Research Article

Some Integral Inequalities of Hermite-Hadamard Type for Convex Functions with Applications to Means

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The authors establish some new inequalities for differentiable convex functions, which are similar to the celebrated Hermite-Hadamard's integral inequality for convex functions, and apply these inequalities to construct inequalities for special means of two positive numbers.

1. Introduction

In [1], the following Hermite-Hadamard type inequalities for differentiable convex functions were proved.

Theorem 1.1 (see [1, Theorem 2.2]). *Let $f : I^\circ \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° , $a, b \in I^\circ$ with $a < b$. If $|f'(x)|$ is convex on $[a, b]$, then*

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{(b-a)(|f'(a)| + |f'(b)|)}{8}. \quad (1.1)$$

Theorem 1.2 (see [1, Theorem 2.3]). *Let $f : I^\circ \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° , $a, b \in I^\circ$ with $a < b$, and let $p > 1$. If the new mapping $|f'(x)|^{p/(p-1)}$ is convex on $[a, b]$, then*

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{2(p+1)^{1/p}} \left[\frac{|f'(a)|^{p/(p-1)} + |f'(b)|^{p/(p-1)}}{2} \right]^{(p-1)/p}. \quad (1.2)$$

In [2], the above inequalities were generalized as follows.

Theorem 1.3 (see [2, Theorems 1 and 2]). *Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be differentiable on I° , $a, b \in I$ with $a < b$, and let $q \geq 1$. If $|f'(x)|^q$ is convex on $[a, b]$, then*

$$\begin{aligned} \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| &\leq \frac{b-a}{4} \left[\frac{|f'(a)|^q + |f'(b)|^q}{2} \right]^{1/q}, \\ \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| &\leq \frac{b-a}{4} \left[\frac{|f'(a)|^q + |f'(b)|^q}{2} \right]^{1/q}. \end{aligned} \quad (1.3)$$

In [3], the above inequalities were further generalized as follows.

Theorem 1.4 (see [3, Theorems 2.3 and 2.4]). *Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be differentiable on I° , $a, b \in I$ with $a < b$, and let $p > 1$. If $|f'(x)|^{p/(p-1)}$ is convex on $[a, b]$, then*

$$\begin{aligned} &\left| \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \right| \\ &\leq \frac{b-a}{16} \left(\frac{4}{p+1} \right)^{1/p} \\ &\quad \times \left\{ \left[|f'(a)|^{p/(p-1)} + 3|f'(b)|^{p/(p-1)} \right]^{(p-1)/p} \right. \\ &\quad \left. + \left[3|f'(a)|^{p/(p-1)} + |f'(b)|^{p/(p-1)} \right]^{(p-1)/p} \right\}, \\ &\left| \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \right| \leq \frac{b-a}{4} \left(\frac{4}{p+1} \right)^{1/p} (|f'(a)| + |f'(b)|). \end{aligned} \quad (1.4)$$

In [4], an inequality similar to the above ones was given as follows.

Theorem 1.5 (see [4, Theorem 3]). *Let $f : [a, b] \rightarrow \mathbb{R}$ be an absolutely continuous mapping on $[a, b]$ whose derivative belongs to $L_p[a, b]$. Then*

$$\left| \frac{1}{3} \left[\frac{f(a) + f(b)}{2} + 2f\left(\frac{a+b}{2}\right) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{1}{6} \left[\frac{2^{q+1} + 1}{3(q+1)} \right]^{1/q} (b-a)^{1/q} \|f'\|_p, \quad (1.5)$$

where $(1/p) + (1/q) = 1$ and $p > 1$.

Recently, the following inequalities were obtained in [5].

Theorem 1.6. Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be differentiable on I° , $a, b \in I$ with $a < b$, and $f' \in L[a, b]$. If $|f'(x)|^q$ for $q \geq 1$ is convex on $[a, b]$, then

$$\begin{aligned} & \left| \frac{1}{6} \left[f(a) + f(b) + 4f\left(\frac{a+b}{2}\right) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{b-a}{12} \left[\frac{2^{q+1}+1}{3(q+1)} \right]^{1/q} \left[\left(\frac{3|f'(a)|^q + |f'(b)|^q}{4} \right)^{1/q} + \left(\frac{|f'(a)|^q + 3|f'(b)|^q}{4} \right)^{1/q} \right], \\ & \left| \frac{1}{6} \left[f(a) + f(b) + 4f\left(\frac{a+b}{2}\right) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{5(b-a)}{72} \left[\left(\frac{61|f'(a)|^q + 29|f'(b)|^q}{90} \right)^{1/q} + \left(\frac{29|f'(a)|^q + 61|f'(b)|^q}{90} \right)^{1/q} \right]. \end{aligned} \quad (1.6)$$

In this paper, we will establish some new Hermite-Hadamard type integral inequalities for differentiable functions and apply them to derive some inequalities of special means.

2. Lemmas

For establishing new integral inequalities of Hermite-Hadamard type, we need the lemmas below.

Lemma 2.1. Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be differentiable on I° , $a, b \in I$ with $a < b$. If $f' \in L[a, b]$ and $\lambda, \mu \in \mathbb{R}$, then

$$\begin{aligned} & \frac{\lambda f(a) + \mu f(b)}{2} + \frac{2-\lambda-\mu}{2} f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \\ & = \frac{b-a}{4} \int_0^1 \left[(1-\lambda-t)f'\left(ta + (1-t)\frac{a+b}{2}\right) + (\mu-t)f'\left(t\frac{a+b}{2} + (1-t)b\right) \right] dt. \end{aligned} \quad (2.1)$$

Proof. Integrating by part and changing variable of definite integral yield

$$\begin{aligned} & \int_0^1 (1-\lambda-t)f'\left(ta + (1-t)\frac{a+b}{2}\right) dt \\ & = -\frac{2}{b-a} \left[(1-\lambda-t)f\left(ta + (1-t)\frac{a+b}{2}\right) \Big|_0^1 + \int_0^1 f\left(ta + (1-t)\frac{a+b}{2}\right) dt \right] \\ & = \frac{2}{b-a} \left[\lambda f(a) + (1-\lambda)f\left(\frac{a+b}{2}\right) \right] - \frac{4}{(b-a)^2} \int_a^{(a+b)/2} f(x) dx. \end{aligned} \quad (2.2)$$

Similarly, we have

$$\begin{aligned}
 & \int_0^1 (\mu - t) f' \left(t \frac{a+b}{2} + (1-t)b \right) dt \\
 &= -\frac{2}{b-a} \left[(\mu - t) f \left(t \frac{a+b}{2} + (1-t)b \right) \right]_0^1 + \int_0^1 f \left(t \frac{a+b}{2} + (1-t)b \right) dt \\
 &= \frac{2}{b-a} \left[(1-\mu) f \left(\frac{a+b}{2} \right) + \mu f(b) \right] - \frac{4}{(b-a)^2} \int_{(a+b)/2}^b f(x) dx.
 \end{aligned} \tag{2.3}$$

Adding these two equations leads to Lemma 2.1. $\square$

Lemma 2.2. For $s > 0$ and $0 \leq \xi \leq 1$, one has

$$\begin{aligned}
 \int_0^1 |\xi - t|^s dt &= \frac{\xi^{s+1} + (1-\xi)^{s+1}}{s+1}, \\
 \int_0^1 t |\xi - t|^s dt &= \frac{\xi^{s+2} + (s+1+\xi)(1-\xi)^{s+1}}{(s+1)(s+2)}.
 \end{aligned} \tag{2.4}$$

Proof. This follows from a straightforward computation of definite integrals. $\square$

3. Some Integral Inequalities of Hermite-Hadamard Type

Now we are in a position to establish some new integral inequalities of Hermite-Hadamard type for differentiable convex functions.

The first main result is Theorem 3.1.

Theorem 3.1. Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function on I° , $a, b \in I$ with $a < b$, $0 \leq \lambda, \mu \leq 1$, and $f' \in L[a, b]$. If $|f'(x)|^q$ for $q \geq 1$ is convex on $[a, b]$, then

$$\begin{aligned}
 & \left| \frac{\lambda f(a) + \mu f(b)}{2} + \frac{2-\lambda-\mu}{2} f \left(\frac{a+b}{2} \right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \\
 & \leq \frac{b-a}{8} \left(\frac{1}{6} \right)^{1/q} \left\{ \left(1 - 2\lambda + 2\lambda^2 \right)^{1-1/q} \right. \\
 & \quad \times \left[\left(4 - 9\lambda + 12\lambda^2 - 2\lambda^3 \right) |f'(a)|^q + \left(2 - 3\lambda + 2\lambda^3 \right) |f'(b)|^q \right]^{1/q} \\
 & \quad + \left(1 - 2\mu + 2\mu^2 \right)^{1-(1/q)} \\
 & \quad \times \left[\left(2 - 3\mu + 2\mu^3 \right) |f'(a)|^q + \left(4 - 9\mu + 12\mu^2 - 2\mu^3 \right) |f'(b)|^q \right]^{1/q} \Big\}.
 \end{aligned} \tag{3.1}$$

Proof. For $q > 1$, by Lemma 2.1, the convexity of $|f'(x)|^q$ on $[a, b]$, and the noted Hölder's integral inequality, we have

$$\begin{aligned}
 & \left| \frac{\lambda f(a) + \mu f(b)}{2} + \frac{2 - \lambda - \mu}{2} f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \\
 & \leq \frac{b-a}{4} \left[\int_0^1 |1 - \lambda - t| \left| f'\left(ta + (1-t)\frac{a+b}{2}\right) \right| dt + \int_0^1 |\mu - t| \left| f'\left(t\frac{a+b}{2} + (1-t)b\right) \right| dt \right] \\
 & \leq \frac{b-a}{4} \left\{ \left(\int_0^1 |1 - \lambda - t| dt \right)^{1-1/q} \left[\int_0^1 |1 - \lambda - t| \left(\frac{1+t}{2} |f'(a)|^q + \frac{1-t}{2} |f'(b)|^q \right) dt \right]^{1/q} \right. \\
 & \quad \left. + \left(\int_0^1 |\mu - t| dt \right)^{1-1/q} \left[\int_0^1 |\mu - t| \left(\frac{t}{2} |f'(a)|^q + \frac{2-t}{2} |f'(b)|^q \right) dt \right]^{1/q} \right\}. \tag{3.2}
 \end{aligned}$$

In virtue of Lemma 2.2, a direct calculation yields

$$\begin{aligned}
 & \int_0^1 |1 - \lambda - t| \left(\frac{1+t}{2} |f'(a)|^q + \frac{1-t}{2} |f'(b)|^q \right) dt \\
 & = \frac{1}{2} (|f'(a)|^q + |f'(b)|^q) \int_0^1 |1 - \lambda - t| dt + \frac{1}{2} (|f'(a)|^q - |f'(b)|^q) \int_0^1 t |1 - \lambda - t| dt \\
 & = \frac{1}{2} (|f'(a)|^q + |f'(b)|^q) \left(\frac{1}{2} - \lambda + \lambda^2 \right) + \frac{1}{12} (|f'(a)|^q - |f'(b)|^q) [(1 - \lambda)^3 + (1 - \lambda)^2 (3 - \lambda)] \\
 & = \frac{1}{12} (4 - 9\lambda + 12\lambda^2 - 2\lambda^3) |f'(a)|^q + \frac{1}{12} (2 - 3\lambda + 2\lambda^3) |f'(b)|^q, \\
 & \int_0^1 |\mu - t| \left(\frac{t}{2} |f'(a)|^q + \frac{2-t}{2} |f'(b)|^q \right) dt \\
 & = \frac{1}{2} (|f'(a)|^q - |f'(b)|^q) \int_0^1 t |\mu - t| dt + |f'(b)|^q \int_0^1 |\mu - t| dt \\
 & = \frac{1}{12} [\mu^3 + (1 - \mu)^2 (2 + \mu)] (|f'(a)|^q - |f'(b)|^q) + \left(\frac{1}{2} - \mu + \mu^2 \right) |f'(b)|^q \\
 & = \frac{1}{12} (2 - 3\mu + 2\mu^3) |f'(a)|^q + \frac{1}{12} (4 - 9\mu + 12\mu^2 - 2\mu^3) |f'(b)|^q. \tag{3.3}
 \end{aligned}$$

Substituting the above two equalities into the inequality (3.2) and utilizing Lemma 2.2 result in the inequality (3.1) for $q > 1$.

For $q = 1$, from Lemmas 2.1 and 2.2 it follows that

$$\begin{aligned}
 & \left| \frac{\lambda f(a) + \mu f(b)}{2} + \frac{2 - \lambda - \mu}{2} f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \\
 & \leq \frac{b-a}{4} \left[\int_0^1 |1 - \lambda - t| \left(\frac{1+t}{2} |f'(a)| + \frac{1-t}{2} |f'(b)| \right) dt \right. \\
 & \quad \left. + \int_0^1 |\mu - t| \left(\frac{t}{2} |f'(a)| + \frac{2-t}{2} |f'(b)| \right) dt \right] \\
 & = \frac{b-a}{48} \left[(4 - 9\lambda + 12\lambda^2 - 2\lambda^3) |f'(a)| + (2 - 3\lambda + 2\lambda^2) |f'(b)| \right. \\
 & \quad \left. + (2 - 3\mu + 2\mu^2) |f'(a)| + (4 - 9\mu + 12\mu^2 - 2\mu^3) |f'(b)| \right],
 \end{aligned} \tag{3.4}$$

which is just equivalent to (3.1) for $q = 1$. Theorem 3.1 is proved. $\square$

If taking $\lambda = \mu$ in Theorem 3.1, we derive the following corollary.

Corollary 3.2. Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be differentiable on I° , $a, b \in I$ with $a < b$, $0 \leq \lambda \leq 1$, and $f' \in L[a, b]$. If $|f'(x)|^q$ is convex on $[a, b]$ for $q \geq 1$, then

$$\begin{aligned}
 & \left| \frac{\lambda}{2} [f(a) + f(b)] + (1 - \lambda) f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \\
 & \leq \frac{b-a}{8} \left(\frac{1}{6} \right)^{1/q} (1 - 2\lambda + 2\lambda^2)^{1-(1/q)} \\
 & \quad \times \left\{ \left[(4 - 9\lambda + 12\lambda^2 - 2\lambda^3) |f'(a)|^q + (2 - 3\lambda + 2\lambda^3) |f'(b)|^q \right]^{1/q} \right. \\
 & \quad \left. + \left[(2 - 3\lambda + 2\lambda^3) |f'(a)|^q + (4 - 9\lambda + 12\lambda^2 - 2\lambda^3) |f'(b)|^q \right]^{1/q} \right\}.
 \end{aligned} \tag{3.5}$$

If letting $\lambda = \mu = 1/2, 2/3, 1/3$, respectively, in Theorem 3.1, we can deduce the inequalities below.

Corollary 3.3. Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be differentiable on I° , $a, b \in I$ with $a < b$, and $f' \in L[a, b]$. If $|f'(x)|^q$ is convex on $[a, b]$ for $q \geq 1$, then

$$\begin{aligned}
 & \left| \frac{1}{2} \left[\frac{f(a) + f(b)}{2} + f\left(\frac{a+b}{2}\right) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\
 & \leq \frac{b-a}{16} \left(\frac{1}{12} \right)^{1/q} \left\{ [9|f'(a)|^q + 3|f'(b)|^q]^{1/q} + [3|f'(a)|^q + 9|f'(b)|^q]^{1/q} \right\},
 \end{aligned}$$

$$\begin{aligned}
& \left| \frac{1}{3} \left[f(a) + f(b) + f\left(\frac{a+b}{2}\right) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\
& \leq \frac{5(b-a)}{72} \left(\frac{1}{180} \right)^{1/q} \left[(74|f'(a)|^q + 16|f'(b)|^q)^{1/q} + (16|f'(a)|^q + 74|f'(b)|^q)^{1/q} \right], \\
& \left| \frac{1}{6} \left[f(a) + f(b) + 4f\left(\frac{a+b}{2}\right) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\
& \leq \frac{5(b-a)}{72} \left(\frac{1}{90} \right)^{1/q} \left\{ [61|f'(a)|^q + 29|f'(b)|^q]^{1/q} + [29|f'(a)|^q + 61|f'(b)|^q]^{1/q} \right\}.
\end{aligned} \tag{3.6}$$

If setting $q = 1$ in Corollary 3.3, then one has the following.

Corollary 3.4. Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be differentiable on I° , $a, b \in I$ with $a < b$, and $f' \in L[a, b]$. If $|f'(x)|$ is convex on $[a, b]$, then

$$\begin{aligned}
& \left| \frac{1}{2} \left[\frac{f(a) + f(b)}{2} + f\left(\frac{a+b}{2}\right) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{16} (|f'(a)| + |f'(b)|), \\
& \left| \frac{1}{3} \left[f(a) + f(b) + f\left(\frac{a+b}{2}\right) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{5(b-a)}{144} (|f'(a)| + |f'(b)|), \\
& \left| \frac{1}{6} \left[f(a) + f(b) + 4f\left(\frac{a+b}{2}\right) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{5(b-a)}{72} (|f'(a)| + |f'(b)|).
\end{aligned} \tag{3.7}$$

The second main result is Theorem 3.5.

Theorem 3.5. Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be differentiable on I° , $a, b \in I$ with $a < b$, $0 \leq \lambda, \mu \leq 1$, and $f' \in L[a, b]$. If $|f'(x)|^q$ is convex on $[a, b]$ for $q \geq 1$, then

$$\begin{aligned}
& \left| \frac{\lambda f(a) + \mu f(b)}{2} + \frac{2-\lambda-\mu}{2} f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \\
& \leq \frac{b-a}{4} \left[\frac{1}{2(q+1)(q+2)} \right]^{1/q} \left\{ \left[(q+3-\lambda)(1-\lambda)^{q+1} + (2q+4-\lambda)\lambda^{q+1} \right] |f'(a)|^q \right. \\
& \quad + \left[(q+1+\lambda)(1-\lambda)^{q+1} + \lambda^{q+2} \right] |f'(b)|^q \\
& \quad + \left[(q+1+\mu)(1-\mu)^{q+1} + \mu^{q+2} \right] |f'(a)|^q \\
& \quad \left. + \left[(q+3-\mu)(1-\mu)^{q+1} + (2q+4-\mu)\mu^{q+1} \right] |f'(b)|^q \right\}^{1/q}.
\end{aligned} \tag{3.8}$$

Proof. For $q > 1$, by the convexity of $|f'(x)|^q$ on $[a, b]$, Lemma 2.1, and Hölder's integral inequality, it follows that

$$\begin{aligned}
 & \left| \frac{\lambda f(a) + \mu f(b)}{2} + \frac{2 - \lambda - \mu}{2} f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \\
 & \leq \frac{b-a}{4} \left[\int_0^1 |1 - \lambda - t| \left| f'\left(ta + (1-t)\frac{a+b}{2}\right) \right| dt \right. \\
 & \quad \left. + \int_0^1 |\mu - t| \left| f'\left(t\frac{a+b}{2} + (1-t)b\right) \right| dt \right] \\
 & \leq \frac{b-a}{4} \left\{ \left(\int_0^1 dt \right)^{1-1/q} \left[\int_0^1 |1 - \lambda - t|^q \left(\frac{1+t}{2} |f'(a)|^q + \frac{1-t}{2} |f'(b)|^q \right) dt \right]^{1/q} \right. \\
 & \quad \left. + \left(\int_0^1 dt \right)^{1-1/q} \left[\int_0^1 |\mu - t|^q \left(\frac{t}{2} |f'(a)|^q + \frac{2-t}{2} |f'(b)|^q \right) dt \right]^{1/q} \right\} \\
 & = \frac{b-a}{4} \left\{ \left[\int_0^1 |1 - \lambda - t|^q \left(\frac{1+t}{2} |f'(a)|^q + \frac{1-t}{2} |f'(b)|^q \right) dt \right]^{1/q} \right. \\
 & \quad \left. + \left[\int_0^1 |\mu - t|^q \left(\frac{t}{2} |f'(a)|^q + \frac{2-t}{2} |f'(b)|^q \right) dt \right]^{1/q} \right\}.
 \end{aligned} \tag{3.9}$$

By Lemma 2.2, we have

$$\begin{aligned}
 & \int_0^1 |1 - \lambda - t|^q \left(\frac{1+t}{2} |f'(a)|^q + \frac{1-t}{2} |f'(b)|^q \right) dt \\
 & = \frac{1}{2} (|f'(a)|^q + |f'(b)|^q) \int_0^1 |1 - \lambda - t|^q dt \\
 & \quad + \frac{1}{2} (|f'(a)|^q - |f'(b)|^q) \int_0^1 t |1 - \lambda - t|^q dt \\
 & = \frac{1}{2(q+1)} (|f'(a)|^q + |f'(b)|^q) [\lambda^{q+1} + (1-\lambda)^{q+1}] \\
 & \quad + \frac{1}{2(q+1)(q+2)} (|f'(a)|^q - |f'(b)|^q) [(1-\lambda)^{q+2} + (q+2-\lambda)\lambda^{q+1}] \\
 & = \frac{1}{2(q+1)(q+2)} \left\{ [(q+3-\lambda)(1-\lambda)^{q+1} + (2q+4-\lambda)\lambda^{q+1}] |f'(a)|^q \right. \\
 & \quad \left. + [(q+1+\lambda)(1-\lambda)^{q+1} + \lambda^{q+2}] |f'(b)|^q \right\},
 \end{aligned}$$

$$\begin{aligned}
& \int_0^1 |\mu - t|^q \left(\frac{t}{2} |f'(a)|^q + \frac{2-t}{2} |f'(b)|^q \right) dt \\
&= \frac{1}{2(q+1)(q+2)} \left\{ \left[(q+1+\mu)(1-\mu)^{q+1} + \mu^{q+2} \right] |f'(a)|^q \right. \\
&\quad \left. + \left[(q+3-\mu)(1-\mu)^{q+1} + (2q+4-\mu)\mu^{q+1} \right] |f'(b)|^q \right\}.
\end{aligned} \tag{3.10}$$

Substituting these two equalities into the inequality (3.9) yields (3.8) for $q > 1$.

For $q = 1$, the proof is the same as the deduction of (3.4). Thus, Theorem 3.5 is proved. $\square$

As the derivation of corollaries of Theorem 3.1, we can obtain the following corollaries of Theorem 3.5.

Corollary 3.6. *Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be differentiable on I° , $a, b \in I$ with $a < b$, $0 \leq \lambda \leq 1$, and $f' \in L[a, b]$. If $|f'(x)|^q$ is convex for $q \geq 1$ on $[a, b]$, then*

$$\begin{aligned}
& \left| \lambda \frac{f(a) + f(b)}{2} + (1-\lambda)f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \\
& \leq \frac{b-a}{4} \left[\frac{1}{2(q+1)(q+2)} \right]^{1/q} \\
& \quad \times \left\{ \left[(q+3-\lambda)(1-\lambda)^{q+1} + (2q+4-\lambda)\lambda^{q+1} \right] |f'(a)|^q \right. \\
& \quad + \left[(q+1+\lambda)(1-\lambda)^{q+1} + \lambda^{q+2} \right] |f'(b)|^q \Big)^{1/q} \\
& \quad + \left(\left[(q+1+\lambda)(1-\lambda)^{q+1} + \lambda^{q+2} \right] |f'(a)|^q \right. \\
& \quad \left. + \left[(q+3-\lambda)(1-\lambda)^{q+1} + (2q+4-\lambda)\lambda^{q+1} \right] |f'(b)|^q \right)^{1/q} \Big\}.
\end{aligned} \tag{3.11}$$

Corollary 3.7. *Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be differentiable on I° , $a, b \in I$ with $a < b$, and $f' \in L[a, b]$. If $|f'(x)|^q$ is convex for $q \geq 1$ on $[a, b]$, then*

$$\begin{aligned}
& \left| \frac{1}{2} \left[\frac{f(a) + f(b)}{2} + f\left(\frac{a+b}{2}\right) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\
& \leq \frac{b-a}{8} \left[\frac{1}{4(q+1)(q+2)} \right]^{1/q} \left\{ \left[(3q+6)|f'(a)|^q + (q+2)|f'(b)|^q \right]^{1/q} \right. \\
& \quad \left. + \left[(q+2)|f'(a)|^q + (3q+6)|f'(b)|^q \right]^{1/q} \right\},
\end{aligned}$$

$$\begin{aligned}
& \left| \frac{1}{6} \left[f(a) + f(b) + 4f\left(\frac{a+b}{2}\right) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\
& \leq \frac{b-a}{12} \left[\frac{1}{18(q+1)(q+2)} \right]^{1/q} \\
& \quad \times \left\{ \left[(11+6q+(3q+8)2^{q+1}) |f'(a)|^q + (1+(3q+4)2^{q+1}) |f'(b)|^q \right]^{1/q} \right. \\
& \quad \left. + \left[(1+(3q+4)2^{q+1}) |f'(a)|^q + (11+6q+(3q+8)2^{q+1}) |f'(b)|^q \right]^{1/q} \right\}.
\end{aligned} \tag{3.12}$$

Corollary 3.8. Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be differentiable on I° , $a, b \in I$ with $a < b$, and $f' \in L[a, b]$. If $|f'(x)|$ is convex on $[a, b]$, then

$$\begin{aligned}
& \left| \frac{1}{2} \left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{16} (|f'(a)| + |f'(b)|), \\
& \left| \frac{1}{6} \left[f(a) + f(b) + 4f\left(\frac{a+b}{2}\right) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{2(b-a)}{27} (|f'(a)| + 32|f'(b)|).
\end{aligned} \tag{3.13}$$

Corollary 3.9. Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be differentiable on I° , $a, b \in I$ with $a < b$, and $f' \in L[a, b]$. If $|f'(x)|^q$ is convex for $q \geq 1$ on $[a, b]$ and

$$\frac{f(a)+f(b)}{2} = f\left(\frac{a+b}{2}\right) \tag{3.14}$$

then

$$\begin{aligned}
& \left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| = \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \\
& \leq \frac{b-a}{16} \left[\frac{1}{4(q+1)(q+2)} \right]^{1/q} \\
& \quad \times \left\{ [(2q+5)|f'(a)|^q + (2q+3)|f'(b)|^q]^{1/q} + [|f'(a)|^q + (4q+7)|f'(b)|^q]^{1/q} \right. \\
& \quad \left. + [(4q+7)|f'(a)|^q + |f'(b)|^q]^{1/q} + [(2q+3)|f'(a)|^q + (2q+5)|f'(b)|^q]^{1/q} \right\}.
\end{aligned} \tag{3.15}$$

In particular, when $q = 1$, one has

$$\begin{aligned} \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| &= \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ &\leq \frac{b-a}{16} (|f'(a)| + |f'(b)|). \end{aligned} \quad (3.16)$$

4. Applications to Means

For two positive numbers $a > 0$ and $b > 0$, define

$$\begin{aligned} A(a, b) &= \frac{a+b}{2}, \quad G(a, b) = \sqrt{ab}, \quad H(a, b) = \frac{2ab}{a+b}, \\ L(a, b) &= \begin{cases} \left[\frac{b^{s+1} - a^{s+1}}{(s+1)(b-a)} \right]^{1/s}, & a \neq b \\ a, & a = b, \end{cases} \\ I(a, b) &= \begin{cases} \frac{1}{e} \left(\frac{b^b}{a^a} \right)^{1/(b-a)}, & a \neq b \\ a, & a = b, \end{cases} \\ H_{\omega, s}(a, b) &= \begin{cases} \left[\frac{a^s + (ab)^{s/2} \omega + b^s}{\omega + 2} \right]^{1/s}, & s \neq 0 \\ \sqrt{ab}, & s = 0 \end{cases} \end{aligned} \quad (4.1)$$

for $0 \leq \omega < \infty$. These means are, respectively, called the arithmetic, geometric, harmonic, generalized logarithmic, identric, and Heronian means of two positive number a and b .

Applying Theorem 3.1 to $f(x) = x^s$ for $s \neq 0$ and $x > 0$ leads to the following inequalities for means.

Theorem 4.1. Let $a, b > 0$, $q \geq 1$, either $s > 1$ and $(s-1)q \geq 1$ or $s < 0$, then

$$\begin{aligned} \left| \frac{A(a^s, b^s) + A^s(a, b)}{2} - L^s(a, b) \right| &\leq \frac{b-a}{16} |s|^q \left[\frac{1}{4(q+1)(q+2)} \right]^{1/q} \\ &\times \left\{ \left[(2q+5)a^{(s-1)q} + (2q+3)b^{(s-1)q} \right]^{1/q} + \left[a^{(s-1)q} + (4q+7)b^{(s-1)q} \right]^{1/q} \right. \\ &\left. + \left[(4q+7)a^{(s-1)q} + b^{(s-1)q} \right]^{1/q} + \left[(2q+3)a^{(s-1)q} + (2q+5)b^{(s-1)q} \right]^{1/q} \right\}. \end{aligned} \quad (4.2)$$

In particular, if $s \geq 2$ or $s < 0$, then

$$\left| \frac{A(a^s, b^s) + A^s(a, b)}{2} - L^s(a, b) \right| \leq \frac{b-a}{8} |s| A(a^{s-1}, b^{s-1}). \quad (4.3)$$

Taking $f(x) = \ln x$ for $x > 0$ in Theorem 3.1 results in the following inequalities for means.

Theorem 4.2. For $a, b > 0$ and $q \geq 1$, one has

$$\begin{aligned} & \left| \frac{\ln G(a, b) + \ln A(a, b)}{2} - \ln I(a, b) \right| \\ & \leq \frac{b-a}{16} \left[\frac{1}{4(q+1)(q+2)} \right]^{1/q} \left[\left(\frac{2q+5}{a^q} + \frac{2q+3}{b^q} \right)^{1/q} + \left(\frac{1}{a^q} + \frac{4q+7}{b^q} \right)^{1/q} \right. \\ & \quad \left. + \left(\frac{4q+7}{a^q} + \frac{1}{b^q} \right)^{1/q} + \left(\frac{2q+3}{a^q} + \frac{2q+5}{b^q} \right)^{1/q} \right]. \end{aligned} \quad (4.4)$$

In particular,

$$\left| \frac{\ln A(a, b) + \ln G(a, b)}{2} - \ln I(a, b) \right| \leq \frac{b-a}{8} \frac{1}{H(a, b)}. \quad (4.5)$$

Finally, we can establish an inequality for Heronian mean as follows.

Theorem 4.3. For $b > a > 0$, $\omega \geq 0$, and $s \geq 4$ or $0 \neq s < 1$, one has

$$\begin{aligned} & \left| \frac{H_{\omega, s}^s(a, b)}{H(a^s, b^s)} + H_{\omega, (s/2)+1}^{\left(\frac{b}{a} + \frac{a}{b}, 1\right)} - H_{\omega, s}^s\left(\frac{L(a^2, b^2)}{G^2(a, b)}, 1\right) \right| \\ & \leq \frac{b-a}{4} |s| \left[\frac{A(a, b)A(a^{2(s-1)}, b^{2(s-1)})}{(\omega+2)G^{2s}(a, b)} + \frac{\omega A(a, b)A(a^{s-2}, b^{s-2})}{2(\omega+2)G^s(a, b)} \right]. \end{aligned} \quad (4.6)$$

Proof. Let $f(x) = (x^s + \omega x^{s/2} + 1)/(\omega + 2)$ for $x > 0$ and $s \notin (1, 4)$. Then

$$\begin{aligned} f'(x) &= \frac{s}{\omega+2} \left(x^{s-1} + \frac{\omega}{2} x^{s/2-1} \right), \\ |f'(x)|'' &= \frac{x^{(s/2)-3} |s|}{\omega+2} \left[\frac{\omega}{8} (s-2)(s-4) + (s-1)(s-2)x^{s/2} \right] \geq 0. \end{aligned} \quad (4.7)$$

In virtue of Corollary 3.3, it follows that

$$\begin{aligned}
 & \left| \frac{1}{2} \left[\frac{f(b/a) + f(a/b)}{2} + f\left(\frac{b/a + a/b}{2}\right) \right] - \frac{1}{b/a - a/b} \int_{a/b}^{b/a} f(x) dx \right| \\
 &= \left| \frac{1}{2} \left\{ \frac{1}{2} \left[\frac{b^s + \omega(ab)^{s/2} + a^s}{a^s(\omega + 2)} + \frac{a^s + \omega(ab)^{s/2} + b^s}{b^s(\omega + 2)} \right] \right. \right. \\
 &\quad \left. \left. + \frac{(b/a + a/b)^s + \omega(b/a + a/b)^{s/2} + 1}{\omega + 2} \right\} \right. \\
 &\quad \left. - \frac{1}{\omega + 2} \left[\frac{(b/a)^{s+1} - (a/b)^{s+1}}{(s+1)(b/a - a/b)} + \omega \frac{(b/a)^{s/2+1} - (a/b)^{s/2+1}}{(s/2+1)(b/a - a/b)} + 1 \right] \right| \\
 &= \left| \frac{H_{\omega,s}^s(a,b)}{H(a^s,b^s)} + H_{\omega,(s/2)+1}^{(s/2)+1}\left(\frac{b}{a} + \frac{a}{b}, 1\right) - H_{\omega,s}^s\left(\frac{L(a^2,b^2)}{G^2(a,b)}, 1\right) \right|.
 \end{aligned} \tag{4.8}$$

On the other hand, we have

$$\begin{aligned}
 & \frac{b/a - a/b}{16} \left[\left| f'\left(\frac{a}{b}\right) \right| + \left| f'\left(\frac{b}{a}\right) \right| \right] \\
 &= \frac{b^2 - a^2}{16(\omega + 2)ab} \left\{ |s| \left[\left(\frac{b}{a}\right)^{s-1} + \left(\frac{a}{b}\right)^{s-1} \right] + \frac{\omega}{2} |s| \left[\left(\frac{b}{a}\right)^{s/2-1} + \left(\frac{a}{b}\right)^{s/2-1} \right] \right\} \\
 &= \frac{(b-a)|s|}{4} \left[\frac{A(a,b)A(a^{2(s-1)}, b^{2(s-1)})}{(\omega + 2)G^{2s}(a,b)} + \frac{\omega A(a,b)A(a^{s-2}, b^{s-2})}{2(\omega + 2)G^s(a,b)} \right].
 \end{aligned} \tag{4.9}$$

The proof is complete. $\square$

Remark 4.4. Some inequalities of Hermite-Hadamard type were also obtained in [6–9] by the authors.

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