Research Article

Progressive Gelfand-Shilov Spaces and Wavelet Transforms

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We discuss progressive Gelfand-Shilov spaces consisting of analytic signals with almost exponential decay in time and frequency variables. It is shown that such signals enjoy an additional localization property. We define wavelet transform and inverse wavelet transform in (progressive) Gelfand-Shilov spaces and study their continuity properties. It is shown that with a slightly faster decay in domain we may control the decay of the wavelet transform independently in each variable.

1. Introduction

In this paper, we study a class of Gelfand-Shilov spaces and their closed subspaces of analytic signals which are almost exponentially localized in time and frequency variables. Since the support of Fourier transform of such signals is in $[0, \infty)$, those subspaces are called progressive Gelfand-Shilov spaces, see Definition 2.1. We use the well-known characterization of Gelfand-Shilov spaces, Theorem 2.6. Note that in contrast to the proofs given in [1–4] Theorem 2.6 can be proved without arguments based on the Fourier transform invariance properties, which is of interest when dealing with progressive Gelfand-Shilov spaces, see Remark 2.7. In particular, we show that progressive Gelfand-Shilov spaces enjoy an additional localization property, Proposition 2.8.

Note that the description of progressive Gelfand-Shilov type spaces given by almost exponential decay in time and frequency variables does not contain explicit regularity conditions.

The study of progressive Gelfand-Shilov spaces is motivated by the fact that they form closed subspaces of the Hardy space $H^2(\mathbb{R})$ which consists of functions in $L^2(\mathbb{R})$ whose

Fourier transforms are supported by the right half-plane. It is well known that Hardy spaces $H^p(\mathbb{R})$, $1 \leq p \leq \infty$, play fundamental role in signal processing, cf. [5–7]. Analytic signal F , associated to a real-valued signal $f \in L^2(\mathbb{R})$ by $F = f + \sqrt{-1}H(f)$, where $H(f)$ is the Hilbert transform of f , belongs to $H^2(\mathbb{R})$, which is the main motivation for the study of analytic signals in the framework of the Hardy space, in particular through the wavelet analysis, for example [5, 7–9].

For the study of (micro) local properties of analytic signals and of the corresponding dual spaces, analyzing wavelets should satisfy certain regularity and decay conditions. It is usually assumed that such wavelets have vanishing moments of certain order. In order to describe local properties of polynomial decay in time-frequency plane, it is sufficient to perform wavelet analysis in the Schwartz space of rapidly decreasing functions [9–11]. Analytic signals, however, might have better localization properties which cannot be captured within that framework. In the present paper, we show how progressive Gelfand-Shilov spaces can be used in this context. Our approach is a starting point for the study of wavelet transforms of (ultra) distributions of almost exponential growth at infinity which are not tempered distributions. This will be the subject of a separate paper. We remark that progressive functions cannot be exponentially localized simultaneously in time and in frequency domain.

In time-frequency analysis, weight functions are commonly used to quantify growth and decay conditions, for example [12]. From that point of view, it is natural to assume that results in certain spaces of ultra-distributions follow directly from the counterparts in the space of tempered distributions, by replacing polynomial type weights of the form $\langle \cdot \rangle^s = (1 + |\cdot|^2)^{s/2}$, $s > 0$, with subexponential weights of the form $e^{a|\cdot|^b}$, $a > 0$, $0 < b < 1$. Although this is true in certain situations, and some results of the paper can be formulated to hold for both polynomial and subexponential weights, this cannot be done in full generality. The reason for this is different behavior of weights and their inverse functions with respect to convolution, translation, and dilation. Even in polynomial case, there is a difference between the power of the Haar module $w_s(x, \xi) = w_s(\xi) = \langle \xi \rangle^s$, $s > 0$, and a weight of the form $w_{s_1, s_2}(x, \xi) = \langle x \rangle^{s_1} \langle \xi \rangle^{s_2}$, $s_1, s_2 > 0$, $(x, \xi) \in \mathbb{R} \times (0, \infty)$. The Haar module type weights are submultiplicative with respect to the action of the “ $ax + b$ ”-group and properties of the wavelet transform can be derived from the general theory of coorbit spaces, as it is done in [13–15]. This is not the case for $w_{s_1, s_2}(x, \xi)$. Here, we give explicit calculations for subexponential weights and give explanations concerning more general situations whenever convenient.

Gelfand-Shilov spaces were introduced in [2] (and called there the spaces of type S) to study the uniqueness of the Cauchy problems of partial differential equations and since then they were observed both from theoretical (e.g. [1, 3, 4]) and applied point of view (e.g. [16–18]). For example, in [18] it is shown that these spaces can be used to construct a noncommutative quantum field theory.

Our results can be compared to [19] where one can find certain global type results. The analyzing wavelets in [19] may not satisfy the admissibility condition (see (4.24)) which is connected to the study of local regularity properties of analyzing functions, while our results describe both local and global behavior of wavelet transforms.

The paper is organized as follows. In Section 2, we recall the characterization of Gelfand-Shilov spaces, Theorem 2.6, and we introduce the progressive Gelfand-Shilov spaces and show a localization property of its elements, Proposition 2.8. In Section 3, we define the wavelet transform and study its decay properties, Propositions 3.2 and 3.3. This, together

with the results from Section 2, give a continuity result, Theorem 3.5. Section 4 is devoted to the inverse wavelet transform, whose decay properties are studied in Propositions 4.1 and 4.2, and a continuity result and the inversion formula are given in Theorem 4.3. In the last section, we give a reformulation of results from Sections 3 and 4 under the assumption that both the analyzing wavelet and the analyzed function have slightly faster decay. This opens the possibility to control the decay of the wavelet transform in each variable separately. We believe that those results are of independent interest.

In order to simplify the exposition, we observe one-dimensional case only. Extension to higher dimensions is straightforward. Throughout the paper, we use standard notation. For example, $(L^p(\mathbb{R}), \|\cdot\|_p)$, $1 \leq p \leq \infty$, are Lebesgue spaces, $\langle f, g \rangle$ denotes the dual pairing of f and g , and the Fourier transform of $f \in L^1(\mathbb{R})$ is normalized to be

$$\mathcal{F}f(\xi) = \widehat{f}(\xi) = \int_{\mathbb{R}} e^{-2\pi i x \xi} f(x) dx, \quad \xi \in \mathbb{R}. \quad (1.1)$$

By D_a we denote the dilation $D_a(\cdot) = (1/a)f(\cdot/a)$, $a \in \mathbb{R}_+ = (0, \infty)$. We adopt the following notation used in the study of pseudo-differential operators (cf. [17, 20]): $\langle \cdot \rangle = (1 + |\cdot|^2)^{1/2}$.

For reader's convenience, below we give the notation for spaces of test functions introduced in this paper and refer to corresponding definitions. Progressive Gelfand-Shilov spaces $S_v^{\mu,+}(\mathbb{R})$ are introduced in Definition 2.1. The range of the wavelet transform over such spaces is contained in $S_v^{\mu}(\mathbb{R} \times \mathbb{R}_+)$, see Definition 3.4, and in $Z_v^{\mu}(\mathbb{R} \times \mathbb{R}_+)$, see Definition 5.1, the latter being introduced for the analysis of decay of wavelet transform with respect to variables a and b separately, for example, Section 5.

2. Progressive Gelfand-Shilov Spaces

Let h, ν , and μ be positive constants such that $\nu + \mu \geq 1$. By $S_{\nu,h}^{\mu}(\mathbb{R})$ we denote the Banach space which consists of all $f \in C^{\infty}(\mathbb{R})$ such that the norm

$$\|f\|_{S_{\nu,h}^{\mu}} = \sup_{\alpha, \beta \in \mathbb{N}_0} \frac{\|x^{\alpha} f^{(\beta)}(x)\|_{L^{\infty}}}{h^{\alpha+\beta} \alpha!^{\nu} \beta!^{\mu}} \quad (2.1)$$

is finite.

The Gelfand-Shilov space $S_{\nu}^{\mu}(\mathbb{R})$ is given by

$$S_{\nu}^{\mu}(\mathbb{R}) = \bigcup_{h>0} S_{\nu,h}^{\mu}(\mathbb{R}), \quad (2.2)$$

and its topology is given by the inductive limit:

$$S_{\nu}^{\mu}(\mathbb{R}) = \text{ind} \lim_{h>0} S_{\nu,h}^{\mu}(\mathbb{R}). \quad (2.3)$$

Definition 2.1. Let h, ν , and μ , be positive constants such that $\nu + \mu \geq 1$. The space $S_{\nu,h}^{\mu,+}(\mathbb{R})$ consists of all $f \in C^{\infty}(\mathbb{R})$ such that $\text{supp } \widehat{f} \in [0, \infty)$ and such that (2.1) is finite.

The progressive Gelfand-Shilov space $S_v^{\mu,+}(\mathbb{R})$ is given by

$$S_v^{\mu,+}(\mathbb{R}) = \bigcup_{h>0} S_{v,h}^{\mu,+}(\mathbb{R}), \quad (2.4)$$

and its topology is given by the inductive limit:

$$S_v^{\mu,+}(\mathbb{R}) = \operatorname{ind} \lim_{h>0} S_{v,h}^{\mu,+}(\mathbb{R}). \quad (2.5)$$

Obviously, $S_v^{\mu,+}(\mathbb{R})$ is a closed subspace of $S_v^{\mu}(\mathbb{R})$.

Remark 2.2. In the study of ultradistributions, it is of interest to observe projective limits of Banach spaces of test functions, apart from the inductive limits. Results of the present paper can be reformulated for progressive Gelfand-Shilov type spaces defined as projective limits of the spaces $S_{v,h}^{\mu,+}(\mathbb{R})$. For the projective limits of $S_{v,h}^{\mu}(\mathbb{R})$, $h > 0$, we refer to [21, 22].

Note also that the decay of $\|x^{\alpha} f^{(\beta)}(x)\|_{L^{\infty}}$ can be controlled by some other choice of sequences $\{M_{\alpha}\}$, $\{N_{\beta}\}$, instead of the Gevrey sequences $M_{\alpha} = \alpha!^{\nu}$ and $N_{\beta} = \beta!^{\mu}$ used in (2.1). We refer to [1, 21–23] for the usual conditions on $\{M_{\alpha}\}$ and $\{N_{\beta}\}$.

For the sake of the clarity of the exposition, in this paper, we observe the inductive limits and the Gevrey sequences $\alpha!^{\nu}$ and $\beta!^{\mu}$, which is not an essential restriction.

It can be shown that multiplication by x and differentiation are continuous operations in $S_v^{\mu,+}(\mathbb{R})$ and that the same is true for translation and dilation by a positive factor, for example [2, Chapter IV]. On the other hand, $S_v^{\mu,+}(\mathbb{R})$ is not closed under modulations. This implies that Short-time Fourier transform, an important tool in local time-frequency analysis (see [24] for the definition and basic properties), is not appropriate for the analysis of $S_v^{\mu,+}(\mathbb{R})$. For that reason, we cannot use the elegant arguments from for example [3, 4] in our analysis.

The following proposition can be used in the proof of Theorem 2.6, and it is of independent interest.

Proposition 2.3. *Let $s > 0$ and $f \in C^{\infty}(\mathbb{R})$. Then, the following conditions are equivalent.*

(a) *There exists $h > 0$ such that*

$$\sup_{x \in \mathbb{R}} |x^{\alpha} f(x)| \leq Ch^{\alpha} \alpha!^s, \quad \forall \alpha \in \mathbb{N}_0, \quad (2.6)$$

for some $C > 0$.

(b) *There exists $k > 0$ such that*

$$\sup_{x \in \mathbb{R}} e^{k|x|^{1/s}} |f(x)| < \infty. \quad (2.7)$$

The proof is omitted, since it can be found in [17, Proposition 6.1.5.], see also [2, Chapter IV]. Alternatively, Proposition 2.3 can be proved by the use of the following auxiliary result, see [25].

Lemma 2.4. *Let $t > 0$ and $\varepsilon > 0$. There exist $C = C(t, \varepsilon) > 0$ such that*

$$C^{-1} e^{(t-\varepsilon)\eta^{1/t}} \leq \sum_{j=0}^{\infty} \frac{\eta^j}{(j!)^t} \leq C e^{(t+\varepsilon)\eta^{1/t}}, \quad \forall \eta \geq 0. \quad (2.8)$$

Remark 2.5. When $s > 0$ and $f \in C^\infty(\mathbb{R})$ by Proposition 2.3, it directly follows that the following conditions are equivalent.

(a) There exists $h > 0$ such that

$$\sup_{\omega > 0} \left| \left(\omega + \frac{1}{\omega} \right)^\alpha \widehat{f}(\omega) \right| \leq Ch^\alpha \alpha!^s, \quad \forall \alpha \in \mathbb{N}_0, \quad (2.9)$$

for some $C > 0$.

(b) There exists $k > 0$ such that

$$\sup_{\omega > 0} e^{k(\omega+1/\omega)^{1/s}} \left| \widehat{f}(\omega) \right| < \infty. \quad (2.10)$$

Next, we recall the characterization of Gelfand-Shilov spaces.

Theorem 2.6. *Let ν and μ be positive constants such that $\nu + \mu \geq 1$ and let $f \in C^\infty(\mathbb{R})$. Then, the following conditions are equivalent,*

(a) $f \in S_\nu^\mu(\mathbb{R})$.

(b) There exist $h > 0$ and $C > 0$ (which depend only on f) such that

$$\sup_x |x^\alpha f(x)| \leq Ch^\alpha \alpha!^\nu, \quad \forall \alpha \in \mathbb{N}_0 \quad \sup_x |f^{(\beta)}(x)| \leq Ch^\beta \beta!^\mu, \quad \forall \beta \in \mathbb{N}_0. \quad (2.11)$$

(c) There exist $h > 0$ and $C > 0$ (which depend only on f) such that

$$\sup_x |x^\alpha f(x)| \leq Ch^\alpha \alpha!^\nu, \quad \forall \alpha \in \mathbb{N}_0 \quad \sup_\omega |\omega^\beta \widehat{f}(\omega)| \leq Ch^\beta \beta!^\mu, \quad \forall \beta \in \mathbb{N}_0. \quad (2.12)$$

(d) There exist $h > 0$ and $C > 0$ (which depend only on f) such that

$$\sup_\omega |\omega^\alpha \widehat{f}^{(\beta)}(\omega)| \leq Ch^{\alpha+\beta} \alpha!^\mu \beta!^\nu, \quad \forall \beta \in \mathbb{N}_0. \quad (2.13)$$

(e) There exist $h > 0$ and $C > 0$ (which depend only on f) such that

$$\sup_x e^{h|x|^{1/\nu}} |f(x)| \leq C, \quad \sup_\omega e^{h|\omega|^{1/\mu}} |\widehat{f}(\omega)| \leq C. \quad (2.14)$$

This theorem is well known, so we omits the proof. It can be proved as in [1, 3, 4] where arguments based on Fourier transform invariance of $S_v^\mu(\mathbb{R})$ ($\mathcal{F}S_v^\mu(\mathbb{R}) = S_v^\mu(\mathbb{R})$) are used. Another proof which follows from direct computations is given in [17, Theorem 6.1.6.], see also [21, 22].

Remark 2.7. By Theorem 2.6 immediately follows the characterization of progressive Gelfand-Shilov spaces $S_v^{\mu,+}(\mathbb{R})$ where the supremum with respect to ω in Theorem 2.6(c), (d), and (e) is taken over $\omega > 0$ instead of $\omega \in \mathbb{R}$.

Moreover, elements in $S_v^{\mu,+}(\mathbb{R})$ satisfy an additional localization property, which is sometimes called "strip localization." We refer to [9] for the definition of polynomial strip localized progressive functions. Let us show that elements from $S_v^{\mu,+}(\mathbb{R})$ are almost exponentially strip localized.

Proposition 2.8. *Let v and μ be positive constants such that $v + \mu \geq 1$ and let $f \in S_v^\mu(\mathbb{R})$ satisfy $\text{supp } \hat{f} \in [0, \infty)$, that is, $f \in S_v^{\mu,+}(\mathbb{R})$. Then, $\sup_{\omega>0} e^{h(\omega+1/\omega)^{1/s}} |\hat{f}(\omega)| < \infty$, where $s = \max\{\mu, v\}$.*

Proof. Note that $f \in S_v^{\mu,+}(\mathbb{R})$ implies $\hat{f}^{(n)}(0) = 0, n \in \mathbb{N}_0$, that is,

$$\int_{\mathbb{R}} x^n f(x) dx = 0, \quad n \in \mathbb{N}_0, \quad (2.15)$$

and $\hat{f}(\omega) = o(\omega^n)$ when $\omega \rightarrow 0$, for every $n \in \mathbb{N}_0$. By Taylor's formula, it follows that

$$\begin{aligned} |\hat{f}(\omega)| &= \left| \sum_{m=0}^{n-1} \frac{\hat{f}^{(m)}(0)}{m!} \omega^m + \frac{\omega^n}{n!} n \int_0^1 (1-\theta)^{n-1} \hat{f}^{(n)}(\theta\omega) d\theta \right| \\ &\leq \frac{\omega^n}{n!} \sup_{\omega>0} |\hat{f}^{(n)}(\omega)| n \int_0^1 (1-\theta)^{n-1} d\theta \\ &\leq \omega^n C h^n n!^s, \end{aligned} \quad (2.16)$$

where we have used Theorem 2.6(d) and $n \int_0^1 (1-\theta)^{n-1} d\theta = 1$, for example [26, page 59]. Therefore,

$$\left| \frac{1}{\omega^n} \hat{f}(\omega) \right| \leq C h^n n!^s, \quad n \in \mathbb{N}_0, \quad \omega > 0, \quad (2.17)$$

which, together with Theorem 2.6(c), implies that there exist $C > 0$ and $h > 0$ such that

$$\begin{aligned} \sup_{\omega>0} \left| \left(\omega + \frac{1}{\omega} \right)^\alpha \hat{f}(\omega) \right| &\leq 2^\alpha \left(\sup_{\omega>0} \frac{1}{\omega^\alpha} |\hat{f}(\omega)| + \sup_{\omega>0} \omega^\alpha |\hat{f}(\omega)| \right) \\ &\leq C h^\alpha \alpha!^s, \quad \forall \alpha \in \mathbb{N}_0. \end{aligned} \quad (2.18)$$

By Remark 2.5 there exist $C > 0$ and $h > 0$ such that

$$\sup_{\omega > 0} e^{h(\omega+1/\omega)^{1/s}} |\hat{f}(\omega)| \leq C \quad (2.19)$$

as claimed. $\square$

3. Wavelet Transform

In this section, we define the wavelet transform and study its decay and continuity properties.

The wavelet transform $W(f, g) = W_g f : L^2(\mathbb{R}) \times L^2(\mathbb{R}) \mapsto L^2(\mathbb{R} \times \mathbb{R}_+)$ of $f \in L^2(\mathbb{R})$ with respect to the wavelet $g \in L^2(\mathbb{R})$ is given by

$$W_g f(b, a) = \left\langle f(x), \frac{1}{a} \bar{g}\left(\frac{x-b}{a}\right) \right\rangle = \int_{\mathbb{R}} f(x) \frac{1}{a} \bar{g}\left(\frac{x-b}{a}\right) dx, \quad (b, a) \in \mathbb{R} \times \mathbb{R}_+, \quad (3.1)$$

and can be extended to any dual pair of spaces provided that duality is meaningful.

In particular, $W_g f$ is well defined when $f, g \in S_v^\mu(\mathbb{R})$. Theorem 2.6 (e) suggests to observe the behavior of f and g in the configuration and in the frequency domain separately in the study of $W_g f$ when $f, g \in S_v^\mu(\mathbb{R})$.

We introduce following notation for the subexponential weight function:

$$m_{h,v}(x) := e^{h|x|^{1/v}}, \quad x \in \mathbb{R}, \quad (3.2)$$

where $h, v \in \mathbb{R}$.

When $h > 0$ and $v > 1$ the weight $m_{h,v}$ is radial, submultiplicative and subconvolutive. Recall, a nonnegative function v is called subconvolutive if $v^{-1} \in L^1(\mathbb{R})$ and

$$v^{-1} * v^{-1} \leq C v^{-1}, \quad (3.3)$$

for some constant $C > 0$.

When $v > 1$, the following lemma follows from [12, Lemma 7.1], see also [27], where it is proved for a general subconvolutive weight. We give here the proof when the weight is subexponential, since similar arguments will be used later on. Moreover, we allow $v = 1$ here.

Lemma 3.1. *Let there be given $v \geq 1$, $h > 0$ and functions f and g such that*

$$|f(x)| \leq C m_{h,v}^{-1}(x), \quad |g(x)| \leq C m_{h,v}^{-1}(x), \quad x \in \mathbb{R} \quad (3.4)$$

for some positive constant C . Then,

$$(f * g)(x) = \int_{\mathbb{R}} f(y) g(x-y) dy \leq \tilde{C} m_{h,v}^{-1}(x), \quad x \in \mathbb{R}, \quad (3.5)$$

for some $\tilde{C} > 0$ and for every $\tilde{h}$ such that $h \leq \tilde{h} > 0$.

Proof. By the elementary inequality $|y|^{1/\nu} \leq |x - y|^{1/\nu} + |x|^{1/\nu}$, $x, y \in \mathbb{R}$, $\nu \geq 1$, it follows that for a given $h > 0$ we have

$$h|x - y|^{1/\nu} \geq \tilde{h}(|y|^{1/\nu} - |x|^{1/\nu}), \quad x, y \in \mathbb{R}, \quad \forall \tilde{h} \text{ such that } h \geq \tilde{h} > 0. \quad (3.6)$$

Therefore,

$$m_{h,\nu}(x - y) \geq m_{\tilde{h},\nu}(y) m_{h,\nu}^{-1}(x). \quad (3.7)$$

By the assumption (3.4), it follows that

$$\begin{aligned} (f * g)(y) &\leq \tilde{C} \int_{\mathbb{R}} m_{h,\nu}^{-1}(x) m_{h,\nu}^{-1}(x - y) dx \leq \tilde{C} \int_{\mathbb{R}} m_{h,\nu}^{-1}(x) \frac{m_{\tilde{h},\nu}(x)}{m_{\tilde{h},\nu}(y)} dx \\ &\leq \tilde{C} \int_{\mathbb{R}} m_{h-\tilde{h},\nu}^{-1}(x) m_{h,\nu}^{-1}(y) dx \leq \tilde{C} m_{h,\nu}^{-1}(y), \end{aligned} \quad (3.8)$$

where we choose $\tilde{h}$ such that $h > \tilde{h} > 0$. The proof is complete. $\square$

Proposition 3.2. *Let there be given $\nu \geq 1$, $h > 0$ and let f and g satisfy (3.4) for some $C > 0$. Then, there exist $\tilde{C} > 0$ such that*

$$|W_g f(b, a)| \leq \tilde{C} m_{h,\nu}^{-1}\left(\frac{b}{1+a}\right), \quad (b, a) \in \mathbb{R} \times \mathbb{R}_+. \quad (3.9)$$

Proof. When $a = 1$ the so-called “voice transform” $W_g f(b, 1)$ is just the convolution between f and g , so by Lemma 3.1 we have

$$|W_g f(b, 1)| \leq C m_{h,\nu}^{-1}(b), \quad (3.10)$$

for every $\tilde{h}$ such that $h > \tilde{h} > 0$. In particular, when $\tilde{h} = h(1+a)^{-1/\nu} = h2^{-1/\nu}$, we obtain

$$|W_g f(b, a)| \leq C m_{h,\nu}^{-1}\left(\frac{b}{1+a}\right), \quad (3.11)$$

for some $C > 0$. If $a > 1$, then

$$\begin{aligned} |W_g f(b, a)| &\leq \frac{\tilde{C}}{a} \int_{\mathbb{R}} m_{h,\nu}^{-1}(x) m_{h,\nu}^{-1}\left(\frac{x-b}{a}\right) dx \\ &\leq \frac{\tilde{C}}{a} \int_{\mathbb{R}} m_{h,\nu}^{-1}(x) \frac{m_{h,\nu}(x/a)}{m_{h,\nu}(b/a)} dx = \frac{\tilde{C}}{a} m_{h,\nu}^{-1}\left(\frac{b}{a}\right) \int_{\mathbb{R}} \frac{m_{h,\nu}(x/a)}{m_{h,\nu}(x)} dx \\ &\leq \frac{\tilde{C}}{a} m_{h,\nu}^{-1}\left(\frac{b}{1+a}\right) \leq \tilde{C} m_{h,\nu}^{-1}\left(\frac{b}{1+a}\right), \quad (b, a) \in \mathbb{R} \times \mathbb{R}_+. \end{aligned} \quad (3.12)$$

When $a \in (0, 1)$, by the symmetry property of the wavelet transform:

$$W_g f(b, a) = \frac{1}{a} W_{\bar{f}} \bar{g}\left(-\frac{b}{a}, \frac{1}{a}\right), \quad (b, a) \in \mathbb{R} \times \mathbb{R}_+ \quad (3.13)$$

(cf. [9, (2.0.1)]) and by the first part of the proof we have:

$$|W_g f(b, a)| \leq \frac{\tilde{C}}{a} \frac{1}{1/a} \frac{1}{e^{h|-b/a/1+1/a|^{1/v}}} = \tilde{C} m_{h,v}^{-1}\left(\frac{b}{1+a}\right), \quad (b, a) \in \mathbb{R} \times \mathbb{R}_+. \quad (3.14)$$

The proof is complete. $\square$

Proposition 3.2 can be compared to [9], where an extra factor $(1+a)^{-1}$ appears, and where the different technique is used for the proof. Note that (sub)exponential growth of $m_{h,v}$ leads to the convergent integral,

$$\int_{\mathbb{R}} \frac{m_{h,v}(x/a)}{m_{h,v}(x)} dx, \quad (3.15)$$

whereas the same type of estimate for a weight of polynomial growth of the form $\langle x \rangle^s$ gives a divergent integral:

$$\int_{\mathbb{R}} \left\langle \frac{x}{a} \right\rangle^s \langle x \rangle^{-s} dx. \quad (3.16)$$

Next, we observe almost exponential decay in the frequency domain and refer to [9, Theorem 12.0.1.] for its polynomial decay counterpart.

Proposition 3.3. *Let there be given $\mu, h > 0$ and let f and g satisfy*

$$|\hat{f}(\omega)| \leq C m_{h,\mu}^{-1}\left(\omega + \frac{1}{\omega}\right), \quad |\hat{g}(\omega)| \leq C m_{h,\mu}^{-1}\left(\omega + \frac{1}{\omega}\right), \quad \omega > 0, \quad (3.17)$$

for some positive constant $C > 0$. Then, there exist $\tilde{h} > 0$ and $\tilde{C} > 0$ such that

$$|W_g f(b, a)| \leq \tilde{C} m_{\tilde{h},\mu}^{-1}\left(a + \frac{1}{a}\right), \quad (b, a) \in \mathbb{R} \times \mathbb{R}_+. \quad (3.18)$$

Proof. By the Parseval's formula, it follows that

$$W_g f(b, a) = \int_{\mathbb{R}} \hat{f}(\omega) e^{ib\omega} \overline{\hat{g}(a\omega)} d\omega, \quad (b, a) \in \mathbb{R} \times \mathbb{R}_+. \quad (3.19)$$

The change of variables $a\omega = \omega'$ gives

$$\begin{aligned} |W_g f(b, a)| &\leq \tilde{C} \int_0^\infty m_{h,\mu}^{-1}\left(a\omega + \frac{1}{a\omega}\right) m_{h,\mu}^{-1}\left(\omega + \frac{1}{\omega}\right) d\omega \\ &= \frac{\tilde{C}}{a} \int_0^\infty m_{h,\mu}^{-1}\left(\omega' + \frac{1}{\omega'}\right) m_{h,\mu}^{-1}\left(\frac{\omega'}{a} + \frac{a}{\omega'}\right) d\omega'. \end{aligned} \quad (3.20)$$

When $a > 1$, we have

$$|W_g f(b, a)| \leq \tilde{C} \int_0^\infty m_{h,\mu}^{-1}\left(\omega' + \frac{1}{\omega'}\right) m_{h,\mu}^{-1}\left(a'\omega' + \frac{1}{a'\omega'}\right) d\omega', \quad (3.21)$$

where $1/a = a'$. Therefore, it is enough to estimate $|W_g f(b, a)|$ when $0 < a \leq 1$.

We split $W_g f$ in three parts:

$$\begin{aligned} |W_g f(b, a)| &\leq \tilde{C} \int_0^1 m_{h,\mu}^{-1}\left(a\omega + \frac{1}{a\omega}\right) m_{h,\mu}^{-1}\left(\omega + \frac{1}{\omega}\right) d\omega \\ &\quad + \tilde{C} \int_1^{1/a^2} m_{h,\mu}^{-1}\left(a\omega + \frac{1}{a\omega}\right) m_{h,\mu}^{-1}\left(\omega + \frac{1}{\omega}\right) d\omega \\ &\quad + \tilde{C} \int_{1/a^2}^\infty m_{h,\mu}^{-1}\left(a\omega + \frac{1}{a\omega}\right) m_{h,\mu}^{-1}\left(\omega + \frac{1}{\omega}\right) d\omega = I_1 + I_2 + I_3. \end{aligned} \quad (3.22)$$

When $0 < a \leq 1$ and $\omega \in (0, 1)$, we have $1/a < a\omega + 1/a\omega$, wherefrom

$$I_1 \leq \tilde{C} m_{h,\mu}^{-1}\left(\frac{1}{a}\right) \int_0^1 m_{h,\mu}^{-1}\left(\omega + \frac{1}{\omega}\right) d\omega \leq \tilde{C} m_{h,\mu}^{-1}\left(\frac{1}{a}\right). \quad (3.23)$$

Since

$$a + \frac{1}{a} \leq \frac{2}{a} \text{ when } a \in (0, 1], \quad (3.24)$$

it follows that

$$I_1 \leq \tilde{C} m_{h,\mu}^{-1}\left(a + \frac{1}{a}\right), \text{ where } \tilde{h} = \frac{h}{2^{1/\mu}}. \quad (3.25)$$

Next, we make the change of variables: $1/\omega = \omega'$ in I_2 and I_3 and then rename ω' into ω .

By the mean value theorem,

$$\begin{aligned}
 I_2 &= \tilde{C} \int_{a^2}^1 m_{h,\mu}^{-1} \left(\omega + \frac{1}{\omega} \right) m_{h,\mu}^{-1} \left(\frac{a}{\omega} + \frac{\omega}{a} \right) \frac{d\omega}{\omega^2} \\
 &= \tilde{C} m_{h,\mu}^{-1} \left(\frac{a}{\theta} + \frac{\theta}{a} \right) \int_{a^2}^1 m_{h,\mu}^{-1} \left(\omega + \frac{1}{\omega} \right) \frac{d\omega}{\omega^2} \\
 &\leq \tilde{C} m_{h,\mu}^{-1} \left(\frac{\theta}{a} \right) \int_0^1 m_{h,\mu}^{-1} \left(\omega + \frac{1}{\omega} \right) \frac{d\omega}{\omega^2} \\
 &\leq \tilde{C} m_{h,\mu}^{-1} \left(\frac{\theta}{a} \right) \leq \tilde{C} m_{h,\mu}^{-1} \left(a + \frac{1}{a} \right),
 \end{aligned} \tag{3.26}$$

where $\theta \in (a^2, 1) \subseteq (0, 1)$, $\tilde{h} = (\theta/2)^{1/\mu}$ and we have used inequalities $\theta/a < a/\theta + \theta/a$ and (3.24).

It remains to estimate I_3 . We have

$$\begin{aligned}
 I_3 &= \tilde{C} \int_0^{a^2} m_{h,\mu}^{-1} \left(\omega + \frac{1}{\omega} \right) m_{h,\mu}^{-1} \left(\frac{a}{\omega} + \frac{\omega}{a} \right) \frac{d\omega}{\omega^2} \\
 &\leq \tilde{C} \int_0^{a^2} m_{h,\mu}^{-1} \left(\omega + \frac{1}{\omega} \right) m_{h,\mu}^{-1} \left(\frac{1}{a} \right) \frac{d\omega}{\omega^2} \\
 &\leq \tilde{C} m_{h,\mu}^{-1} \left(\frac{1}{a} \right) \int_0^1 m_{h,\mu}^{-1} \left(\omega + \frac{1}{\omega} \right) \frac{d\omega}{\omega^2} \\
 &\leq \tilde{C} m_{h,\mu}^{-1} \left(\frac{1}{a} \right) \leq \tilde{C} m_{h,\mu}^{-1} \left(a + \frac{1}{a} \right),
 \end{aligned} \tag{3.27}$$

where $\tilde{h} = h/2^{1/\mu}$ and we have used inequalities $1/a < a/\omega + \omega/a$ and (3.24).

This completes the proof. $\square$

We end the section with a continuity result when $f, g \in S_v^{\mu,+}(\mathbb{R})$. We refer to [9, Theorem 19.0.1] for the case of progressive functions of polynomial decay, and to [19] for the (Gelfand-Shilov) spaces of type S . First, we introduce the space $S_v^\mu(\mathbb{R} \times \mathbb{R}_+)$.

Definition 3.4. Let there be given positive constants h, v and μ . The space $S_{v,h}^\mu(\mathbb{R} \times \mathbb{R}_+)$ consists of all $\tau \in C^\infty(\mathbb{R} \times \mathbb{R}_+)$ such that

$$|\tau(b, a)| \leq C m_{h,v}^{-1} \left(\frac{b}{1+a} \right) m_{h,\mu}^{-1} \left(a + \frac{1}{a} \right), \quad (b, a) \in \mathbb{R} \times \mathbb{R}_+, \tag{3.28}$$

for some $C > 0$.

Then $S_v^\mu(\mathbb{R} \times \mathbb{R}_+) = \bigcup_{h>0} S_{v,h}^\mu(\mathbb{R} \times \mathbb{R}_+)$, and the topology is defined by the inductive limit:

$$S_v^\mu(\mathbb{R} \times \mathbb{R}_+) = \text{ind} \lim_{h>0} S_{v,h}^\mu(\mathbb{R} \times \mathbb{R}_+). \tag{3.29}$$

Theorem 3.5. *Let there be given $\nu \geq 1$ and $\mu > 0$ and let $f, g \in S_{\nu}^{\mu,+}(\mathbb{R})$. Then, the wavelet transform $W : (f, g) \mapsto W_g f$ is a continuous map from $S_{\nu}^{\mu,+}(\mathbb{R}) \times S_{\nu}^{\mu,+}(\mathbb{R})$ into $S_{\nu}^s(\mathbb{R} \times \mathbb{R}_+)$, where $s = \max \{\nu, \mu\}$.*

Proof. Since f and g are smooth, $W_g f(b, a) \in C^{\infty}(\mathbb{R} \times \mathbb{R}_+)$. Proposition 2.8 implies that

$$|\widehat{f}(\omega)| \leq C m_{h,s}^{-1}\left(\omega + \frac{1}{\omega}\right), \quad |\widehat{g}(\omega)| \leq C m_{h,s}^{-1}\left(\omega + \frac{1}{\omega}\right), \quad \omega > 0, \quad (3.30)$$

for $s = \max \{\nu, \mu\}$ and for some positive constants h and C .

By Propositions 3.2 and 3.3, it follows that there exist positive constants C and h such that

$$|W_g f(b, a)| \leq C m_{h/2,\nu}^{-1}\left(\frac{b}{1+a}\right) m_{h/2,s}^{-1}\left(a + \frac{1}{a}\right), \quad (b, a) \in \mathbb{R} \times \mathbb{R}_+, \quad (3.31)$$

which proves the continuity. $\square$

Note that f and g in Theorem 3.5 belong to the same progressive Gelfand-Shilov space, while in [19] different decay conditions are imposed on f and g . Since the analyzing wavelet g in [19] may not satisfy the admissibility condition (4.24), the results given there are of global nature, and the inversion formula cannot be derived.

4. Inverse Wavelet Transform

In this section, we study the inverse wavelet transform, or the wavelet synthesis operator M_g of function $\tau \in S_{\nu}^{\mu}(\mathbb{R} \times \mathbb{R}_+)$ with respect to the wavelet $g \in L^1(\mathbb{R})$ given by

$$(M_g \tau)(x) = \int_0^{+\infty} \frac{da}{a} \int_{\mathbb{R}} \tau(b, a) \frac{1}{a} g\left(\frac{x-b}{a}\right) db, \quad x \in \mathbb{R}. \quad (4.1)$$

Our first result is analogous to Proposition 3.2 (see also [9, Theorem 18.2.1]).

Proposition 4.1. *Let $\nu \geq 1$ and $\mu > 0$. If $\tau \in S_{\nu}^{\mu}(\mathbb{R} \times \mathbb{R}_+)$ and if g satisfies*

$$|g(x)| \leq C m_{h,\nu}^{-1}(x), \quad x \in \mathbb{R}, \quad (4.2)$$

for some positive constants h and C , then

$$|M_g \tau(x)| \leq \tilde{C} m_{h,\nu}^{-1}(x), \quad x \in \mathbb{R}, \quad (4.3)$$

for some positive constants $\tilde{h}$ and $\tilde{C}$.

Proof. We have

$$\begin{aligned}
 |(M_g \tau)(x)| &\leq \int_0^{+\infty} \frac{da}{a} \int_{\mathbb{R}} |\tau(b, a)| \frac{1}{a} \left| g\left(\frac{x-b}{a}\right) \right| db \\
 &\leq C \int_0^{+\infty} \frac{da}{a^2} \int_{\mathbb{R}} m_{h,\nu}^{-1}\left(\frac{b}{1+a}\right) m_{h,\mu}^{-1}\left(a + \frac{1}{a}\right) m_{h,\nu}^{-1}\left(\frac{x-b}{a}\right) db \\
 &\leq C(I_1(x) + I_2(x)), \quad x \in \mathbb{R},
 \end{aligned} \tag{4.4}$$

where in I_1 the integration with respect to a is taken over $(0, 1)$ and in I_2 over $(1, \infty)$.

By (3.6), it follows that for any $\tilde{h} \in (0, h)$ we have

$$I_1(x) \leq \int_0^1 m_{h,\mu}^{-1}\left(a + \frac{1}{a}\right) \frac{da}{a^2} \int_{\mathbb{R}} m_{h,\nu}^{-1}\left(\frac{b}{1+a}\right) m_{h,\nu}^{-1}(x) m_{\tilde{h},\nu}(b) db, \quad x \in \mathbb{R}. \tag{4.5}$$

Since

$$\int_{\mathbb{R}} m_{h,\nu}^{-1}\left(\frac{b}{1+a}\right) m_{\tilde{h},\nu}(b) db < \infty, \tag{4.6}$$

when $\tilde{h} < h2^{-1/\nu} \leq h(1+a)^{-1/\nu}$ and since

$$\int_0^1 m_{h,\mu}^{-1}\left(a + \frac{1}{a}\right) \frac{da}{a^2} < \infty, \tag{4.7}$$

it follows that

$$I_1(x) \leq C m_{h,\nu}^{-1}(x), \quad x \in \mathbb{R}, \tag{4.8}$$

for some $C > 0$.

Next, we estimate I_2 . We have

$$\begin{aligned}
 I_2(x) &= \int_1^{+\infty} m_{h,\mu}^{-1}\left(a + \frac{1}{a}\right) da \int_{\mathbb{R}} \frac{1}{a} m_{h,\nu}^{-1}\left(\frac{b}{1+a}\right) \frac{1}{a} m_{h,\nu}^{-1}\left(\frac{x-b}{a}\right) db \\
 &\leq \int_1^{+\infty} m_{h,\mu}^{-1}\left(a + \frac{1}{a}\right) da \int_{\mathbb{R}} \frac{1}{a} m_{h,\nu}^{-1}\left(\frac{b}{2a}\right) \frac{1}{a} m_{h,\nu}^{-1}\left(\frac{x-b}{2a}\right) db \\
 &= \int_1^{+\infty} m_{h,\mu}^{-1}\left(a + \frac{1}{a}\right) da \int_{\mathbb{R}} D_a\left(m_{h,\nu}^{-1}\left(\frac{b}{2}\right)\right) D_a\left(m_{h,\nu}^{-1}\left(\frac{x-b}{2}\right)\right) db \\
 &= \int_1^{+\infty} m_{h,\mu}^{-1}\left(a + \frac{1}{a}\right) \left(D_a\left(m_{h,\nu}^{-1}\left(\frac{b}{2}\right)\right) * D_a\left(m_{h,\nu}^{-1}\left(\frac{b}{2}\right)\right)\right)(x) da \\
 &= \int_1^{+\infty} m_{h,\mu}^{-1}\left(a + \frac{1}{a}\right) D_a\left(\left(m_{h,\nu}^{-1}\left(\frac{b}{2}\right) * m_{h,\nu}^{-1}\left(\frac{b}{2}\right)\right)(x)\right) da, \quad x \in \mathbb{R}.
 \end{aligned} \tag{4.9}$$

By Lemma 3.1,

$$I_2(x) \leq C \int_1^{+\infty} m_{h,\mu}^{-1} \left(a + \frac{1}{a} \right) D_a \left(m_{h,\nu}^{-1}(x) \right) da, \quad x \in \mathbb{R}, \quad (4.10)$$

for every $\tilde{h} \in (0, h)$. After the change of variables $a' = 1/a$ and renaming a' to a , by the mean value theorem, we obtain

$$\begin{aligned} I_2(x) &\leq \int_0^1 C m_{h,\mu}^{-1} \left(a + \frac{1}{a} \right) m_{h,\nu}^{-1}(ax) \frac{da}{a} \\ &= C m_{h,\nu}^{-1}(\theta x) \int_0^1 m_{h,\mu}^{-1} \left(a + \frac{1}{a} \right) \frac{da}{a}, \quad x \in \mathbb{R}, \end{aligned} \quad (4.11)$$

for some $\theta \in (0, 1)$. Therefore, there are $C > 0$ and $\tilde{h} > 0$ such that $I_2(x) \leq C m_{h,\nu}^{-1}(x)$, $x \in \mathbb{R}$, and (4.3) is proved. $\square$

The next result can be compared to [9, Theorem 18.1.1]. Here, the assumption on the decay of g leads to a more precise estimate for $\widehat{M_g \tau}$ than the one given in [9].

Proposition 4.2. *Let ν and μ be positive constants. If $\tau \in S_\nu^\mu(\mathbb{R} \times \mathbb{R}_+)$ and if g satisfies*

$$|\widehat{g}(\omega)| \leq C m_{h,\mu}^{-1} \left(\omega + \frac{1}{\omega} \right), \quad \omega > 0, \quad (4.12)$$

for some positive constants h and C , then

$$\left| \widehat{M_g \tau}(\omega) \right| \leq \tilde{C} m_{h,\mu}^{-1} \left(\omega + \frac{1}{\omega} \right), \quad \omega > 0, \quad (4.13)$$

for some positive constants $\tilde{h}$ and $\tilde{C}$.

Proof. The Fourier transform of $M_g \tau$ is given by

$$\widehat{M_g \tau}(\omega) = \int_0^{+\infty} \frac{da}{a} \int_{\mathbb{R}} \tau(b, a) \widehat{g}(a\omega) e^{-ib\omega} db, \quad \omega > 0. \quad (4.14)$$

Therefore,

$$\begin{aligned} \left| \widehat{M_g \tau}(\omega) \right| &\leq C \int_0^{+\infty} \frac{1}{a} m_{h,\mu}^{-1} \left(a + \frac{1}{a} \right) m_{h,\mu}^{-1} \left(a\omega + \frac{1}{a\omega} \right) da \int_{\mathbb{R}} m_{h,\nu}^{-1} \left(\frac{b}{1+a} \right) db \\ &\leq C_1 (I_1(\omega) + I_2(\omega)), \quad \omega > 0, \end{aligned} \quad (4.15)$$

where

$$\begin{aligned} I_1(\omega) &= \int_0^1 \frac{1+a}{a} m_{h,\mu}^{-1}\left(a + \frac{1}{a}\right) m_{h,\mu}^{-1}\left(a\omega + \frac{1}{a\omega}\right) da, \\ I_2(\omega) &= \int_1^{+\infty} \frac{1+a}{a} m_{h,\mu}^{-1}\left(a + \frac{1}{a}\right) m_{h,\mu}^{-1}\left(a\omega + \frac{1}{a\omega}\right) da, \quad \omega > 0, \end{aligned} \quad (4.16)$$

and we have made the change of variables $b/(1+a) = b'$.

We use similar arguments as in proof of Proposition 3.3 to estimate $I_1(\omega)$ when $0 < \omega \leq 1$. This gives

$$I_1(\omega) \leq m_{h,\mu}^{-1}\left(\frac{1}{\omega}\right) \int_0^1 \frac{2}{a} m_{h,\mu}^{-1}\left(a + \frac{1}{a}\right) da \leq m_{h,\mu}^{-1}\left(\frac{1}{2}\left(\omega + \frac{1}{\omega}\right)\right), \quad \omega \in (0, 1], \quad (4.17)$$

where we have used the inequality $(\omega + 1/\omega)/2 < 1/\omega, \omega \in (0, 1]$.

When $\omega > 1$ by the mean value theorem, we obtain

$$\begin{aligned} I_1(\omega) &\leq C m_{h,\mu}^{-1}\left(\theta\omega + \frac{1}{\theta\omega}\right) \int_0^1 \frac{2}{a} m_{h,\mu}^{-1}\left(a + \frac{1}{a}\right) da \\ &\leq C m_{h,\mu}^{-1}(\theta\omega) \leq C m_{h,\mu}^{-1}\left(\omega + \frac{1}{\omega}\right), \quad \omega \in (1, \infty), \end{aligned} \quad (4.18)$$

where $\theta \in (0, 1)$ and $\tilde{h} = (\theta/2)^{1/\mu}h$.

It remains to estimate $I_2(\omega)$, $\omega > 0$.

When $\omega \in (0, 1)$, by change of variables $1/a = a'$ and by the mean value theorem, it follows that

$$\begin{aligned} I_2(\omega) &= \int_0^1 \frac{1+a'}{(a')^2} m_{h,\mu}^{-1}\left(a' + \frac{1}{a'}\right) m_{h,\mu}^{-1}\left(a'\omega + \frac{a'}{\omega}\right) da' \\ &\leq \int_0^1 \frac{2}{(a')^2} m_{h,\mu}^{-1}\left(a' + \frac{1}{a'}\right) m_{h,\mu}^{-1}\left(a'\left(\omega + \frac{1}{\omega}\right)\right) da' \\ &= m_{h\theta^{1/\mu},\mu}^{-1}\left(\omega + \frac{1}{\omega}\right) \int_0^1 \frac{2}{(a')^2} m_{h,\mu}^{-1}\left(a' + \frac{1}{a'}\right) da' \\ &\leq C m_{h,\mu}^{-1}\left(\omega + \frac{1}{\omega}\right), \quad \omega \in (0, 1), \end{aligned} \quad (4.19)$$

where $\theta \in (0, 1)$ and $\tilde{h} = \theta^{1/\mu}h$.

In a similar way, as in the proof of Proposition 3.3, it follows that

$$I_2(\omega) \leq C m_{h2^{-1/\mu},\mu}^{-1}\left(\omega + \frac{1}{\omega}\right), \quad \omega \in (1, \infty), \quad (4.20)$$

and the proof is complete. $\square$

Finally, we have the following result, which can be compared to the inversion formulas from [14, 15] which hold for general coorbit spaces. As mentioned in the introduction, our approach is different due to the fact that the weight $m_{h,\nu}$ is not submultiplicative on the “ $ax + b$ ”-group.

Theorem 4.3. *Let $\nu \geq 1$ and $\mu > 0$. The inverse wavelet transform $M : (g, \tau) \mapsto M_g \tau$ is a continuous map from $S_\nu^{\mu,+}(\mathbb{R}) \times S_\nu^\mu(\mathbb{R} \times \mathbb{R}_+)$ into $S_\nu^{s,+}(\mathbb{R})$, where $s = \max\{\nu, \mu\}$.*

Moreover, if $g \neq 0$, then for any $f \in S_\nu^{s,+}(\mathbb{R})$ the following inversion formula holds:

$$f(x) = \frac{1}{c_g} \int_0^\infty \frac{da}{a} \int_{\mathbb{R}} W_g f(b, a) \frac{1}{a} g\left(\frac{x-b}{a}\right) db, \quad (4.21)$$

where $c_g = \int_0^\infty |\widehat{g}(\omega)|^2 (d\omega/\omega)$.

Proof. For the first part of Theorem 4.3, it is enough to observe that Proposition 2.8 implies that

$$|\widehat{g}(\omega)| \leq C m_{h,s}^{-1}\left(\omega + \frac{1}{\omega}\right), \quad \omega > 0, \quad (4.22)$$

for some positive constants h and C , and for $s = \max\{\nu, \mu\}$.

The continuity now follows from Propositions 4.1 and 4.2 and Theorem 2.6(e), since

$$\left| \widehat{M_g \tau}(\omega) \right| \leq \tilde{C} m_{h,s}^{-1}\left(\omega + \frac{1}{\omega}\right) \leq \tilde{C} m_{h,s}^{-1}(\omega), \quad \omega > 0. \quad (4.23)$$

From the the proof of Proposition 2.8, it follows that g satisfies *the admissibility condition*:

$$0 < \left| \int_0^\infty |\widehat{g}(\omega)|^2 \frac{d\omega}{\omega} \right| < \infty. \quad (4.24)$$

Since

$$\frac{1}{c_g} \int_0^\infty \frac{da}{a} \int_{\mathbb{R}} W_g f(b, a) \frac{1}{a} g\left(\frac{x-b}{a}\right) db = \frac{1}{c_g} M_g(W_g f)(x), \quad x \in \mathbb{R}, \quad (4.25)$$

and by Theorem 3.5, we have that $W_g f \in S_\nu^s(\mathbb{R} \times \mathbb{R}_+)$, from the first part of the proof it follows that $M_g(W_g f) \in S_\nu^{s,+}(\mathbb{R})$.

The proof is now the same as the proof of [9, Theorem 14.0.2] if we take g to be its own reconstruction wavelet and observe $S_\nu^{s,+}(\mathbb{R})$ instead of the space of polynomially strip localized functions. $\square$

5. Concluding Remarks

We first note that under the assumptions of Proposition 3.2 we have that for any given $M > 0$ there exist $\tilde{h} > 0$ and $\tilde{C} > 0$ such that

- (i) $|W_g f(b, a)| \leq \tilde{C} e^{-\tilde{h}|b|^{1/\nu}}, \quad (b, a) \in \mathbb{R} \times (0, M];$
- (ii) $|W_g f(b, a)| \leq \tilde{C} D_a(e^{-\tilde{h}|\cdot|^{1/\nu}})(b), \quad (b, a) \in \mathbb{R} \times [M, \infty).$

Note that those estimates imply that we are not able to separate variables b and a in a straightforward manner and lead to (3.28) in Definition 3.4. It is probably more natural to have a possibility to analyze decay with respect to variables b and a separately. Therefore, we have the following definition.

Definition 5.1. Let there be given positive constants h, ν , and μ . The space $Z_{\nu, h}^{\mu}(\mathbb{R} \times \mathbb{R}_+)$ consists of all $\tau \in C^{\infty}(\mathbb{R} \times \mathbb{R}_+)$ such that

$$|\tau(b, a)| \leq C m_{h, \nu}^{-1}(b) m_{h, \mu}^{-1}\left(a + \frac{1}{a}\right), \quad (b, a) \in \mathbb{R} \times \mathbb{R}_+, \quad (5.1)$$

for some $C > 0$.

Then $Z_{\nu}^{\mu}(\mathbb{R} \times \mathbb{R}_+) = \bigcup_{h>0} Z_{\nu, h}^{\mu}(\mathbb{R} \times \mathbb{R}_+)$, and the topology is given by the inductive limit:

$$Z_{\nu}^{\mu}(\mathbb{R} \times \mathbb{R}_+) = \text{ind} \lim_{h>0} Z_{\nu, h}^{\mu}(\mathbb{R} \times \mathbb{R}_+). \quad (5.2)$$

Next, we reformulate results from Sections 3 and 4 in terms of $Z_{\nu}^{\mu}(\mathbb{R} \times \mathbb{R}_+)$ since we believe that this might be of independent interest.

Theorem 5.2. Let $\nu > 1$ and $\mu > 0$, let $f, g \in S_{\nu-\varepsilon}^{\mu,+}(\mathbb{R})$, with $\varepsilon > 0$ chosen such that $\nu - \varepsilon \geq 1$. Then, the wavelet transform $W : (f, g) \mapsto W_g f$ is a continuous map from $S_{\nu-\varepsilon}^{\mu,+}(\mathbb{R}) \times S_{\nu-\varepsilon}^{\mu,+}(\mathbb{R})$ into $Z_{\nu}^s(\mathbb{R} \times \mathbb{R}_+)$, where $s = \max \{\nu, \mu\}$.

Proof. It follows from the proof of Theorem 3.5 and the fact that for every positive N, M , and ε there is a constant $C > 0$ such that

$$e^{-N|\cdot|^{1/(v-\varepsilon)}} \leq C e^{-M|\cdot|^{1/\nu}}, \quad (5.3)$$

wherefrom

$$|W_g f(b, a)| \leq \frac{\tilde{C}}{e^{\tilde{h}|b/(1+a)|^{1/(v-\varepsilon)}}} \leq \frac{\tilde{C}}{e^{k|b|^{1/\nu}}}, \quad (b, a) \in \mathbb{R} \times \mathbb{R}_+, \quad (5.4)$$

for arbitrary $k > 0$. The theorem is proved. $\square$

The inverse wavelet transform M_g of $\tau \in Z_{\nu}^{\mu}(\mathbb{R} \times \mathbb{R}_+)$ with respect to the wavelet $g \in L^1(\mathbb{R})$ given by

$$(M_g \tau)(x) = \int_0^{+\infty} \frac{da}{a} \int_{\mathbb{R}} \tau(b, a) \frac{1}{a} g\left(\frac{x-b}{a}\right) db, \quad x \in \mathbb{R}. \quad (5.5)$$

Propositions 4.1 and 4.2 still hold true with $S_{\nu}^{\mu}(\mathbb{R} \times \mathbb{R}_+)$ replaced by $Z_{\nu}^{\mu}(\mathbb{R} \times \mathbb{R}_+)$, while Theorem 4.3 takes the following form.

Theorem 5.3. Let $\nu \geq 1$ and $\mu > 0$. The inverse wavelet transform $M : (g, \tau) \mapsto M_g \tau$ is a continuous map from $S_{\nu}^{\mu,+}(\mathbb{R}) \times Z_{\nu}^{\mu}(\mathbb{R} \times \mathbb{R}_+)$ into $S_{\nu}^{s,+}(\mathbb{R})$, where $s = \max \{\nu, \mu\}$.

Moreover, if $f, g \in S_{\nu-\varepsilon}^{\mu,+}(\mathbb{R})$ for some $\varepsilon > 0$ such that $\nu - \varepsilon > 1$ and $g \neq 0$, then

$$f(x) = \frac{1}{c_g} \int_0^{\infty} \frac{da}{a} \int_{\mathbb{R}} W_g f(b, a) \frac{1}{a} g\left(\frac{x-b}{a}\right) db, \quad (5.6)$$

where $c_g = \int_0^{\infty} |\widehat{g}(\omega)|^2 (d\omega/\omega)$.

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