpha}}. \end{aligned} \quad (4.17)$$

The last inequality is also from the same proof of $J_1(x)$. Inserting this into $J_3(x)$ leads to $J_3(x) \leq C \|f\|_{\text{BMO}_{L_\alpha}}^2$. Now, it remains to show $J_4(x) \leq c \|f\|_{\text{BMO}_{L_\alpha}}^2$. Using (b) of Proposition 2.9, by the standard argument it easily follows that

$$J_4(x) \lesssim \int_1^\infty \frac{e^{-t/10}}{t} \left| \int_0^\infty \frac{1}{t} e^{-|x-y|^2/2t^2} |f(y)| dy \right|^2 dt \lesssim \|f\|_{\text{BMO}_{L_\alpha}}^2. \quad (4.18)$$

To complete the proof of (4.6), we need to show that

$$J_5(x) = \int_{20\rho_{L_\alpha}(x_k)}^\infty |I_3^t f(x)|^2 \frac{dt}{t} \leq c \|f\|_{\text{BMO}_{L_\alpha}}^2. \quad (4.19)$$

We also consider two cases of $x_k \leq 1$ and $x_k > 1$. When $x_k \leq 1$, repeating the above argument for $J_3(x)$ and using (a) of Proposition 2.10, we have $J_5 \leq c\|f\|_{\text{BMO}_{L_\alpha}}^2$. When $x_k > 1$, using (a) of Proposition 2.10 again, for $t \geq 20\rho_{L_\alpha}(x_k)$, we obtain

$$\begin{aligned} |I_3^t(x)| &\lesssim \frac{1}{x_k t} \int_0^\infty \frac{1}{t} e^{-c|y-x|^2/t^2} |f(y)| dy \\ &\lesssim \frac{1}{tx_k} \sum_{j=0}^\infty 2^{-j(N-1)} \frac{1}{2^j t} \int_{\{y>0, |y-x|<2^j t\}} |f(y)| dy \\ &\lesssim \frac{\rho_{L_\alpha}(x_k)}{t} \|f\|_{\text{BMO}_{L_\alpha}}, \end{aligned} \quad (4.20)$$

which shows that (4.19) holds.

Next, we come to prove assertion (2). By (4.6) and Lemma 2.5, we only need to show

$$\left\| \left(\int_0^{20\rho_{L_\alpha}(x_k)} |Q_t f(x)|^2 \frac{dt}{t} \right)^{1/2} \right\|_{\text{BMO}(B_k^*)} \leq C \|f\|_{\text{BMO}_{L_\alpha}}. \quad (4.21)$$

Consider any ball $B = B_r(x_0) \subset B_k^*$. By Lemma 2.11, we have

$$\frac{1}{|B|} \int_B \left(\int_0^r |Q_t f(x)|^2 \frac{dt}{t} \right)^{1/2} dx \leq \left(\frac{1}{|B|} \int_B \int_0^r |Q_t f(x)|^2 \frac{dt}{t} dx \right)^{1/2} \leq C \|f\|_{\text{BMO}_{L_\alpha}}. \quad (4.22)$$

Therefore, by Lemma 2.5 and Corollary 2.3, it suffices to prove

$$\frac{1}{|B|} \int_B \left| \left(\int_r^{20\rho_{L_\alpha}(x_k)} |Q_t f(x)|^2 \frac{dt}{t} \right)^{1/2} - \left(\int_r^{20\rho_{L_\alpha}(x_k)} |Q_t f(x_0)|^2 \frac{dt}{t} \right)^{1/2} \right| dx \leq C \|f\|_{\text{BMO}_{L_\alpha}}. \quad (4.23)$$

To prove (4.23), we first claim that, for all $f \in \text{BMO}_{L_\alpha}$, $x \in B_k^*$, and $t \leq 20\rho_{L_\alpha}(x_k)$,

$$|Q_t f(x)| \leq C \|f\|_{\text{BMO}_{L_\alpha}}. \quad (4.24)$$

We shall split into three different estimates:

$$\int_{X_1^t(x)} |f(y)| |Q_t(x, y)| dy \leq C \|f\|_{\text{BMO}_{L_\alpha}}, \quad (4.25)$$

$$\int_{X_2^t(x)} |f(y)| |Q_t(x, y) - P_t(x, y)| dy \leq C \|f\|_{\text{BMO}_{L_\alpha}}, \quad (4.26)$$

$$\left| \int_{X_2^t(x)} f(y) P_t(x, y) dy \right| \leq C \|f\|_{\text{BMO}_{L_\alpha}}. \quad (4.27)$$

Let us first treat (4.25). Since $y \leq c(\rho_{L_\alpha}(x_k)^2/x_k)$, when $y \in X_1^t(x)$, notice that $x \sim x_k$ when $x \in B_k^*$, using (a) of Proposition 2.9, and recalling the definition of $\rho_{L_\alpha}(x)$, we have

$$\int_{X_1^t(x)} |f(y)| |Q_t(x, y)| dy \leq C \int_0^{c\rho_{L_\alpha}(x_k)^2/x_k} \frac{t}{x_k^2} |f(y)| dy \leq C \frac{t}{x_k} \|f\|_{\text{BMO}_{L_\alpha}} \leq C \|f\|_{\text{BMO}_{L_\alpha}}. \quad (4.28)$$

For (4.26), using (b) of Proposition 2.9, for $x \in B_k^*$, the left side of (4.26) is controlled by

$$\begin{aligned} & \frac{1}{x_k} \int_{x_k/2}^{2x_k} \frac{t}{x_k} |f(y)| dy + \int_{(0, \infty) \cap (x_k/2, 2x_k)^c} \frac{1}{t} e^{-|x-y|^2/2t^2} |f(y)| dy \\ & \lesssim \frac{t}{x_k} \|f\|_{\text{BMO}_{L_\alpha}} + \frac{t}{x_k} \int_0^\infty \frac{1}{t} e^{-|x-y|^2/4t^2} |f(y)| dy \\ & \lesssim \frac{t}{x_k} \|f\|_{\text{BMO}_{L_\alpha}} + \left(\frac{t}{x_k} \right)^{\frac{1}{2}} \|f\|_{\text{BMO}_{L_\alpha}}. \end{aligned} \quad (4.29)$$

The third inequality is from the same argument for dealing with (3.21) in the proof of Theorem 3.2.

For (4.27), we write

$$\left| \int_{X_2^t(x)} f(y) P_t(x, y) dy \right| \leq \left| \int_{X_2^t(x)} |f(y) - f_{B_t(x)}| |P_t(x, y)| dy \right| + \left| \int_{X_2^t(x)} f_{B_t(x)} P_t(x, y) dy \right|. \quad (4.30)$$

By (a), (c) of Proposition 2.10 and the fact that $|f_{B_t(x)}| \lesssim (1 + \log 20\rho_{L_\alpha}(x_k)/t) \|f\|_{\text{BMO}_{L_\alpha}}$, we have

$$\begin{aligned} \left| \int_{X_2^t(x)} f_{B_t(x)} P_t(x, y) dy \right| & \leq \left| \int_{-\infty}^\infty f_{B_t(x)} P_t(x, y) dy \right| + \left| \int_{-\infty}^{e^{t^2}(1-e^{-2t^2})/x} f_{B_t(x)} P_t(x, y) dy \right| \\ & \lesssim \frac{t^2}{\rho_{L_\alpha}(x_k)^2} \left(1 + \log \frac{20\rho_{L_\alpha}(x_k)}{t} \right) \|f\|_{\text{BMO}_{L_\alpha}} \\ & \quad + \int_{-\infty}^{(1-e^{-2t^2})e^{t^2}/x} \frac{1}{t} e^{-(x-y)^2/16t^2} dy \left(1 + \log \frac{20\rho_{L_\alpha}(x_k)}{t} \right) \|f\|_{\text{BMO}_{L_\alpha}} \\ & \lesssim \int_{-\infty}^{(1-e^{-2t^2})e^{t^2}/tx-x/t} e^{(-1/16)z^2} dz \left(1 + \log \frac{20\rho_{L_\alpha}(x_k)}{t} \right) \|f\|_{\text{BMO}_{L_\alpha}} \\ & \quad + \|f\|_{\text{BMO}_{L_\alpha}} \end{aligned}$$

$$\begin{aligned}
&\lesssim e^{-\frac{(1/20)((1-e^{-2t^2})e^{t^2}/tx - \frac{x}{t})}{t}} \left(1 + \log \frac{20\rho_{L_\alpha}(x_k)}{t}\right) \|f\|_{\text{BMO}_{L_\alpha}} \\
&\quad + \|f\|_{\text{BMO}_{L_\alpha}} \\
&\lesssim \frac{t^2}{\rho_{L_\alpha}(x_k)^2} \left(1 + \log \frac{20\rho_{L_\alpha}(x_k)}{t}\right) \|f\|_{\text{BMO}_{L_\alpha}} + \|f\|_{\text{BMO}_{L_\alpha}} \\
&\lesssim \|f\|_{\text{BMO}_{L_\alpha}}.
\end{aligned} \tag{4.31}$$

The third and fourth inequalities come from the fact $(1 - e^{-2t^2})e^{t^2}/xt \lesssim x/t$ and $x \sim x_k$, when $t \leq 20\rho_{L_\alpha}(x_k)$ and $x \in B_k^*$. It is notable that, for $t \leq 20\rho_{L_\alpha}(x_k)$ and $x \in B_k^*$, the inequality above also implies

$$\left| \int_{X_2^t(x)} P_t(x, y) dy \right| \leq C \frac{t}{\rho_{L_\alpha}(x_k)}. \tag{4.32}$$

On the other hand, by (a) of Proposition 2.10 and the simple fact that $|f_{2^j B_t(x)} - f_{B_t(x)}| \leq Cj\|f\|_{\text{BMO}_{L_\alpha}}$, we obtain

$$\begin{aligned}
\left| \int_{X_2^t(x)} |f(y) - f_{B_t(x)}| |P_t(x, y)| dy \right| &\lesssim \int_0^\infty \frac{1}{t} e^{-|y-x|^2/10t^2} |f(y) - f_{B_t(x)}| dy \\
&\lesssim \sum_{j=0}^\infty 2^{-j(N-1)} \frac{1}{2^j t} \int_{\{y>0, |y-x|\leq 2^j t\}} |f(y) - f_{2^j B_t(x)}| dy \\
&\quad + \sum_{j=0}^\infty 2^{-j(N-1)} |f_{2^j B_t(x)} - f_{B_t(x)}| \\
&\lesssim \sum_{j=0}^\infty j 2^{-j(N-1)} \|f\|_{\text{BMO}_{L_\alpha}} \lesssim \|f\|_{\text{BMO}_{L_\alpha}}.
\end{aligned} \tag{4.33}$$

From (4.24), we deduce that

$$\begin{aligned}
&\left| \left(\int_r^{20\rho_{L_\alpha}(x_k)} |Q_t f(x)|^2 \frac{dt}{t} \right)^{1/2} - \left(\int_r^{20\rho_{L_\alpha}(x_k)} |Q_t f(x_0)|^2 \frac{dt}{t} \right)^{1/2} \right| \\
&\lesssim \left(\int_r^{20\rho_{L_\alpha}(x_k)} |Q_t f(x) - Q_t f(x_0)|^2 \frac{dt}{t} \right)^{1/2} \\
&\lesssim \|f\|_{\text{BMO}_{L_\alpha}}^{1/2} \left(\int_r^{20\rho_{L_\alpha}(x_k)} |Q_t f(x) - Q_t f(x_0)| \frac{dt}{t} \right)^{1/2}.
\end{aligned} \tag{4.34}$$

By (4.28) and (4.29), for $z \in B_k^*$, we have

$$\begin{aligned} \int_r^{20\rho_{L_\alpha}(x_k)} \int_{X_1^t(z)} |f(y)| |Q_t(z, y)| dy \frac{dt}{t} &\leq C \|f\|_{\text{BMO}_{L_\alpha}}, \\ \int_r^{20\rho_{L_\alpha}(x_k)} \int_{X_2^t(z)} |f(y)| |Q_t(z, y) - P_t(z, y)| dy \frac{dt}{t} &\leq C \|f\|_{\text{BMO}_{L_\alpha}}. \end{aligned} \quad (4.35)$$

To finish the proof of (4.23), it suffices to show

$$\int_r^{20\rho_{L_\alpha}(x_k)} \left| \int_{X_2^t(x)} (P_t(x, y) - P_t(x_0, y)) f(y) dy \right| \frac{dt}{t} \leq C \|f\|_{\text{BMO}_{L_\alpha}}, \quad (4.36)$$

$$\int_r^{20\rho_{L_\alpha}(x_k)} \left| \int_{X_2^t(x)} P_t(x_0, y) f(y) dy - \int_{X_2^t(x_0)} P_t(x_0, y) f(y) dy \right| \frac{dt}{t} \leq C \|f\|_{\text{BMO}_{L_\alpha}}. \quad (4.37)$$

It is easy to check (4.37). Indeed, without loss of generality, we assume $x < x_0$. According to (a) of Proposition 2.10,

$$\begin{aligned} &\left| \int_{X_2^t(x)} P_t(x_0, y) f(y) dy - \int_{X_2^t(x_0)} P_t(x_0, y) f(y) dy \right| \\ &\lesssim \int_{(1-e^{-2t^2})/e^{t^2}x_0}^{(1-e^{-2t^2})/e^{t^2}x} \frac{1}{t} |f(y)| dy \frac{tr}{x_0^2} \|f\|_{\text{BMO}_{L_\alpha}}. \end{aligned} \quad (4.38)$$

Inserting this into (4.37) gives the desired result. Moreover, from the above we also have proved

$$\left| \int_{X_2^t(x)} P_t(x_0, y) f(y) dy - \int_{X_2^t(x_0)} P_t(x_0, y) dy \right| \leq C \frac{t}{\rho_{L_\alpha}(x_0)}. \quad (4.39)$$

For $r \leq t \leq 20\rho_{L_\alpha}(x_k)$, we write

$$\begin{aligned} &\left| \int_{X_2^t(x)} (P_t(x, y) - P_t(x_0, y)) f(y) dy \right| \\ &\leq \left| \int_{X_2^t(x)} (P_t(x, y) - P_t(x_0, y)) (f(y) - f_B) dy \right| + |f_B| \left| \int_{X_2^t(x)} (P_t(x, y) - P_t(x_0, y)) dy \right| \\ &= S_1^t(x) + S_2^t(x). \end{aligned} \quad (4.40)$$

By (b) of Proposition 2.10, we obtain

$$\begin{aligned}
 S_1^t(x) &\lesssim \int_{\{y>0, |y-x_0|>r\}} \frac{r}{t} \frac{1}{t} e^{-c|y-x_0|^2/t^2} |f(y) - f_B| dy + \frac{1}{t} \int_{\{y>0, |y-x_0|\leq r\}} |f(y) - f_B| dy \\
 &\lesssim \sum_{j=0}^{\infty} \int_{2^{j+1}B} \frac{r}{t^{1/2}} \frac{1}{(2^j r)^{3/2}} |f(y) - f_{2^{j+1}B} + f_{2^{j+1}B} - f_B| dy + \frac{r}{t} \|f\|_{\text{BMO}_{L_\alpha}} \\
 &\lesssim \left(\frac{r}{t}\right)^{1/2} \sum_{j=0}^{\infty} 2^{-j/2} j \|f\|_{\text{BMO}_{L_\alpha}} + \frac{r}{t} \|f\|_{\text{BMO}_{L_\alpha}} \lesssim \left(\frac{r}{t} + \left(\frac{r}{t}\right)^{1/2}\right) \|f\|_{\text{BMO}_{L_\alpha}}.
 \end{aligned} \tag{4.41}$$

From the inequality above, we deduce that

$$\int_r^{20\rho_{L_\alpha}(x_k)} S_1^t(x) \frac{dt}{t} \leq \int_r^{20\rho_{L_\alpha}(x_k)} \left(\frac{r}{t} + \left(\frac{r}{t}\right)^{1/2}\right) \frac{dt}{t} \|f\|_{\text{BMO}_{L_\alpha}} \leq C \|f\|_{\text{BMO}_{L_\alpha}}. \tag{4.42}$$

On the other hand, by using (b) of Proposition 2.10 again, we also have

$$\left| \int_{X_2^t(x)} (P_t(x, y) - P_t(x_0, y)) dy \right| \leq C \frac{r}{t}. \tag{4.43}$$

This together with (4.32), (4.39), and $|f_B| \lesssim (1 + \log(\rho_{L_\alpha}(x_0)/r)) \|f\|_{\text{BMO}_{L_\alpha}}$ gives that

$$\begin{aligned}
 \int_r^{20\rho_{L_\alpha}(x_k)} S_2^t(x) \frac{dt}{t} &\lesssim \int_r^{20\rho_{L_\alpha}(x_k)} \left(\frac{r}{t}\right)^{2/3} \left(\frac{t}{\rho_{L_\alpha}(x_k)}\right)^{1/3} \left(1 + \log \frac{\rho_{L_\alpha}(x_0)}{r}\right) \frac{dt}{t} \|f\|_{\text{BMO}_{L_\alpha}} \\
 &\lesssim \int_r^{20\rho_{L_\alpha}(x_k)} \left(\frac{r}{t}\right)^{1/3} \frac{dt}{t} \|f\|_{\text{BMO}_{L_\alpha}} \\
 &\lesssim \|f\|_{\text{BMO}_{L_\alpha}}.
 \end{aligned} \tag{4.44}$$

Thus we finish the proof of (4.36) and, hence, finish the proof of Theorem 4.1. $\square$

Acknowledgments

The authors would like to express their gratitude to the referee for many valuable suggestions and revisions which improve the presentation of this paper. Supported by National Natural Science Foundation of China under Grant no. 10990012.

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