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Research Article

Positive Periodic Solutions for Second-Order Ordinary Differential Equations with Derivative Terms and Singularity in Nonlinearities

Yongxiang Li and Xiaoyu Jiang

Department of Mathematics, Northwest Normal University, Lanzhou 730070, China

Correspondence should be addressed to Yongxiang Li, liyxnwnu@163.com

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The existence results of positive ω-periodic solutions are obtained for the second-order ordinary differential equation $u''(t) = f(t, u(t), u'(t)), t \in \mathbb{R}$ where, $f : \mathbb{R} \times (0, \infty) \times \mathbb{R} \to \mathbb{R}$ is a continuous function, which is ω-periodic in t and $f(t, u, v)$ may be singular at $u = 0$. The discussion is based on the fixed point index theory in cones.

1. Introduction

In this paper, we discuss the existence of positive ω-periodic solutions of the second-order ordinary differential equation with first-order derivative term in the nonlinearity

$$u''(t) = f\big(t, u(t), u'(t)\big), \quad t \in \mathbb{R}, \tag{1.1}$$

where the nonlinearity $f : \mathbb{R} \times (0, \infty) \times \mathbb{R} \to \mathbb{R}$ is a continuous function, which is ω-periodic in t and $f(t, u, v)$ may be singular at $u = 0$.

The existence problems of periodic solutions for nonlinear second-order ordinary differential equations have attracted many authors' attention and concern, and most works are on the special equation

$$u''(t) = f(t, u(t)), \quad t \in \mathbb{R}, \tag{1.2}$$

that does not contain explicitly first-order derivative term in nonlinearity. Many theorems and methods of nonlinear functional analysis have been applied to the periodic problems of

(1.2). These theorems and methods are mainly the upper and lower solutions method and monotone iterative technique [1–4], the continuation method of topological degree [5–7], variational method and critical point theory, [8–10] and so forth.

In recent years, the fixed point theorems of cone mapping, especially the fixed point theorem of Krasnoselskii's cone expansion or compression type, have been extensively applied to two-point boundary value problems of second-order ordinary differential equations, and some results of existence and multiplicity of positive solutions have been obtained, see [11–15]. Lately, the authors of [16–18] have also applied the Krasnoselskii's fixed point theorem to periodic problems of second-order nonlinear ordinary differential equations, and obtained existence results of positive periodic solutions. In these works, the new discovered positivity of Green function of the corresponding linear second-order periodic boundary value problems plays an important role. The positivity guarantees that the integral operators of the second-order periodic problems are cone-preserving in the cone

$$K_0 = \{u \in C[0,\omega] \mid u(t) \geq \sigma\|u\|, t \in [0,\omega]\} \tag{1.3}$$

in the Banach space $C[0,\omega]$, where $\sigma > 0$ is a constant. Hence the fixed point theorems of cone mapping can be applied to the second-order periodic problems. For more precise results using the theory of the fixed point index in cones to discuss the existence of positive periodic solutions of second-order ordinary differential equation, see [19–22]. However, all of these works are on the special second-order equation (1.2), and few people consider the existence of the positive periodic solutions for the general second-order equation (1.1) that explicitly contains the first order derivative term.

The purpose of this paper is to extend the results of [16–22] to the general second-order equation (1.1). We will use the theory of the fixed point index in cones to discuss the existence of positive periodic solutions of (1.1). For the periodic problem of (1.1), since the corresponding integral operator has no definition on the cone K_0 in $C[0,\omega]$, the argument methods used in [16–22] are not applicable. We will use a completely different method to treat (1.1). Our main results will be given in Section 3. Some preliminaries to discuss (1.1) are presented in Section 2.

2. Preliminaries

Let $C_\omega(\mathbb{R})$ denote the Banach space of all continuous ω-periodic function $u(t)$ with norm $\|u\|_C = \max_{0\leq t\leq\omega}|u(t)|$. Let $C^1_\omega(\mathbb{R})$ be the Banach space of all continuous differentiable ω-periodic function $u(t)$ with the norm

$$\|u\|_{C^1} = \|u\|_C + \|u'\|_C. \tag{2.1}$$

Generally, $C^n_\omega(\mathbb{R})$ denotes the nth-order continuous differentiable ω-periodic function space for $n \in \mathbb{N}$. Let $C^+_\omega(\mathbb{R})$ be the cone of all nonnegative functions in $C_\omega(\mathbb{R})$.

Let $M \in (0,\ \pi^2/\omega^2)$ be a constant. For $h \in C_\omega(\mathbb{R})$, we consider the linear second-order differential equation

$$u''(t) + Mu(t) = h(t), \quad t \in \mathbb{R}. \tag{2.2}$$

The ω-periodic solutions of (2.2) are closely related with the linear second-order boundary value problem

$$\begin{aligned} u''(t) + Mu(t) = 0, \quad & 0 \le t \le \omega, \\ u(0) - u(\omega) = 0, \quad & u'(0) - u'(\omega) = 1, \end{aligned} \tag{2.3}$$

see [19]. It is easy to see that Problem (2.3) has a unique solution, which is explicitly given by

$$U(t) = \frac{\cos\beta(t-\omega/2)}{2\beta\sin(\beta\omega/2)}, \quad 0 \le t \le \omega, \tag{2.4}$$

where $\beta = \sqrt{M}$. We have the following Lemma.

Lemma 2.1. *Let $M \in (0, \pi^2/\omega^2)$. Then for every $h \in C_\omega(\mathbb{R})$, the linear equation* (2.2) *has a unique ω-periodic solution $u(t)$, which is given by*

$$u(t) = \int_{t-\omega}^{t} U(t-s)h(s)ds := Sh(t), \quad t \in \mathbb{R}. \tag{2.5}$$

Moreover, $S : C_\omega(\mathbb{R}) \to C^1_\omega(\mathbb{R})$ is a completely continuous linear operator.

Proof. Taking the derivative in (2.5) and using the boundary condition of $U(t)$, we obtain that

$$\begin{aligned} u''(t) &= \big(U'(0) - U'(\omega)\big)h(t) + \int_{t-\omega}^{t} U''(t-s)h(s)ds \\ &= h(t) - M\int_{t-\omega}^{t} U(t-s)h(s)ds \\ &= h(t) - Mu(t). \end{aligned} \tag{2.6}$$

Therefore, $u(t)$ satisfies (2.2). Let $\tau = s + \omega$; it follows from (2.5) that

$$\begin{aligned} u(t) &= \int_{t}^{t+\omega} U(t+\omega-\tau)h(\tau-\omega)d\tau \\ &= \int_{t}^{t+\omega} U(t+\omega-\tau)h(\tau)d\tau = u(t+\omega). \end{aligned} \tag{2.7}$$

Hence, $u(t)$ is an ω-periodic solution of (2.2). From the maximum principle for second-order periodic boundary value problems [4], it is easy to see that $u(t)$ is the unique ω-periodic solution of (2.2).

From (2.5) and (2.6), we easily see that $S : C_\omega(\mathbb{R}) \to C^2_\omega(\mathbb{R})$ is a linear bounded operator. By the compactness of the embedding $C^2_\omega(\mathbb{R}) \hookrightarrow C^1_\omega(\mathbb{R})$, $S : C_\omega(\mathbb{R}) \to C^1_\omega(\mathbb{R})$ is a completely continuous operator. □

Since $U(t) > 0$ for every $t \in [0,\omega]$, by (2.5), if $h \in C_\omega^+(\mathbb{R})$ and $h(t) \not\equiv 0$, then the ω-periodic solution of (2.2) $u(t) > 0$ for every $t \in \mathbb{R}$, and we term it the positive ω-periodic solution. Let

$$\begin{gathered}\overline{U} = \max_{0\le t\le\omega} U(t) = \frac{1}{2\beta\sin(\beta\omega/2)}, \qquad \underline{U} = \min_{0\le t\le\omega} U(t) = \frac{\cos(\beta\omega/2)}{2\beta\sin(\beta\omega/2)},\\ \overline{U}_1 = \max_{0\le t\le\omega}\left|U'(t)\right| = \max_{0\le t\le\omega}\frac{\left|\sin\beta(t-\omega/2)\right|}{2\sin(\beta\omega/2)} = \frac{1}{2},\\ \sigma = \frac{\underline{U}}{\overline{U}} = \cos\frac{\beta\omega}{2}, \qquad C_0 = \frac{\overline{U}_1}{\underline{U}} = \beta\tan\frac{\beta\omega}{2}.\end{gathered} \tag{2.8}$$

Define the cone K in $C_\omega^1(\mathbb{R})$ by

$$K = \left\{u \in C_\omega^1(\mathbb{R}) \mid u(t) \ge \sigma\|u\|_C, \left|u'(t)\right| \le C_0|u(t)|, t \in \mathbb{R}\right\}. \tag{2.9}$$

We have the following Lemma.

Lemma 2.2. *Let $M \in (0,\pi^2/\omega^2)$. Then for every $h \in C_\omega^+(\mathbb{R})$, the positive ω-periodic solution of* (2.2) *$u = Sh \in K$. Namely, $S(C_\omega^+(\mathbb{R})) \subset K$.*

Proof. Let $h \in C_\omega^+(\mathbb{R})$, $u = Sh$. For every $t \in \mathbb{R}$, from (2.5) it follows that

$$u(t) = \int_{t-\omega}^{t} U(t-s)h(s)ds \le \overline{U}\int_{t-\omega}^{t} h(s)ds = \overline{U}\int_0^\omega h(s)ds, \tag{2.10}$$

and therefore,

$$\|u\|_C \le \overline{U}\int_0^\omega h(s)ds. \tag{2.11}$$

Using (2.5), we obtain that

$$u(t) = \int_{t-\omega}^{t} U(t-s)h(s)ds \ge \underline{U}\int_{t-\omega}^{t} h(s)ds = \underline{U}\int_0^\omega h(s)ds \ge \sigma\|u\|_C. \tag{2.12}$$

For every $t \in \mathbb{R}$, since

$$u'(t) = \int_{t-\omega}^{t} U'(t-s)h(s)ds, \tag{2.13}$$

we have

$$\begin{aligned} \left|u'(t)\right| &\le \int_{t-\omega}^{t} \left|U'(t-s)\right| h(s)ds \le \overline{U}_1 \int_{t-\omega}^{t} h(s)ds \\ &= \overline{U}_1 \int_0^{\omega} h(s)ds = C_0\, \underline{U} \int_0^{\omega} h(s)ds \le C_0 u(t). \end{aligned} \tag{2.14}$$

Hence, $u \in K$. □

Now we consider the nonlinear equation (1.1). Hereafter, we assume that the nonlinearity f satisfies the following condition.

(F0) There exists $M \in (0, \pi^2/\omega^2)$ such that

$$f(t,x,y) + Mx \ge 0, \qquad x > 0,\ t, y \in \mathbb{R}. \tag{2.15}$$

Let $f_1(t,x,y) = f(t,x,y) + Mx$, then $f_1(t,x,y) \ge 0$ for $x > 0$, $t, y \in \mathbb{R}$, and (1.1) is rewritten to

$$u''(t) + Mu(t) = f_1\left(t, u(t), u'(t)\right), \quad t \in \mathbb{R}. \tag{2.16}$$

For $u \in K$, if $u \neq 0$, then $\|u\|_C > 0$ and by the definition of K, $u(t) \ge \sigma\|u\|_C > 0$ for every $t \in \mathbb{R}$. Hence

$$F(u)(t) := f_1\left(t, u(t), u'(t)\right), \qquad t \in \mathbb{R} \tag{2.17}$$

is well defined, and we can define the integral operator $A : K \setminus \{0\} \to C^1_\omega(\mathbb{R})$ by

$$Au(t) = \int_{t-\omega}^{t} U(t-s) f_1\left(s, u(s), u'(s)\right) ds = (S \circ F)(t). \tag{2.18}$$

By the definition of operator S, the positive ω-periodic solution of (1.1) is equivalent to the nontrivial fixed point of A. From Assumption (F0), Lemmas 2.1 and 2.2, we easily see the following Lemma.

Lemma 2.3. *$A(K \setminus \{0\}) \subset K$, and $A : K \setminus \{0\} \to K$ is completely continuous.*

We will find the nonzero fixed point of A by using the fixed point index theory in cones. Since the singularity of f at $x = 0$ implies that A has no definition at $u = 0$, the fixed point index theory in the cone K cannot be directly applied to A. We need to make some Preliminaries.

We recall some concepts and conclusions on the fixed point index in [23, 24]. Let E be a Banach space and $K \subset E$ a closed convex cone in E. Assume Ω is a bounded open subset of E with boundary $\partial\Omega$, and $K \cap \Omega \neq \emptyset$. Let $A : K \cap \overline{\Omega} \to K$ be a completely continuous mapping. If $Au \neq u$ for any $u \in K \cap \partial\Omega$, then the fixed point index $i(A, K \cap \Omega, K)$ has a definition. One important fact is that if $i(A, K \cap \Omega, K) \neq 0$, then A has a fixed point in $K \cap \Omega$. The following two lemmas are needed in our argument.

Lemma 2.4 (see [24]). *Let Ω be a bounded open subset of E with $\theta \in \Omega$ and $A : K \cap \overline{\Omega} \to K$ a completely continuous mapping. If $\lambda Au \neq u$ for every $u \in K \cap \partial\Omega$ and $0 < \lambda \leq 1$, then $i(A, K \cap \Omega, K) = 1$.*

Lemma 2.5 (see [24]). *Let Ω be a bounded open subset of E and $A : K \cap \overline{\Omega} \to K$ a completely continuous mapping. If there exists an $e \in K \setminus \{\theta\}$ such that $u - Au \neq \tau e$ for every $u \in K \cap \partial\Omega$ and $\tau \geq 0$, then $i(A, K \cap \Omega, K) = 0$.*

We use Lemmas 2.4 and 2.5 to show the following fixed-point theorem in cones which is applicable to the operator A defined by (2.18).

Theorem 2.6. *Let E be a Banach space and $K \subset E$ a closed convex cone. Assume Ω_1 and Ω_2 are bounded open subsets of E with $\theta \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$. Let $A : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$ be a completely continuous mapping. If A satisfies the following conditions:*

(1) *$\lambda Au \neq u$ for $u \in K \cap \partial\Omega_1$, $0 < \lambda \leq 1$;*

(2) *there exists $e \in K \setminus \{\theta\}$ such that $u - Au \neq \tau e$ for $u \in K \cap \partial\Omega_2$, $\tau \geq 0$, or the following conditions:*

(3) *there exists $e \in K \setminus \{\theta\}$ such that $u - Au \neq \tau e$ for $u \in K \cap \partial\Omega_1$, $\tau \geq 0$;*

(4) *$\lambda Au \neq u$ for $u \in K \cap \partial\Omega_2$, $0 < \lambda \leq 1$,*

then A has a fixed-point in $K \cap (\Omega_2 \setminus \overline{\Omega}_1)$.

Proof. By Dugundji's extension theorem, the operator $A : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$ can be extended into a completely continuous operator from $K \cap \overline{\Omega}_2$ to K, says $\tilde{A} : K \cap \overline{\Omega}_2 \to K$.

If A satisfies conditions (1) and (2) of Theorem 2.6, then $\tilde{A}$ also satisfies them. By Lemmas 2.4 and 2.5, respectively, we have

$$i\left(\tilde{A}, K \cap \Omega_1, K\right) = 1, \qquad i\left(\tilde{A}, K \cap \Omega_2, K\right) = 0. \tag{2.19}$$

By the additivity of the fixed point index, we have

$$i\left(\tilde{A}, K \cap \left(\Omega_2 \setminus \overline{\Omega}_1\right), K\right) = i\left(\tilde{A}, K \cap \Omega_2, K\right) - i\left(\tilde{A}, K \cap \Omega_1, K\right) = -1. \tag{2.20}$$

Hence $\tilde{A}$ has a fixed-point in $K \cap (\Omega_2 \setminus \overline{\Omega}_1)$. Since $\tilde{A}$ is an extension of A, it follows that A has a fixed-point in $K \cap (\Omega_2 \setminus \overline{\Omega}_1)$.

If A satisfies conditions (3) and (4) of Theorem 2.6, with a similar count, we obtain that

$$i\left(\tilde{A}, K \cap \left(\Omega_2 \setminus \overline{\Omega}_1\right), K\right) = 1. \tag{2.21}$$

This means that $\tilde{A}$ has a fixed-point in $K \cap (\Omega_2 \setminus \overline{\Omega}_1)$. Hence, A has a fixed-point in $K \cap (\Omega_2 \setminus \overline{\Omega}_1)$. □

Theorem 2.6 is an improvement of the fixed point theorem of Krasnoselskii's cone expansion or compression. We will use it to discuss the existence of positive ω-periodic solutions of (1.1) in the next section.

3. Main Results

We consider the the existence of positive ω-periodic solutions of (1.1). Let $f \in C(\mathbb{R}\times(0,\infty)\times\mathbb{R})$ satisfy Assumption (F0) and $f(t,x,y)$ be ω-periodic in t. Let C_0 be the constant defined by (2.8) and $I = [0,\omega]$. To be convenient, we introduce the notations

$$\begin{aligned} f_0 &= \liminf_{x\to 0^+} \min_{|y|\le C_0|x|, t\in I} \left(\frac{f(t,x,y)}{x}\right), \\ f^0 &= \limsup_{x\to 0^+} \max_{|y|\le C_0|x|, t\in I} \left(\frac{f(t,x,y)}{x}\right), \\ f_\infty &= \liminf_{x\to +\infty} \min_{|y|\le C_0|x|, t\in I} \left(\frac{f(t,x,y)}{x}\right), \\ f^\infty &= \limsup_{x\to +\infty} \max_{|y|\le C_0|x|, t\in I} \left(\frac{f(t,x,y)}{x}\right). \end{aligned} \tag{3.1}$$

Our main results are as follows.

Theorem 3.1. *Let $f : \mathbb{R}\times(0,\infty)\times\mathbb{R} \to \mathbb{R}$ be continuous and $f(t,x,y)$ be ω-periodic in t. If f satisfies Assumption* (F0) *and the condition*

(F1) $f^0 < 0, f_\infty > 0,$

then (1.1) *has at least one positive ω-periodic solution.*

Theorem 3.2. *Let $f : \mathbb{R}\times(0,\infty)\times\mathbb{R} \to \mathbb{R}$ be continuous and $f(t,x,y)$ be ω-periodic in t. If f satisfies Assumption* (F0) *and the conditions*

(F2) $f_0 > 0, f^\infty < 0,$

then (1.1) *has at least one positive ω-periodic solution.*

Noting that 0 is an eigenvalue of the associated linear eigenvalue problems of (1.1) with periodic boundary condition, if one inequality concerning comparison with 0 in (F1) or (F2) of Theorem 3.1 or Theorem 3.2 is not true, the existence of periodic solution to (1.1) cannot be guaranteed. Hence, the 0 is the optimal value in condition (F1) and (F2).

In Theorem 3.1, the condition (F1) allows $f(t,x,y)$ to have superlinear growth on x and y. For example,

$$f(t,x,y) = x^2 + y^2 - \frac{1}{4}\frac{\pi^2}{\omega^2}\left(2+\sin\frac{\pi t}{\omega}\right)x \tag{3.2}$$

satisfies (F0) with $M = (3/4)(\pi^2/\omega^2)$ and (F1) with $f^0 = -(1/4)(\pi^2/\omega^2)$ and $f_\infty = +\infty$.

In Theorem 3.2, the condition (F2) allows that $f(t,x,y)$ has singularity at $x = 0$. For example,

$$f(t,x,y) = \frac{xy^2 + 2 + \sin(\pi t/\omega)}{x^3} - \frac{\pi^2}{2\omega^2}\, x \tag{3.3}$$

satisfies (F0) with $M = \pi^2/2\omega^2$, and (F2) with $f_0 = +\infty$ and $f^\infty = -\pi^2/2\omega^2$. The existence of periodic solutions for singular ordinary differential equations has been studied by several authors, see [20, 25, 26]. But the equations considered by these authors do not contain derivative term $u'(t)$.

Proof of Theorem 3.1. Choose the working space $E = C^1_\omega(\mathbb{R})$. Let $K \subset C^1_\omega(\mathbb{R})$ be the closed convex cone in $C^1_\omega(\mathbb{R})$ defined by (2.9) and $A : K \setminus \{0\} \to K$ the operator defined by (2.18). Then the positive ω-periodic solution of (1.1) is equivalent to the nontrivial fixed point of A. Let $0 < r < R < +\infty$ and set

$$\Omega_1 = \left\{u \in C^1_\omega(\mathbb{R}) \mid \|u\|_{C^1} < r\right\}, \qquad \Omega_2 = \left\{u \in C^1_\omega(\mathbb{R}) \mid \|u\|_{C^1} < R\right\}. \tag{3.4}$$

We show that the operator A has a fixed point in $K \cap (\Omega_2 \setminus \overline{\Omega}_1)$ by Theorem 2.6 when r is small enough and R large enough.

By $f^0 < 0$ and the definition of f^0, there exist $\varepsilon \in (0, M)$ and $\delta > 0$, such that

$$f(t,x,y) \le -\varepsilon x, \quad t \in [0,\omega], |y| \le C_0, 0 < x \le \delta. \tag{3.5}$$

Let $r \in (0,\delta)$. We now prove that A satisfies the Condition (1) of Theorem 2.6, namely, $\lambda\, Au \neq u$ for every $u \in K \cap \partial\Omega_1$ and $0 < \lambda \le 1$. In fact, if there exist $u_0 \in K \cap \partial\Omega_1$ and $0 < \lambda_0 \le 1$ such that $\lambda_0\, Au_0 = u_0$, then by definition of A and Lemma 2.1, $u_0 \in C^2_\omega(\mathbb{R})$ satisfies the differential equation

$$u_0''(t) + Mu_0(t) = \lambda_0 f_1(t, u_0(t), u_0'(t)), \quad t \in \mathbb{R}. \tag{3.6}$$

Since $u_0 \in K \cap \partial\Omega_1$, by the definitions of K and Ω_1, we have

$$|u_0'(t)| \le C_0 u_0(t), \quad 0 < \sigma\|u_0\|_C \le u_0(t) \le \|u_0\|_{C^1} = r < \delta, \quad t \in \mathbb{R}. \tag{3.7}$$

Hence from (3.5) it follows that

$$f(t, u_0(t), u_0'(t)) \le -\varepsilon u_0(t), \quad t \in \mathbb{R}. \tag{3.8}$$

By this, (3.6), and the definition of f_1 we have

$$u_0''(t) + Mu_0(t) \le \lambda_0(Mu_0(t) - \varepsilon u_0(t)) \le (M - \varepsilon)u_0(t), \quad t \in \mathbb{R}. \tag{3.9}$$

Integrating both sides of this inequality from 0 to ω and using the periodicity of u_0, we obtain that

$$M\int_0^{\omega} u_0(t)dt \le (M-\varepsilon)\int_0^{\omega} u_0(t)dt. \tag{3.10}$$

Since $\int_0^{\omega} u_0(t)dt \ge \omega\ \sigma\|u_0\|_C > 0$, it follows that $M \le M-\varepsilon$, which is a contradiction. Hence the Condition (1) of Theorem 2.6 holds.

On the other hand, since $f_\infty > 0$, by the definition of f_∞, there exist $\varepsilon_1 > 0$ and $H > 0$ such that

$$f(t,x,y) \ge \varepsilon_1 x, \quad t\in[0,\omega], |y| \le C_0 x, x \ge H. \tag{3.11}$$

Define a function $g:(0,\infty)\to\mathbb{R}^+$ by

$$g(x) = \max\left\{\frac{f(t,x,y)+Mx}{x} \mid t\in[0,\omega], |y|\le x\right\}. \tag{3.12}$$

Then $g:(0,\infty)\to\mathbb{R}^+$ is continuous. By (3.5) and Assumption (F0),

$$0 \le g(x) \le \varepsilon + M, \quad 0 < x \le \delta. \tag{3.13}$$

This implies that

$$C_1 := \sup\{xg(x) \mid 0 < x \le H\} < +\infty. \tag{3.14}$$

Hence for every $t\in[0,\omega]$, $0<x\le H$, and $|y|\le C_0x$, we have

$$\begin{aligned} |f(t,x,y)-\varepsilon_1 x| &\le |f(t,x,y)+Mx| + |(M+\varepsilon_1)x| \\ &= (f(t,x,y)+Mx) + (M+\varepsilon_1)x \\ &\le xg(x) + (M+\varepsilon_1)x \\ &\le C_1 + (M+\varepsilon_1)H := C_2. \end{aligned} \tag{3.15}$$

Combining this with (3.11), it follows that

$$f(t,x,y) \ge \varepsilon_1 x - C_2, \quad t\in[0,\omega], |y|\le C_0 x, x>0. \tag{3.16}$$

Choose $e(t)\equiv 1$. Clearly, $e\in K\setminus\{\theta\}$. We show that A satisfies the Condition (2) of Theorem 2.6 if R is large enough, namely, $u - Au \ne \tau e$ for every $u\in K\cap\partial\Omega_2$ and $\tau\ge 0$. In fact, if there exist $u_1\in K\cap\partial\Omega_2$ and $\tau_1\ge 0$ such that $u_1 - Au_1 = \tau_1 e$, since $u_1-\tau_1 e = Au_1$, by definition of A and Lemma 2.1, $u_1\in C^2_\omega(\mathbb{R})$ satisfies the differential equation

$$u_1''(t) + M(u_1(t)-\tau_1) = f_1(t,u_1(t),u_1'(t)), \quad t\in\mathbb{R}. \tag{3.17}$$

From (3.17) and (3.16), it follows that

$$\begin{aligned} u_1''(t) &= f\left(t, u_1(t), u_1'(t)\right) + M\tau_1 \\ &\geq f\left(t, u_1(t), u_1'(t)\right) \geq \varepsilon_1 u_1(t) - C_2, \quad t \in \mathbb{R}. \end{aligned} \tag{3.18}$$

Integrating this inequality on $[0, \omega]$ and using the periodicity of u_1, we get that

$$\int_0^\omega u_1(t)dt \leq \frac{C_2}{\varepsilon_1}. \tag{3.19}$$

Since $u_1 \in K \cap \partial\Omega_2$, by the definition of K, we have

$$u_1(t) \geq \sigma \|u_1\|_C, \quad \left|u_1'(t)\right| \leq C_0 u_1(t), \quad t \in \mathbb{R}. \tag{3.20}$$

By the first inequality of (3.20), we have

$$\int_0^\omega u_1(t)dt \geq \omega\sigma\|u_1\|_C. \tag{3.21}$$

From this and (3.19), it follows that

$$\|u_1\|_C \leq \frac{1}{\omega\sigma}\int_0^\omega u_1(t)dt \leq \frac{C_2}{\omega\sigma\varepsilon_1}. \tag{3.22}$$

By this and the second inequality of (3.20), we have

$$\|u_1\|_{C^1} = \|u_1\|_C + \left\|u_1'\right\|_C \leq \|u_1\|_C + C_0\|u_1\|_C \leq \frac{(1+C_0)C_2}{\omega\sigma\varepsilon_1} := \overline{R}. \tag{3.23}$$

Therefore, choose $R > \max\{\overline{R}, \delta\}$, then A satisfies the Condition (2) of Theorem 2.6.

Now by the first part of Theorem 2.6, A has a fixed point in $K \cap (\Omega_2 \setminus \overline{\Omega}_1)$, which is a positive ω-periodic solution of (1.1). □

Proof of Theorem 3.2. Let $\Omega_1, \Omega_2 \subset C_\omega^1(\mathbb{R})$ be defined by (3.4). We use Theorem 2.6 to prove that the operator A has a fixed point in $K \cap (\Omega_2 \setminus \overline{\Omega}_1)$ if r is small enough and R large enough.

By $f_0 > 0$ and the definition of f_0, there exist $\varepsilon > 0$ and $\delta > 0$, such that

$$f(t, x, y) \geq \varepsilon x, \quad t \in [0, \omega], \left|y\right| \leq C_0 x, 0 < x \leq \delta. \tag{3.24}$$

Let $r \in (0, \delta)$ and $e(t) \equiv 1$. We prove that A satisfies the Condition (3) of Theorem 2.6, namely, $u - Au \neq \tau e$ for every $u \in K \cap \partial\Omega_1$ and $\tau \geq 0$. In fact, if there exist $u_0 \in K \cap \partial\Omega_1$ and $\tau_0 \geq 0$ such that $u_0 - Au_0 = \tau_0 e$, since $u_0 - \tau_0 e = Au_0$, by definition of A and Lemma 2.1, $u_0 \in C_\omega^2(\mathbb{R})$ satisfies the differential equation

$$u_0''(t) + M(u_0(t) - \tau_0) = f_1\left(t, u_0(t), u_0'(t)\right), \quad t \in \mathbb{R}. \tag{3.25}$$

Since $u_0 \in K \cap \partial\Omega_1$, by the definitions of K and Ω_1, u_0 satisfies (3.7). From (3.7), and (3.24) it follows that

$$f\left(t, u_0(t), u_0'(t)\right) \geq \varepsilon u_0(t), \quad t \in \mathbb{R}. \tag{3.26}$$

By this, (3.25), and the definition of f_1, we have

$$u_0''(t) + Mu_0(t) = f_1\left(t, u_0(t), u_0''(t)\right) + M\tau_0 \geq (M + \varepsilon)u_0(t), \quad t \in \mathbb{R}. \tag{3.27}$$

Integrating this inequality on $[0, \omega]$ and using the periodicity of $u_0(t)$, we obtain that

$$M \int_0^\omega u_0(t)dt \geq (M + \varepsilon) \int_0^\omega u_0(t)dt. \tag{3.28}$$

Since $\int_0^\omega u_0(t)dt \geq \omega\sigma\|u_0\|_C > 0$, from this inequality it follows that $M \geq M + \varepsilon$, which is a contradiction. Hence A satisfies the Condition (3) of Theorem 2.6.

Since $f^\infty < 0$, by the definition of f^∞, there exist $\varepsilon_1 \in (0, M)$ and $H > 0$ such that

$$f\left(t, x, y\right) \leq -\varepsilon_1 x, \quad t \in [0, \omega], \left|y\right| \leq C_0 x, x \geq H. \tag{3.29}$$

Choosing $R > \max\{(1 + C_0)H/\sigma, \delta\}$, we show that A satisfies the Condition (4) of Theorem 2.6, namely, $\lambda Au \neq u$ for every $u \in K \cap \partial\Omega_2$ and $0 < \lambda \leq 1$. In fact, if there exist $u_1 \in K \cap \partial\Omega_2$ and $0 < \lambda_1 \leq 1$ such that $\lambda_1 Au_1 = u_1$, then by the definition of A and Lemma 2.1, $u_1 \in C_\omega^2(\Omega)$ satisfies the differential equation

$$u_1''(t) + Mu_1(t) = \lambda_1 f_1\left(t, u_1(t), u_1'(t)\right), \quad t \in \mathbb{R}. \tag{3.30}$$

Since $u_1 \in K \cap \partial\Omega_2$, by the definition of K, u_1 satisfies (3.20). By the second inequality of (3.20), we have

$$\|u_1\|_{C^1} = \|u_1\|_C + \left\|u_1'\right\|_C \leq \|u_1\|_C + C_0\|u_1\|_C = (1 + C_0)\|u_1\|_C. \tag{3.31}$$

Consequently,

$$\|u_1\|_C \geq \frac{1}{(1 + C_0)} \|u_1\|_{C^1}. \tag{3.32}$$

By (3.32) and the first inequality of (3.20), we have

$$u_1(t) \geq \sigma\|u_1\|_C \geq \frac{\sigma}{(1 + C_0)} \|u_1\|_{C^1} = \frac{\sigma}{(1 + C_0)} R > H, \quad t \in \mathbb{R}. \tag{3.33}$$

From this, the second inequality of (3.20) and (3.29), it follows that

$$f\left(t, u_1(t), u_1'(t)\right) \leq -\varepsilon_1 u_1(t), \quad t \in \mathbb{R}. \tag{3.34}$$

By this and (3.30), we have

$$u_1''(t) + Mu_1(t) \le \lambda_1(M\ u_1(t) - \varepsilon_1 u_1(t)) \le (M - \varepsilon_1)u_1(t), \quad t \in \mathbb{R}. \tag{3.35}$$

Integrating this inequality on $[0,\omega]$ and using the periodicity of $u_1(t)$, we obtain that

$$M\int_0^\omega u_1(t)dt \le (M - \varepsilon_1)\int_0^\omega u_1(t)dt. \tag{3.36}$$

Since $\int_0^\omega u_1(t)\ dt \ge \omega\sigma\|u_1\|_C > 0$, from this inequality it follows that $M \le M - \varepsilon_1$, which is a contradiction. This means that A satisfies the Condition (4) of Theorem 2.6.

By the second part of Theorem 2.6, A has a fixed point in $K \cap (\Omega_2 \setminus \overline{\Omega}_1)$, which is a positive ω-periodic solution of (1.1). □

Example 3.3. Consider the second-order differential equation

$$u'' = a_1(t)u + a_2(t)u^2 + a_3(t)\left(u'\right)^2 u, \quad t \in \mathbb{R}, \tag{3.37}$$

where $a_i(t) \in C_\omega(\mathbb{R}),\ i = 1,2,3$. If $-\pi^2/\ \omega^2 < a_1(t) < 0$ and $a_2(t), a_3(t) > 0$ for $t \in [0,\omega]$, then $f(t,x,y) = a_1(t)x + a_2(t)x^2 + a_3xy^2$ satisfies the conditions (F0) and (F1). By Theorem 3.1, (3.37) has at least one positive ω-periodic solution.

Example 3.4. Consider the singular differential equation:

$$u'' = a(t)u + \frac{b(t)u + c(t)(u')^2}{u^2}, \quad t \in \mathbb{R}, \tag{3.38}$$

where $a(t), b(t), c(t) \in C_\omega(\mathbb{R})$. If $-\pi^2/\omega^2 < a(t) < 0$ and $b(t), c(t) > 0$ for $t \in [0,\omega]$, then $f(t,x,y) = a(t)x + (b(t)x + c(t)y^2)/x^2$ satisfies the conditions (F0) and (F2). By Theorem 3.2, the (3.38) has a positive ω-periodic solution.

4. Remarks

Our discussion on the existence of the positive ω-periodic solutions to (1.1) is applicable to the following ordinary differential equation:

$$-u''(t) = f\left(t, u(t), u'(t)\right), \quad t \in \mathbb{R}, \tag{4.1}$$

where the nonlinearity $f : \mathbb{R} \times (0,\infty) \times \mathbb{R} \to \mathbb{R}$ is continuous and $f(t,x,y)$ is ω-periodic in t. For (4.1), we need the following assumption.

(F0)* There exists $M > 0$ such that

$$f\left(t,x,y\right) + Mx \ge 0, \quad x > 0,\ t,y \in \mathbb{R}. \tag{4.2}$$

Similarly to Lemma 2.1, we have the following conclusion.

Lemma 4.1. *Let $M > 0$ be a constant. Then for every $h \in C_\omega(\mathbb{R})$, the linear second order differential equation*

$$-u''(t) + Mu(t) = h(t), \quad t \in \mathbb{R}, \tag{4.3}$$

has a unique ω-periodic solution $u(t)$, which is given by

$$u(t) = \int_{t-\omega}^{t} V(t-s)h(s)ds \quad t \in \mathbb{R}, \tag{4.4}$$

where $V(t)$ is the unique solution of the linear second-order boundary value problem

$$\begin{gathered} -u''(t) + Mu(t) = 0, \quad 0 \le t \le \omega, \\ u(0) - u(\omega) = 0, \qquad u'(0) - u'(\omega) = -1, \end{gathered} \tag{4.5}$$

which is explicitly given by

$$V(t) = \frac{\cosh \beta(t - \omega/2)}{2\beta \sinh(\beta\omega/2)}, \quad 0 \le t \le \omega, \tag{4.6}$$

with $\beta = \sqrt{M}$.

Since

$$\begin{gathered} \overline{V} := \max_{0\le t\le\omega} V(t) = \frac{\cosh(\beta\omega/2)}{2\beta\sinh(\beta\omega/2)}, \qquad \underline{V} := \min_{0\le t\le\omega} V(t) = \frac{1}{2\beta\sinh(\beta\omega/2)}, \\ \overline{V}_1 := \max_{0\le t\le\omega} \left|V'(t)\right| = \max_{0\le t\le\omega} \frac{\left|\sinh\beta(t-\omega/2)\right|}{2\sinh(\beta\omega/2)} = \frac{1}{2}, \end{gathered} \tag{4.7}$$

we renew to define σ and C_0 by

$$\sigma = \frac{\underline{V}}{\overline{V}} = \frac{1}{\cosh(\beta\omega/2)}, \qquad C_0 = \frac{\overline{V}_1}{\underline{V}} = \beta \sinh\frac{\beta\omega}{2}. \tag{4.8}$$

Now, using the similar arguments to Theorems 3.1 and 3.2, we can obtain the following results.

Theorem 4.2. *Let $f : \mathbb{R} \times (0,\infty) \times \mathbb{R} \to \mathbb{R}$ be continuous and $f(t,x,y)$ be ω-periodic in t. If f satisfies Assumption* (F0)* *and the condition*

(F1) $f^0 < 0$, $f_\infty > 0$,

then (4.1) *has at least one positive ω-periodic solution.*

Theorem 4.3. *Let $f : \mathbb{R} \times (0, \infty) \times \mathbb{R} \to \mathbb{R}$ be continuous and $f(t, x, y)$ be ω-periodic in t. If f satisfies Assumption* (F0)* *and the conditions*

(F2) $f_0 > 0$, $f^\infty < 0$,

then (4.1) *has at least one positive ω-periodic solution.*

Theorems 4.2 and 4.3 improve and extend some results in References [18, 19, 22].

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