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Research Article

λ-Statistical Convergence of Sequences of Functions in Intuitionistic Fuzzy Normed Spaces

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We study λ-statistically convergent sequences of functions in intuitionistic fuzzy normed spaces. We define concept of λ-statistical pointwise convergence and λ-statistical uniform convergence in intuitionistic fuzzy normed spaces and we give some basic properties of these concepts.

1. Introduction and Some Definitions

Fuzzy logic was introduced by Zadeh [1]. Since then, the importance of fuzzy logic has come increasingly to the present. There are many applications of fuzzy logic in the field of science and engineering, for example, population dynamics [2], chaos control [3, 4], computer programming [5], nonlinear dynamical systems [6], fuzzy topology [7], and so forth. The concept of intuitionistic fuzzy set, as a generalization of fuzzy logic, was introduced by Atanassov [8] in 1983.

In the literature, t-norm and t-conorm were defined by Schweizer and Sklar [9]. The norms on intuitionistic fuzzy sets are introduced firstly in [10]. Recently Park [11] has introduced the concept of intuitionistic fuzzy metric space and in [12], Saadati and Park introduced intuitionistic fuzzy normed spaces and concept of convergence of a sequence in intuitionistic fuzzy normed spaces. In light of these developments, intuitionistic fuzzy analogues of many concepts in classical analysis were studied by many authors [13–17], and so forth.

The concept of statistical convergence was introduced by Fast [18] and Steinhaus [19] independently. Mursaleen defined λ-statistical convergence in [20]. Also the concept of statistical convergence was studied in intuitionistic fuzzy normed space in [21]. Quite recently, Karakaya et al. [22] defined and studied statistical convergence of sequences of functions in intuitionistic fuzzy normed spaces. Mohiuddine and Lohani [23] defined and studied λ-statistical convergence in intuitionistic fuzzy normed spaces.

In this paper, we will study concept λ-statistical convergence for sequences of functions and investigate some basic properties related to the concept in intuitionistic fuzzy normed space.

We first recall some basic notions and definitions of intuitionistic fuzzy normed spaces.

Definition 1.1 (see [12]). Let $*$ be a continuous t-norm, let $\diamond$ be a continuous t-conorm, and X be a linear space over the field IF ($\mathbb{R}$ or $\mathbb{C}$). If μ and ν are fuzzy sets on $X \times (0,\infty)$ satisfying the following conditions, the five-tuple $(X,\mu,\nu,*,\diamond)$ is said to be an intuitionistic fuzzy normed space and (μ,ν) is called an intuitionistic fuzzy norm. For every $x,y \in X$ and $s,t > 0$,

(i) $\mu(x,t) + \nu(x,t) \leq 1$,

(ii) $\mu(x,t) > 0$,

(iii) $\mu(x,t) = 1 \Leftrightarrow x = 0$,

(iv) $\mu(ax,t) = \mu(x,t/|a|)$ for each $a \neq 0$,

(v) $\mu(x,t) * \mu(y,s) \leq \mu(x+y,t+s)$,

(vi) $\mu(x,\cdot) : (0,\infty) \to [0,1]$ is continuous,

(vii) $\lim_{t\to\infty}\mu(x,t) = 1$ and $\lim_{t\to 0}\mu(x,t) = 0$,

(viii) $\nu(x,t) < 1$,

(ix) $\nu(x,t) = 0 \Leftrightarrow x = 0$,

(x) $\nu(ax,t) = \nu(x,t/|a|)$ for each $a \neq 0$,

(xi) $\nu(x,t) \diamond \nu(y,s) \geq \nu(x+y,t+s)$,

(xii) $\nu(x,\cdot) : (0,\infty) \to [0,1]$ is continuous,

(xiii) $\lim_{t\to\infty}\nu(x,t) = 0$ and $\lim_{t\to 0}\nu(x,t) = 1$.

Definition 1.2 (see [19, 24]). Let $K \subset \mathbb{N}$ and $K_n = \{k \in K : k \leq n\}$. Then, the natural density is defined by $\delta(K) = \lim_{n\to\infty}(|K_n|/n)$, where $|K_n|$ denotes the cardinality of K_n. A sequence $x = (x_k)$ is said to be statistically convergent to the number L if for every $\varepsilon > 0$, the set $N(\varepsilon)$ has asymptotic density zero, where

$$N(\varepsilon) = \{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}. \tag{1.1}$$

This case is stated by $st - \lim x = L$.

Definition 1.3 (see [22]). Let $(X,\mu,\nu,*,\diamond)$ and $(Y,\mu',\nu',*,\diamond)$ be two intuitionistic fuzzy normed linear spaces and let $f_k : (X,\mu,\nu,*,\diamond) \to (Y,\mu',\nu',*,\diamond)$ be sequences of functions. If for each $x \in X$ and for all $\varepsilon > 0$, $t > 0$,

$$\delta\left(\left\{k \in \mathbb{N} : \mu'\left(f_k(x) - f(x),t\right) \leq 1-\varepsilon \text{ or } \nu'\left(f_k(x) - f(x),t\right) \geq \varepsilon\right\}\right) = 0, \tag{1.2}$$

then we say that the sequence (f_k) is pointwise statistically convergent to f with respect to intuitionistic fuzzy norm (μ, ν) and we write it $st_{\mu,\nu} - f_k \to f$.

Definition 1.4 (see [20]). Let $\lambda = (\lambda_n)$ be a nondecreasing sequence of positive numbers tending to ∞ such that

$$\lambda_{n+1} \leq \lambda_n, \qquad \lambda_1 = 0. \tag{1.3}$$

Let $K \subset \mathbb{N}$. The number

$$\delta_\lambda(K) = \lim_{n\to\infty} \frac{1}{\lambda_n} |\{k \in I_n : k \in K\}| \tag{1.4}$$

is said to be λ-density of K, where $I_n = [n - \lambda_n + 1, n]$.

If $\lambda_n = n$, then λ-density is reduced to asymptotic density. A sequence $x = (x_k)$ is said to be λ-statistically convergent to the number L if for every $\varepsilon > 0$, the set $N(\varepsilon)$ has λ-density zero, where

$$N(\varepsilon) = \{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}. \tag{1.5}$$

This case is stated by $st_\lambda - \lim x = L$.

Definition 1.5 (see [23]). Let $(X, \mu, \nu, *, \diamond)$ be an intuitionistic fuzzy normed space. Then, a sequence (x_k) is said to be λ-statistically convergent to $L \in X$ with respect to intuitionistic fuzzy norm (μ, ν) provided that for every $\varepsilon > 0$ and $t > 0$,

$$\delta_\lambda\left(\left\{k \in \mathbb{N} : \mu(x_k - L, t) \leq 1 - \varepsilon \text{ or } \nu(x_k - L, t) \geq \varepsilon\right\}\right) = 0, \tag{1.6}$$

or equivalently

$$\delta_\lambda\left(\left\{k \in \mathbb{N} : \mu(x_k - L, t) > 1 - \varepsilon,\ \nu(x_k - L, t) < \varepsilon\right\}\right) = 1. \tag{1.7}$$

This case is stated by $st^\lambda_{\mu,\nu} - \lim x = L$.

2. λ-Statistical Convergence of Sequence of Functions in Intuitionistic Fuzzy Normed Spaces

In this section, we define pointwise λ-statistical and uniformly λ-statistical convergent sequences of functions in intuitionistic fuzzy normed spaces. Also, we give the λ-statistical analog of the Cauchy convergence criterion for pointwise and uniformly λ-statistical convergent in intuitionistic fuzzy normed space. We investigate relations of these concepts with continuity. Let us start definition of pointwise λ-statistical convergence in intuitionistic fuzzy normed spaces.

Definition 2.1. Let $(X,\mu,\nu,*,\diamond)$ and $(Y,\mu',\nu',*,\diamond)$ be two intuitionistic fuzzy normed linear spaces over the same field IF and let $f_k : (X,\mu,\nu,*,\diamond) \to (Y,\mu',\nu',*,\diamond)$ be sequences of functions. If for each $x \in X$ and for all $\varepsilon > 0$, $t > 0$,

$$\delta_\lambda\left(\left\{k \in \mathbb{N} : \mu'\left(f_k(x) - f(x),t\right) \le 1-\varepsilon \text{ or } \nu'\left(f_k(x) - f(x),t\right) \ge \varepsilon\right\}\right) = 0, \tag{2.1}$$

or equivalently

$$\delta_\lambda\left(\left\{k \in \mathbb{N} : \mu'\left(f_k(x) - f(x),t\right) > 1-\varepsilon,\ \nu'\left(f_k(x) - f(x),t\right) < \varepsilon\right\}\right) = 1, \tag{2.2}$$

then we say that the sequence (f_k) is pointwise λ-statistically convergent with respect to intuitionistic fuzzy norm (μ,ν) and we write it $st^{\lambda}_{\mu,\nu} - f_k \to f$.

This means that for every $\delta > 0$, there is integer N such that, for all $n > N(\varepsilon,\delta,t,x)$ and for every $\varepsilon > 0$,

$$\frac{1}{\lambda_n}\left|\left\{\begin{array}{l} k \in I_n : \mu'\left(f_k(x) - f(x),t\right) \le 1-\varepsilon \text{ or} \\ \nu'\left(f_k(x) - f(x),t\right) \ge \varepsilon \text{ for each } x \in X \end{array}\right\}\right| < \delta. \tag{2.3}$$

Remark 2.2. Let $f_k : (X,\mu,\nu,*,\diamond) \to (Y,\mu',\nu',*,\diamond)$ be sequences of functions. If $\lambda_n = n$, since λ-density is reduced to asymptotic density, then (f_k) is pointwise statistically convergent on X with respect to (μ,ν), that is, $st_{\mu,\nu} - f_k \to f$.

Lemma 2.3. *Let $f_k : (X,\mu,\nu,*,\diamond) \to (Y,\mu',\nu',*,\diamond)$ be sequences of functions. Then for every $\varepsilon > 0$ and $t > 0$, the following statements are equivalent.*

(i) *Consider $st^{\lambda}_{\mu,\nu} - f_k \to f$.*

(ii) *For each $x \in X$*

$$\delta_\lambda\left\{k \in \mathbb{N} : \mu'\left(f_k(x) - f(x),t\right) \le 1-\varepsilon\right\} = \delta_\lambda\left\{k \in \mathbb{N} : \nu'\left(f_k(x) - f(x),t\right) \ge \varepsilon\right\} = 0. \tag{2.4}$$

(iii) *For each $x \in X$*

$$\delta_\lambda\left\{k \in \mathbb{N} : \mu'\left(f_k(x) - f(x),t\right) > 1-\varepsilon,\ \nu'\left(f_k(x) - f(x),t\right) < \varepsilon\right\} = 1. \tag{2.5}$$

(iv) *For each $x \in X$*

$$\delta_\lambda\left\{k \in \mathbb{N} : \mu'\left(f_k(x) - f(x),t\right) > 1-\varepsilon\right\} = \delta_\lambda\left\{k \in \mathbb{N} : \nu'\left(f_k(x) - f(x),t\right) < \varepsilon\right\} = 1. \tag{2.6}$$

(v) *For each $x \in X$*

$$st_\lambda - \lim \mu'\left(f_k(x) - f(x),t\right) = 1,\ st_\lambda - \lim \nu'\left(f_k(x) - f(x),t\right) = 0. \tag{2.7}$$

Example 2.4. Let $(\mathbb{R}, |\cdot|)$ denote the space of real numbers with the usual norm, and let $a * b = a \cdot b$ and $a \diamond b = \min\{a + b, 1\}$ for $a, b \in [0,1]$. For all $x \in \mathbb{R}$ and every $t > 0$, consider

$$\mu(x,t) = \frac{t}{t + |x|}, \qquad \nu(x,t) = \frac{|x|}{t + |x|}. \tag{2.8}$$

In this case $(\mathbb{R}, \mu, \nu, *, \diamond)$ is intuitionistic fuzzy normed space. (Also, $([0,1], \mu, \nu, *, \diamond)$ is intuitionistic fuzzy normed space.) Let $f_k : [0,1] \to \mathbb{R}$ be sequences of functions whose terms are given by

$$f_k(x) = \begin{cases} x^k + 1 & \text{if } k \in \left[n - \sqrt{\lambda_n} + 1, n\right],\ x \in \left[0, \frac{1}{2}\right) \\ 0 & \text{if } k \notin \left[n - \sqrt{\lambda_n} + 1, n\right],\ x \in \left[0, \frac{1}{2}\right) \\ x^k + \frac{1}{2} & \text{if } k \in \left[n - \sqrt{\lambda_n} + 1, n\right],\ x \in \left[\frac{1}{2}, 1\right) \\ 1 & \text{if } k \notin \left[n - \sqrt{\lambda_n} + 1, n\right],\ x \in \left[\frac{1}{2}, 1\right) \\ 2 & \text{if } x = 1. \end{cases} \tag{2.9}$$

(f_k) is pointwise λ-statistically convergent on $[0,1]$ with respect to intuitionistic fuzzy norm (μ, ν). It is fact that for each $x \in [0, 1/2)$, and since

$$K(\varepsilon, t) = \left\{k \in I_n : \mu\left(f_k(x) - f(x), t\right) \le 1 - \varepsilon \text{ or } \nu\left(f_k(x) - f(x), t\right) \ge \varepsilon\right\}, \tag{2.10}$$

hence we have

$$\begin{aligned} K(\varepsilon, t) &= \left\{k \in I_n : \frac{t}{t + \left|f_k(x) - 0\right|} \le 1 - \varepsilon \text{ or } \frac{\left|f_k(x) - 0\right|}{t + \left|f_k(x) - 0\right|} \ge \varepsilon\right\} \\ &= \left\{k \in I_n : \left|f_k(x)\right| \ge \frac{\varepsilon t}{1 - \varepsilon}\right\} \\ &= \left\{k \in I_n : f_k(x) = x^k + 1\right\}, \\ |K(\varepsilon, t)| &\le \sqrt{\lambda_n}. \end{aligned} \tag{2.11}$$

Thus, for each $x \in [0, 1/2)$, since

$$\delta_\lambda(K(\varepsilon, t)) = \lim_{n \to \infty} \frac{|K(\varepsilon, t)|}{\lambda_n} = \lim_{n \to \infty} \frac{\sqrt{\lambda_n}}{\lambda_n} = 0, \tag{2.12}$$

f_k is λ-statistically convergent to 0 with respect to intuitionistic fuzzy norm (μ, ν).

If we take $x \in [1/2, 1)$, then we have

$$\begin{aligned} K'(\varepsilon, t) &= \left\{ k \in I_n : \frac{t}{t + |f_k(x) - 1|} \leq 1 - \varepsilon \text{ or } \frac{|f_k(x) - 1|}{t + |f_k(x) - 1|} \geq \varepsilon \right\} \\ &= \left\{ k \in I_n : |f_k(x) - 1| \geq \frac{\varepsilon t}{1 - \varepsilon} \right\} \\ &= \left\{ k \in I_n : f_k(x) = x^k + \frac{1}{2} \right\}, \\ |K'(\varepsilon, t)| &\leq \sqrt{\lambda_n}. \end{aligned} \tag{2.13}$$

Thus $x \in [1/2, 1)$, then f_k is λ-statistically convergent to 1 with respect to intuitionistic fuzzy norm (μ, ν).

If we take $x = 1$, it can be seen easily that

$$\begin{aligned} K''(\varepsilon, t) &= \left\{ k \in I_n : \frac{t}{t + |f_k(x) - 2|} \leq 1 - \varepsilon \text{ or } \frac{|f_k(x) - 2|}{t + |f_k(x) - 2|} \geq \varepsilon \right\} \\ &= \left\{ k \in I_n : 0 \geq \frac{\varepsilon t}{1 - \varepsilon} \right\} \\ &= \emptyset, \\ |K''(\varepsilon, t)| &= 0, \\ \delta_\lambda(K''(\varepsilon, t)) &= \lim_{n \to \infty} \frac{|K''(\varepsilon, t)|}{\lambda_n} = \lim_{n \to \infty} \frac{0}{\lambda_n} = 0. \end{aligned} \tag{2.14}$$

Thus, at $x = 1$, f_k is λ-statistically convergent to 2 with respect to intuitionistic fuzzy norm (μ, ν).

Consequently, since $f_k(x)$ is λ-statistically convergent to different points with respect to intuitionistic fuzzy norm (μ, ν) for each $x \in X$, (f_k) is pointwise λ-statistically intuitionistic fuzzy convergent on $[0, 1]$.

Theorem 2.5. *Let $(X, \mu, \nu, *, \diamond)$ be an intuitionistic fuzzy normed space and let $f_k : (X, \mu, \nu, *, \diamond) \to (Y, \mu', \nu', *, \diamond)$ be sequences of functions. If sequence (f_k) is pointwise intuitionistic fuzzy convergent on X to a function f with respect to (μ, ν), then (f_k) is pointwise λ-statistically convergent with respect to intuitionistic fuzzy norm (μ, ν).*

Proof. Let (f_k) be pointwise intuitionistic fuzzy convergent in X. In this case, $(f_k(x))$ is convergent with respect to (μ', ν') for each $x \in X$. Then for every $\varepsilon > 0$ and $t > 0$, there is number $k_0 \in \mathbb{N}$ such that

$$\mu'(f_k(x) - f(x), t) > 1 - \varepsilon, \qquad \nu'(f_k(x) - f(x), t) < \varepsilon \tag{2.15}$$

for all $k \geq k_0$ and for each $x \in X$. Hence for each $x \in X$, the set

$$\{k \in \mathbb{N} : \mu'(f_k(x) - f(x), t) \leq 1 - \varepsilon \text{ or } \nu'(f_k(x) - f(x), t) \geq \varepsilon\} \tag{2.16}$$

has finite numbers of terms. Since finite subset of $\mathbb{N}$ has λ-density 0, hence

$$\delta_\lambda(\{k \in \mathbb{N} : \mu'(f_k(x) - f(x), t) \leq 1 - \varepsilon \text{ or } \nu'(f_k(x) - f(x), t) \geq \varepsilon\}) = 0. \tag{2.17}$$

That is, $st^\lambda_{\mu,\nu} - f_k \to f$. □

Theorem 2.6. *Let (f_k) and (g_k) be two sequences of functions from intuitionistic fuzzy normed space $(X, \mu, \nu, *, \diamond)$ to $(Y, \mu', \nu', *, \diamond)$. If $st^\lambda_{\mu,\nu} - f_k \to f$ and $st^\lambda_{\mu,\nu} - g_k \to g$, then $st^\lambda_{\mu,\nu} - (\alpha f_k + \beta g_k) \to \alpha f + \beta g$ where $\alpha, \beta \in IF$ ($\mathbb{R}$ or $\mathbb{C}$).*

Proof. The proof is clear for $\alpha = 0$ and $\beta = 0$. Now let $\alpha \neq 0$ and $\beta \neq 0$. Since $st^\lambda_{\mu,\nu} - f_k \to f$ and $st^\lambda_{\mu,\nu} - g_k \to g$, for each $x \in X$ if we define

$$\begin{aligned} A_1 &= \left\{k \in \mathbb{N} : \mu'\left(f_k(x) - f(x), \frac{t}{2|\alpha|}\right) \leq 1 - \varepsilon \text{ or } \nu'\left(f_k(x) - f(x), \frac{t}{2|\alpha|}\right) \geq \varepsilon\right\}, \\ A_2 &= \left\{k \in \mathbb{N} : \mu'\left(g_k(x) - g(x), \frac{t}{2|\beta|}\right) \leq 1 - \varepsilon \text{ or } \nu'\left(g_k(x) - g(x), \frac{t}{2|\beta|}\right) \geq \varepsilon\right\}, \end{aligned} \tag{2.18}$$

then $\delta_\lambda(A_1) = 0$ and $\delta_\lambda(A_2) = 0$. Since $\delta_\lambda(A_1) = 0$ and $\delta_\lambda(A_2) = 0$, if we state A by $(A_1 \cup A_2)$, then

$$\delta_\lambda(A) = 0. \tag{2.19}$$

Hence $A_1 \cup A_2 \neq \mathbb{N}$ and there exists $k_0 \in \mathbb{N}$ such that

$$\begin{aligned} \mu'\left(f_{k_0}(x) - f(x), \frac{t}{2|\alpha|}\right) &> 1 - \varepsilon, \qquad \nu'\left(f_{k_0}(x) - f(x), \frac{t}{2|\alpha|}\right) < \varepsilon, \\ \mu'\left(g_{k_0}(x) - g(x), \frac{t}{2|\beta|}\right) &> 1 - \varepsilon, \qquad \nu'\left(g_{k_0}(x) - g(x), \frac{t}{2|\beta|}\right) < \varepsilon. \end{aligned} \tag{2.20}$$

Let

$$\begin{aligned} B = \{k \in \mathbb{N} : &\mu'((\alpha f_k + \beta g_k)(x) - (\alpha f(x) + \beta g(x)), t) > 1 - \varepsilon, \\ &\nu'((\alpha f_k + \beta g_k)(x) - (\alpha f(x) + \beta g(x)), t) < \varepsilon\}. \end{aligned} \tag{2.21}$$

We will show that for each $x \in X$

$$A^c \subset B. \tag{2.22}$$

Let $k_0 \in A^c$. In this case

$$\begin{aligned} \mu'\left(f_{k_0}(x) - f(x), \frac{t}{2|\alpha|}\right) &> 1-\varepsilon, \qquad \nu'\left(f_{k_0}(x) - f(x), \frac{t}{2|\alpha|}\right) < \varepsilon, \\ \mu'\left(g_{k_0}(x) - g(x), \frac{t}{2|\beta|}\right) &> 1-\varepsilon, \qquad \nu'\left(g_{k_0}(x) - g(x), \frac{t}{2|\beta|}\right) < \varepsilon. \end{aligned} \tag{2.23}$$

Using those mentioned previously, we have

$$\begin{aligned} \mu'\left((\alpha f_{k_0} + \beta g_{k_0})(x) - (\alpha f(x) + \beta g(x)), t\right) &\geq \mu'\left(\alpha f_{k_0}(x) - \alpha f(x), \frac{t}{2}\right) \\ &\quad * \mu'\left(\beta g_{k_0}(x) - \beta g(x), \frac{t}{2}\right) \\ &= \mu'\left(f_{k_0}(x) - f(x), \frac{t}{2|\alpha|}\right) \\ &\quad * \mu'\left(g_{k_0}(x) - g(x), \frac{t}{2|\beta|}\right) \\ &> (1-\varepsilon) * (1-\varepsilon) \\ &= (1-\varepsilon), \\ \nu'\left((\alpha f_{k_0} + \beta g_{k_0})(x) - (\alpha f(x) + \beta g(x)), t\right) &\leq \nu'\left(\alpha f_{k_0}(x) - \alpha f(x), \frac{t}{2}\right) \\ &\quad \diamond \nu'\left(\beta g_{k_0}(x) - \beta g(x), \frac{t}{2}\right) \\ &= \nu'\left(f_{k_0}(x) - f(x), \frac{t}{2|\alpha|}\right) \\ &\quad \diamond \nu'\left(g_{k_0}(x) - g(x), \frac{t}{2|\beta|}\right) \\ &< \varepsilon \diamond \varepsilon \\ &= \varepsilon. \end{aligned} \tag{2.24}$$

This implies that

$$A^c \subset B. \tag{2.25}$$

Since $B^c \subset A$ and $\delta_\lambda(A) = 0$, hence

$$\delta_\lambda(B^c) = 0. \tag{2.26}$$

That is

$$\delta_\lambda\left(\left\{\begin{array}{c} k\in\mathbb{N}:\mu'\big((\alpha f_k+\beta g_k)(x)-(\alpha f(x)+\beta g(x)),t\big)\le 1-\varepsilon,\\ \nu'\big((\alpha f_k+\beta g_k)(x)-(\alpha f(x)+\beta g(x)),t\big)\ge\varepsilon \end{array}\right\}\right)=0,$$
$$st^{\lambda}_{\mu,\nu}-(\alpha f_k+\beta g_k)\longrightarrow \alpha f+\beta g. \tag{2.27}$$

□

Definition 2.7. Let $f_k : (X,\mu,\nu,*,\diamond) \to (Y,\mu',\nu',*,\diamond)$ be sequences of functions. (f_k) is a pointwise λ-statistical Cauchy sequence in intuitionistic fuzzy normed space provided that for every $\varepsilon>0$ and $t>0$ there exists a number $N=N(x,\varepsilon,t)$ such that

$$\delta_\lambda\left(\left\{\begin{array}{c} k\in\mathbb{N}:\mu'\big(f_k(x)-f_N(x),t\big)\le 1-\varepsilon \text{ or}\\ \nu'\big(f_k(x)-f_N(x),t\big)\ge\varepsilon \text{ for each } x\in X \end{array}\right\}\right)=0. \tag{2.28}$$

Theorem 2.8. *Let $f_k : (X,\mu,\nu,*,\diamond) \to (Y,\mu',\nu',*,\diamond)$ be a sequence of functions. If (f_k) is a pointwise λ-statistical convergent sequence with respect to intuitionistic fuzzy norm (μ,ν), then (f_k) is a pointwise λ-statistical Cauchy sequence with respect to intuitionistic fuzzy norm (μ,ν).*

Proof. Suppose that $st^{\lambda}_{\mu,\nu}-f_k\to f$ and let $\varepsilon>0$, $t>0$. For a given $\varepsilon>0$, choose $s>0$ such that $(1-\varepsilon)*(1-\varepsilon)>1-s$ and $\varepsilon\diamond\varepsilon<s$. If we state, respectively, $A_x(\varepsilon,t)$ and $A^c_x(\varepsilon,t)$ by

$$\left\{k\in\mathbb{N}:\mu'\left(f_k(x)-f(x),\frac{t}{2}\right)\le 1-\varepsilon \text{ or } \nu'\left(f_k(x)-f(x),\frac{t}{2}\right)\ge\varepsilon\right\},$$
$$\left\{k\in\mathbb{N}:\mu'\left(f_k(x)-f(x),\frac{t}{2}\right)>1-\varepsilon,\ \nu'\left(f_k(x)-f(x),\frac{t}{2}\right)<\varepsilon\right\} \tag{2.29}$$

for each $x\in X$. Then, we have

$$\delta_\lambda(A_x(\varepsilon,t))=0, \tag{2.30}$$

which implies that

$$\delta_\lambda(A^c_x(\varepsilon,t))=1. \tag{2.31}$$

Let $N\in A^c_x(\varepsilon,t)$. Then

$$\mu'\left(f_N(x)-f(x),\frac{t}{2}\right)>1-\varepsilon,\qquad \nu'\left(f_N(x)-f(x),\frac{t}{2}\right)<\varepsilon. \tag{2.32}$$

We want to show that there exists a number $N=N(x,\varepsilon,t)$ such that

$$\delta_\lambda\left(\left\{\begin{array}{c} k\in\mathbb{N}:\mu'\big(f_k(x)-f_N(x),t\big)\le 1-s \text{ or}\\ \nu'\big(f_k(x)-f_N(x),t\big)\ge s \text{ for each } x\in X \end{array}\right\}\right)=0. \tag{2.33}$$

Therefore, define for each $x \in X$,

$$B_x(\varepsilon,t) = \{k \in \mathbb{N} : \mu'(f_k(x) - f_N(x),t) \leq 1 - s \text{ or } \nu'(f_k(x) - f_N(x),t) \geq s\}. \tag{2.34}$$

We have to show that

$$B_x(\varepsilon,t) \subset A_x(\varepsilon,t). \tag{2.35}$$

Suppose that

$$B_x(\varepsilon,t) \not\subseteq A_x(\varepsilon,t). \tag{2.36}$$

In this case $B_x(\varepsilon,t)$ has at least one different element which $A_x(\varepsilon,t)$ does not have. Let $k \in B_x(\varepsilon,t) \setminus A_x(\varepsilon,t)$ Then we have

$$\mu'(f_k(x) - f_N(x),t) \leq 1 - s, \qquad \mu'\left(f_k(x) - f(x), \frac{t}{2}\right) > 1 - \varepsilon, \tag{2.37}$$

in particular $\mu'(f_N(x) - f(x), t/2) > 1 - \varepsilon$. In this case

$$\begin{aligned} 1 - s &\geq \mu'(f_k(x) - f_N(x),t) \\ &\geq \mu'\left(f_k(x) - f(x), \frac{t}{2}\right) * \mu'\left(f_N(x) - f(x), \frac{t}{2}\right) \\ &\geq (1-\varepsilon) * (1-\varepsilon) > 1 - s, \end{aligned} \tag{2.38}$$

which is not possible. On the other hand

$$\nu'(f_k(x) - f_N(x),t) \geq s, \qquad \nu'(f_k(x) - f(x),t) < \varepsilon, \tag{2.39}$$

in particular $\nu'(f_N(x) - f(x),t) < \varepsilon$. In this case

$$\begin{aligned} s &\leq \nu'(f_k(x) - f_N(x),t) \\ &\leq \nu'\left(f_k(x) - f(x), \frac{t}{2}\right) \diamond \nu'\left(f_N(x) - f(x), \frac{t}{2}\right) \\ &< \varepsilon \diamond \varepsilon < s, \end{aligned} \tag{2.40}$$

which is not possible. Hence $B_x(\varepsilon,t) \subset A_x(\varepsilon,t)$. Therefore, by $\delta_\lambda(A_x(\varepsilon,t)) = 0$, $\delta_\lambda(B_x(\varepsilon,t)) = 0$. That is, (f_k) is a pointwise λ-statistical Cauchy sequence with respect to intuitionistic fuzzy norm (μ,ν). □

In the following, we introduce uniformly λ-statistical convergence in intuitionistic fuzzy normed spaces.

Definition 2.9. Let $(X,\mu,\nu,*,\diamond)$ and $(Y,\mu',\nu',*,\diamond)$ be two intuitionistic fuzzy normed linear spaces over the same field IF and let $f_k : (X,\mu,\nu,*,\diamond) \to (Y,\mu',\nu',*,\diamond)$ be a sequence of functions. If for all $x \in X$ and for all $\varepsilon > 0$, $t > 0$,

$$\delta_\lambda\left(\left\{k \in \mathbb{N} : \mu'\left(f_k(x) - f(x),t\right) \le 1-\varepsilon \text{ or } \nu'\left(f_k(x) - f(x),t\right) \ge \varepsilon\right\}\right) = 0, \tag{2.41}$$

or equivalently

$$\delta_\lambda\left(\left\{k \in \mathbb{N} : \mu'\left(f_k(x) - f(x),t\right) > 1-\varepsilon,\ \nu'\left(f_k(x) - f(x),t\right) < \varepsilon\right\}\right) = 1, \tag{2.42}$$

then we say that the sequence f_k is uniformly λ-statistically convergent with respect to intuitionistic fuzzy norm (μ,ν) and we write it $st^\lambda_{\mu,\nu} - f_k \rightrightarrows f$.

Remark 2.10. If $st^\lambda_{\mu,\nu} - f_k \rightrightarrows f$, then $st^\lambda_{\mu,\nu} - f_k \to f$.

Lemma 2.11. *Let $f_k : (X,\mu,\nu,*,\diamond) \to (Y,\mu',\nu',*,\diamond)$ be a sequence of functions. Then for every $\varepsilon > 0$ and $t > 0$, the following statements are equivalent:*

(i) *Consider $st^\lambda_{\mu,\nu} - f_k \rightrightarrows f$.*

(ii) *For all $x \in X$*

$$\delta_\lambda\left\{k \in \mathbb{N} : \mu'\left(f_k(x) - f(x),t\right) \le 1-\varepsilon\right\} = \delta_\lambda\left\{k \in \mathbb{N} : \nu'\left(f_k(x) - f(x),t\right) \ge \varepsilon\right\} = 0. \tag{2.43}$$

(iii) *For all $x \in X$*

$$\delta_\lambda\left\{k \in \mathbb{N} : \mu'\left(f_k(x) - f(x),t\right) > 1-\varepsilon,\ \nu'\left(f_k(x) - f(x),t\right) < \varepsilon\right\} = 1. \tag{2.44}$$

(iv) *For all $x \in X$*

$$\delta_\lambda\left\{k \in \mathbb{N} : \mu'\left(f_k(x) - f(x),t\right) > 1-\varepsilon\right\} = \delta_\lambda\left\{k \in \mathbb{N} : \nu'\left(f_k(x) - f(x),t\right) < \varepsilon\right\} = 1. \tag{2.45}$$

(v) *For all $x \in X$*

$$st_\lambda - \lim \mu'\left(f_k(x) - f(x),t\right) = 1, \qquad st_\lambda - \lim \nu'\left(f_k(x) - f(x),t\right) = 0. \tag{2.46}$$

Example 2.12. Let $(\mathbb{R},\mu,\nu,*,\diamond)$ be as Example 2.4. Let $f_k : [0,1] \to \mathbb{R}$ be a sequence of functions whose terms are given by

$$f_k(x) = \begin{cases} x^k + 1, & \text{if } n - \sqrt{\lambda_n} + 1 \le k \le n \\ 0, & \text{otherwise.} \end{cases} \tag{2.47}$$

Since $K(\varepsilon,t)=\{k\in I_n:\mu(f_k(x)-f(x),t)\le 1-\varepsilon \text{ or } \nu(f_k(x)-f(x),t)\ge\varepsilon\}$, hence we have

$$\begin{aligned}
K(\varepsilon,t)&=\left\{k\in I_n:\frac{t}{t+\left|f_k(x)-0\right|}\le 1-\varepsilon \text{ or } \frac{\left|f_k(x)-0\right|}{t+\left|f_k(x)-0\right|}\ge\varepsilon\right\}\\
&=\left\{k\in I_n:\left|f_k(x)\right|\ge\frac{\varepsilon t}{1-\varepsilon}\right\}\\
&=\left\{k\in I_n:f_k(x)=x^k+1\right\},\\
|K(\varepsilon,t)|&\le\sqrt{\lambda_n}.
\end{aligned}\tag{2.48}$$

Thus, for all $x\in[0,1]$, since

$$\delta_\lambda(K(\varepsilon,t))=\lim_{n\to\infty}\frac{|K(\varepsilon,t)|}{\lambda_n}=\lim_{n\to\infty}\frac{\sqrt{\lambda_n}}{\lambda_n}=0,\tag{2.49}$$

f_k is uniformly λ-statistically convergent to 0 with respect to intuitionistic fuzzy norm (μ,ν).

Definition 2.13. Let $f_k:(X,\mu,\nu,*,\diamond)\to(Y,\mu',\nu',*,\diamond)$ be sequences of functions. The sequence (f_k) is a uniformly λ-statistical Cauchy sequence in intuitionistic fuzzy normed space provided that for every $\varepsilon>0$ and $t>0$ there exists a number $N=N(\varepsilon,t)$ such that

$$\delta_\lambda\left(\left\{\begin{matrix}k\in\mathbb{N}:\mu'\left(f_k(x)-f_N(x),t\right)\le 1-\varepsilon \text{ or}\\ \nu'\left(f_k(x)-f_N(x),t\right)\ge\varepsilon\ \forall x\in X\end{matrix}\right\}\right)=0.\tag{2.50}$$

Definition 2.14. Let $(X,\mu,\nu,*,\diamond)$ and $(Y,\mu',\nu',*,\diamond)$ be two intuitionistic fuzzy normed spaces, and let F be a family of functions from X to Y. The family F is intuitionistic fuzzy equicontinuous at a point $x_0\in X$ if for every $\varepsilon>0$ and $t>0$, there exists a $\delta>0$ such that $\mu'(f(x_0)-f(x),t)>1-\varepsilon$ and $\nu'(f(x_0)-f(x),t)<\varepsilon$ for all $f\in F$ and all x such that $\mu'(x_0-x,t)>1-\delta$ and $\nu'(x_0-x,t)<\delta$. The family is intuitionistic fuzzy equicontinuous if it is equicontinuous at each point of X. (For continuity, δ may depend on ε, x_0 and f; for equicontinuity, δ must be independent of f.)

Theorem 2.15. *Let $(X,\mu,\nu,*,\diamond)$, $(Y,\mu',\nu',*,\diamond)$ be intuitionistic fuzzy normed space. Assume that $st^\lambda_{\mu,\nu}-f_k\to f$ on X where functions $f_k:(X,\mu,\nu,*,\diamond)\to(Y,\mu',\nu',*,\diamond)$, $k\in\mathbb{N}$ are intuitionistic fuzzy equicontinuous on X and $f:X\to Y$. Then f is continuous on X.*

Proof. Let $x_0\in X$ be an arbitrary point. By the intuitionistic fuzzy equicontinuity of f_k's, for every $\varepsilon>0$ and $t>0$ there exist $\delta=\delta(x_0,\varepsilon,t/3)>0$ such that

$$\mu'\left(f_k(x_0)-f_k(x),\frac{t}{3}\right)>1-\varepsilon,\qquad \nu'\left(f_k(x_0)-f_k(x),\frac{t}{3}\right)<\varepsilon\tag{2.51}$$

for every $k \in \mathbb{N}$ and all x such that $\mu'(x_0 - x, t) > 1 - \delta$ and $\nu'(x_0 - x, t) < \delta$. Let $x \in B(x_0, \delta, t)$ ($B(x_0, \delta, t)$ stands for an open ball in $(X, \mu, v, *, \diamond)$ with center x_0 and radius δ) be fixed. Since $st^{\lambda}_{\mu,\nu} - f_k \to f$ on X, for each $x \in X$, if we state, respectively, $A_x(\varepsilon, t)$ and $B_x(\varepsilon, t)$ by the sets

$$A_x(\varepsilon, t) = \left\{ \begin{array}{c} k \in \mathbb{N} : \mu'\left(f_k(x) - f(x), \dfrac{t}{3}\right) \leq 1 - \varepsilon \text{ or} \\ \nu'\left(f_k(x) - f(x), \dfrac{t}{3}\right) \geq \varepsilon \text{ for each } x \in X \end{array} \right\},$$
$$B_x(\varepsilon, t) = \left\{ \begin{array}{c} k \in \mathbb{N} : \mu'\left(f_k(x_0) - f(x_0), \dfrac{t}{3}\right) \leq 1 - \varepsilon \text{ or} \\ \nu'\left(f_k(x_0) - f(x_0), \dfrac{t}{3}\right) \geq \varepsilon \text{ for each } x \in X \end{array} \right\}, \tag{2.52}$$

then $\delta_\lambda(A_x(\varepsilon, t)) = 0$ and $\delta_\lambda(B_x(\varepsilon, t)) = 0$; hence $\delta_\lambda(A_x(\varepsilon, t) \cup B_x(\varepsilon, t)) = 0$ and $A_x(\varepsilon, t) \cup B_x(\varepsilon, t)$ is different from $\mathbb{N}$. Thus, there exists $\exists m \in \mathbb{N}$ such that

$$\mu'\left(f_m(x) - f(x), \frac{t}{3}\right) > 1 - \varepsilon, \qquad \nu'\left(f_m(x) - f(x), \frac{t}{3}\right) < \varepsilon,$$
$$\mu'\left(f_m(x_0) - f(x_0), \frac{t}{3}\right) > 1 - \varepsilon, \qquad \nu'\left(f_m(x_0) - f(x_0), \frac{t}{3}\right) < \varepsilon. \tag{2.53}$$

Now, we will show that f is intuitionistic fuzzy continuous at x_0. Since $st^{\lambda}_{\mu,\nu} - f_k \to f$ and for every $k \in \mathbb{N}$ f_k's are continuous, f_m is also continuous for $m \in \mathbb{N}$, we have

$$\begin{aligned}
\mu'(f(x) - f(x_0), t) &= \mu'(f(x) - f_m(x) + f_m(x) - f_m(x_0) + f_m(x_0) - f(x_0), t) \\
&\geq \mu'\left(f(x) - f_m(x), \frac{t}{3}\right) \\
&\quad * \mu'\left(f_m(x) - f_k(x_0), \frac{t}{3}\right) * \mu'\left(f_m(x_0) - f(x_0), \frac{t}{3}\right) \\
&> (1 - \varepsilon) * (1 - \varepsilon) * (1 - \varepsilon) \\
&= 1 - \varepsilon, \\
\nu'(f(x) - f(x_0), t) &= \nu'(f(x) - f_m(x) + f_m(x) - f_m(x_0) + f_m(x_0) - f(x_0), t) \\
&\leq \nu'\left(f(x) - f_m(x), \frac{t}{3}\right) \\
&\quad \diamond \nu'\left(f_m(x) - f_m(x_0), \frac{t}{3}\right) \diamond \nu'\left(f_m(x_0) - f(x_0), \frac{t}{3}\right) \\
&< \varepsilon \diamond \varepsilon \diamond \varepsilon \\
&= \varepsilon.
\end{aligned} \tag{2.54}$$

Thus, the proof is completed. □

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