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Research Article

Approximating Polynomials for Functions of Weighted Smirnov-Orlicz Spaces

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Let G_0 and G_∞ be, respectively, bounded and unbounded components of a plane curve Γ satisfying Dini's smoothness condition. In addition to partial sum of Faber series of f belonging to weighted Smirnov-Orlicz space $E_{M,\omega}(G_0)$, we prove that interpolating polynomials and Poisson polynomials are near best approximant for f. Also considering a weighted fractional moduli of smoothness, we obtain direct and converse theorems of trigonometric polynomial approximation in Orlicz spaces with Muckenhoupt weights. On the bases of these approximation theorems, we prove direct and converse theorems of approximation, respectively, by algebraic polynomials and rational functions in weighted Smirnov-Orlicz spaces $E_{M,\omega}(G_0)$ and $E_{M,\omega}(G_\infty)$.

1. Introduction

Let G_0 and G_∞ be, respectively, bounded and unbounded components of a closed rectifiable curve Γ of complex plane $\mathbb{C}$. Without loss of generality we may suppose that $0 \in G_0$. By Riemann conformal mapping theorem [1, page 26], if Γ is connected Jordan curve that consists of more than one point, there exists a conformal mapping $\varphi_0 : \mathbb{D} \to G_0$ of complex unit disc $\mathbb{D} := \{w \in \mathbb{C} : |w| = 1\}$ onto G_0. Let $\gamma_r := \varphi_0(\{w \in \mathbb{C} : |w| = r\})$ for a given $r \in (0,1)$. We denote by $E^p(G_0)$, $1 \le p \le \infty$, Smirnov's classes of analytic functions $f : G_0 \to \mathbb{C}$ satisfying

$$\begin{gathered}\sup_{r\in(0,1)} \int_{\gamma_r} \left|f(z)\right|^p |dz| < c, \quad \text{if } 1 \le p < \infty, \\ \max_{z\in G_0} \left|f(z)\right| < C, \quad \text{if } p = \infty,\end{gathered} \tag{1.1}$$

where positive constant c is independent of r.

It is well known that $E^p(G_0) \subset E^1(G_0)$ for every $1 \leq p < \infty$ and every function $f \in E^1(G_0)$ has a nontangential boundary values a.e. on Γ, the boundary function belongs to Lebesgue space $L^1(\Gamma)$ on Γ. If $1 \leq p < \infty$, then $E^p(G_0)$ is a Banach space with the norm

$$\|f\|_{p,\Gamma} := \left\{ \frac{1}{2\pi} \int_\Gamma |f(z)|^p |dz| \right\}^{1/p}. \tag{1.2}$$

Smirnov classes $E^p(G_\infty)$, $1 \leq p \leq \infty$, of analytic functions $f : G_\infty \to \mathbb{C}$ can be defined similarly and $E^p(G_\infty)$ are fulfilling the same above properties to that of $E^p(G_0)$.

A smooth Jordan curve Γ will be called *Dini-smooth*, if the function $\theta(s)$, the angle between the tangent line and the positive real axis expressed as a function of arclength s, has modulus of continuity $\Omega(\theta, s)$ satisfying the Dini condition

$$\int_0^\delta \frac{\Omega(\theta, s)}{s} ds < \infty, \quad \delta > 0. \tag{1.3}$$

A Jordan curve Γ will be called *Radon curve*, if $\theta(s)$ has bounded variation and it does not contain cusp point.

Main approximation problems in the spaces $E^p(G_0)$, $1 \leq p \leq \infty$, were dealt with by several mathematicians so far. Walsh and Russell gave [2] results in $E^p(G_0)$, $1 < p < \infty$, for algebraic polynomial approximation orders in case of analytic boundary. Al'per proved [3] direct and converse approximation theorems by algebraic polynomials in $E^p(G_0)$, $1 < p < \infty$, for Dini-smooth boundary. Kokilashvili improved [4] to Al'per's direct and converse results of algebraic polynomial approximation, and then considering Regular curves that Cauchy's Singular Integral Operator is bounded (corners are permitted), he obtained [5] improved direct and converse approximation theorems in Smirnov spaces $E^p(G_0)$, $1 < p < \infty$. Andersson proved [6] that Kokilashvili's results also holds in $E^1(G_0)$. When the boundary is a regular curve, approximation of functions of $E^p(G_0)$, $1 < p < \infty$, by partial sum of Faber series was obtained by Israfilov in [7, 8]. These results are generalized to Muckenhoupt weighted Smirnov's spaces in [9–12]. Approximation properties of Faber series in so-called weighted and unweighted Smirnov-Orlicz spaces are investigated in [13–20]. Most of the above results use the partial sums of Faber series as approximation tool. Interpolating polynomials [16] and Poisson polynomials [21] can be also considered as an approximating polynomial. In the present paper we obtain that in addition to partial sums of Faber series of f belonging to weighted Smirnov-Orlicz space $E_{M,\omega}(G_0)$, interpolating polynomials and Poisson polynomials are near best approximant for f. Also considering a weighted fractional moduli of smoothness, we obtain in Section 2 direct and converse theorems of trigonometric polynomial approximation in Orlicz spaces with Muckenhoupt weights. On the bases of these approximation theorems we prove in Section 3 direct and converse theorems of approximation, respectively, by algebraic polynomials and rational functions in weighted Smirnov-Orlicz spaces $E_{M,\omega}(G_0)$ and $E_{M,\omega}(G_\infty)$.

Throughout the work, we will denote by c, C, the constants that are different in different places.

2. Approximation Theorems in Weighted Orlicz Space

A function Φ is called *Young function* if Φ is even, continuous, nonnegative in $\mathbb{R}$, increasing on $(0,\infty)$ such that

$$\Phi(0) = 0, \quad \lim_{x\to\infty} \Phi(x) = \infty. \tag{2.1}$$

A Young function Φ is said to satisfy Δ_2 *condition* ($\Phi \in \Delta_2$) if there is a constant $c > 0$ such that

$$\Phi(2x) \le c\Phi(x) \tag{2.2}$$

for all $x \in \mathbb{R}$.

Two Young functions Φ and Φ_1 are said to be *equivalent* if there are $c, C > 0$ such that

$$\Phi_1(cx) \le \Phi(x) \le \Phi_1(Cx), \quad \forall x > 0. \tag{2.3}$$

A function $M : [0,\infty) \to [0,\infty)$ is said to be *quasiconvex* if there exist a convex Young function Φ and a constant $c \ge 1$ such that

$$\Phi(x) \le M(x) \le \Phi(cx), \quad \forall x \ge 0, \tag{2.4}$$

holds.

A nonnegative function ω defined on $\mathsf{T} := [0, 2\pi]$ will be called weight if ω is measurable and a.e. positive. Let M be a quasiconvex Young function. We denote by $\widetilde{L}_{M,\omega}(\mathsf{T})$ the class of Lebesgue measurable functions $f : \mathsf{T} \to \mathbb{R}$ satisfying the condition

$$\int_{\mathsf{T}} M(|f(x)|)\omega(x)dx < \infty. \tag{2.5}$$

The linear span of the *weighted Orlicz class* $\widetilde{L}_{M,\omega}(\mathsf{T})$, denoted by $L_{M,\omega}(\mathsf{T})$, becomes a normed space with the *Orlicz* norm

$$\|f\|_{M,\omega} := \sup\left\{\int_{\mathsf{T}} |f(x)g(x)|\omega(x)dx : \int_{\mathsf{T}} \widetilde{M}(|g|)\omega(x)dx \le 1\right\}, \tag{2.6}$$

where $\widetilde{M}(y) := \sup_{x\ge 0}(xy - M(x)), y \ge 0$, is the *complementary function* of M.

If M is quasiconvex and $\widetilde{M}$ is its complementary function, then *Young's inequality* holds

$$xy \le M(x) + \widetilde{M}(y), \quad x, y \ge 0. \tag{2.7}$$

For a quasiconvex function M we define the indice $p(M)$ of M as

$$\frac{1}{p(M)} := \inf\{p : p > 0,\ M^p \text{ is quasiconvex}\},$$
$$p'(M) := \frac{p(M)}{p(M)-1}. \tag{2.8}$$

The indice $p(M)$ was first defined and used by Gogatishvili and Kokilashvili in [22] to obtain weighted inequalities for maximal function. We note that the indice $p(M)$ is much more convenient than Gustavsson and Peetre's lower index and Boyd's upper index. If $\omega \in A_{p(M)}$, then it can be easily seen that $L_{M,\omega}(\mathsf{T}) \subset L^1(\mathsf{T})$ and $L_{M,\omega}(\mathsf{T})$ becomes a Banach space with the Orlicz norm. The Banach space $L_{M,\omega}(\mathsf{T})$ is called *weighted Orlicz space*.

We define the *Luxemburg* functional as

$$\|f\|_{(M),\omega} := \inf\left\{\tau > 0 : \int_{\mathsf{T}} M\left(\frac{|f(x)|}{\tau}\right)\omega(x)dx \le 1\right\}. \tag{2.9}$$

There exist [23, page 23] constants $c, C > 0$ such that

$$c\|f\|_{(M),\omega} \le \|f\|_{M,\omega} \le C\|f\|_{(M),\omega}. \tag{2.10}$$

For a weight ω we denote by $L^p(\mathsf{T},\omega)$ the class of measurable functions on T such that $\omega^{1/p}f$ belongs to Lebesgue space $L^p(\mathsf{T})$ on T. We set $\|f\|_{p,\omega} := \|\omega^{1/p}f\|_p$ for $f \in L^p(\mathsf{T},\omega)$.

A 2π-periodic weight function ω belongs to the Muckenhoupt class A_p, $1 < p < \infty$, if

$$\left(\frac{1}{|J|}\int_J \omega(x)dx\right)\left(\frac{1}{|J|}\int_J \omega^{-1/(p-1)}(x)dx\right)^{p-1} \le c \tag{2.11}$$

with a finite constant c independent of J, where J is any subinterval of T and $|J|$ denotes the length of J.

We will denote by $QC_2^\theta(0,1)$ a class of functions g satisfying Δ_2 condition such that g^θ is quasiconvex for some $\theta \in (0,1)$.

In the present section we consider the trigonometric polynomial approximation problems for functions and its fractional derivatives in the spaces $L_{M,\omega}(\mathsf{T})$, $\omega \in A_{p(M)}$, where $M \in QC_2^\theta(0,1)$. We prove a Jackson type direct theorem and a converse theorem of trigonometric approximation with respect to the fractional order moduli of smoothness in weighted Orlicz spaces with Muckenhoupt weights. In the particular case, we obtain a constructive characterization of Lipschitz class in these spaces.

In weighted Lebesgue and Orlicz spaces with Muckenhoupt weights, these results were investigated in [24–29]. For more general doubling weights, some of these problems were investigated in [30]. Jackson and converse inequalities were proved for Lebesgue spaces with Freud weight in [31]. For a general discussion of weighted polynomial approximation, we can refer to the books [32, 33].

Let $b_0 = 0$, $a_k, b_k \in \mathbb{R}$, $c_k = (a_k - ib_k)/2$, $c_{-k} = (a_k + ib_k)/2$, $c_0 = a_0/2$

$$f(x) \sim \sum_{k=-\infty}^{\infty} c_k e^{ikx} = \sum_{k=0}^{\infty} (a_k \cos kx + b_k \sin kx), \tag{2.12}$$

$$\widetilde{f}(x) \sim \sum_{k=1}^{\infty} (a_k \sin kx - b_k \cos kx) \tag{2.13}$$

be the *Fourier* and the *conjugate Fourier series* of $f \in L^1(\mathsf{T})$, respectively. Putting $A_k(x) := c_k e^{ikx}$ in (2.12), we define for $n = 0, 1, 2, \ldots$

$$\begin{aligned} S_n(f) := S_n(x, f) &:= \sum_{k=0}^{n} (A_k(x) + A_{-k}(x)) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx), \\ R_n^{\langle\alpha\rangle}(f, x) &:= \sum_{k=0}^{n} \left(1 - \left(\frac{k}{n+1}\right)^{\alpha}\right)(A_k(x) + A_{-k}(x)), \quad \alpha \in \mathbb{R}^+, \\ \Theta_m^{\langle\alpha\rangle} &:= \frac{1}{1 - ((m+1)/(2m+1))^r} R_{2m}^{\langle\alpha\rangle} - \frac{1}{((2m+1)/(m+1))^r - 1} R_m^{\langle\alpha\rangle}, \\ &\qquad m = 1, 2, 3, \ldots. \end{aligned} \tag{2.14}$$

For a given $f \in L^1(\mathsf{T})$, assuming

$$\int_{\mathsf{T}} f(x)\,dx = 0, \tag{2.15}$$

we define αth *fractional* ($\alpha \in \mathbb{R}^+$) *integral* of f as [34, v.2, page 134]

$$I_\alpha(x, f) := \sum_{k \in \mathbb{Z}^*} c_k (ik)^{-\alpha} e^{ikx}, \tag{2.16}$$

where

$$(ik)^{-\alpha} := |k|^{-\alpha} e^{(-1/2)\pi i \alpha \operatorname{sign} k} \tag{2.17}$$

as principal value.

Let $\alpha \in \mathbb{R}^+$ be given. We define *fractional derivative* of a function $f \in L^1(\mathsf{T})$, satisfying (2.15), as

$$f^{(\alpha)}(x) := \frac{d^{[\alpha]+1}}{dx^{[\alpha]+1}} I_{1+\alpha-[\alpha]}(x, f) \tag{2.18}$$

provided the right hand side exists.

Setting $x, t \in \mathsf{T}$, $r \in \mathbb{R}^+$, $M \in QC_2^\theta(0,1)$, $\omega \in A_{p(M)}$, and $f \in L_{M,\omega}(\mathsf{T})$, we define

$$\begin{aligned}\sigma_t^r f(x) &:= (I - \sigma_t)^r f(x) \\ &= \sum_{k=0}^{\infty} (-1)^k \binom{r}{k} \frac{1}{(2t)^k} \int_{-t}^{t} \cdots \int_{-t}^{t} f(x + u_1 + \cdots u_k) du_1 \cdots du_k,\end{aligned} \tag{2.19}$$

where $\binom{r}{k} := r(r-1)\cdots(r-k+1)/k!$ for $k \geq 1$ and $\binom{r}{0} := 1$ are Binomial coefficients, $\sigma_t f(x) := (1/2t)\int_{-t}^{t} f(x+u)du$ is Steklov's mean operator, and I is identity operator.

Theorem A (see [23, page 278, Theorem 6.7.1]). *One suppose that L is anyone of the operators S_n, σ_h, and $\widetilde{f}$. If $M \in QC_2^\theta(0,1)$, $\omega \in A_{p(M)}$, and $f \in L_{M,\omega}(\mathsf{T})$, then there exists a constant $c > 0$ such that*

$$\int_{\mathsf{T}} M(|Lf(t)|)\omega(t)dt \leq c \int_{\mathsf{T}} M(|f(t)|)\omega(t)dt \tag{2.20}$$

holds.

Since modular inequality implies the norm inequality, under the conditions of Theorem A, we obtain from (2.20) that

$$\|Lf\|_{M,\omega} \leq c\|f\|_{M,\omega} \tag{2.21}$$

with a constant $c > 0$ independent of f.

By [35, page 14, (1.51)], there exists a constant c depending only on r such that

$$\left|\binom{r}{k}\right| \leq \frac{c}{k^{r+1}}, \quad k = 1, 2, \ldots, \tag{2.22}$$

we have

$$\sum_{k=0}^{\infty} \left|\binom{r}{k}\right| < \infty \tag{2.23}$$

and therefore

$$\|\sigma_t^r f\|_{M,\omega} \leq c\|f\|_{M,\omega} < \infty \tag{2.24}$$

provided $f \in L_{M,\omega}(\mathsf{T})$, $\omega \in A_{p(M)}$, where $M \in QC_2^\theta(0,1)$.

Let $M \in QC_2^{\theta}(0,1)$. For $r \in \mathbb{R}^+$, we define the *fractional modulus of smoothness of index* r for $f \in L_{M,\omega}(\mathbb{T})$, $\omega \in A_{p(M)}$ as

$$\Omega_{M,\omega}^{r}(f,\delta) := \sup_{0<h_i,t<\delta} \left\| \prod_{i=1}^{[r]} (I-\sigma_{h_i})(I-\sigma_t)^{r-[r]} f \right\|_{M,\omega}, \tag{2.25}$$

where $[x]$ denotes the integer part of a real number x.

Since the operator σ_t is bounded in $L_{M,\omega}(\mathbb{T})$, $\omega \in A_{p(M)}$, where $M \in QC_2^{\theta}(0,1)$, we have by (2.24) that

$$\Omega_{M,\omega}^{r}(f,\delta) \le c\|f\|_{M,\omega}, \tag{2.26}$$

where the constant $c > 0$, dependent only on r and M.

Remark 2.1. The modulus of smoothness $\Omega_{M,\omega}^{r}(f,\delta)$, *where* $r \in \mathbb{R}^+$, $M \in QC_2^{\theta}(0,1)$, $\omega \in A_{p(M)}$, $f \in L_{M,\omega}(\mathbb{T})$ has the following properties:

(i) $\Omega_{M,\omega}^{r}(f,\delta)$ *is nonnegative, nondecreasing function of* $\delta \ge 0$ *and subadditive,*

(ii) $\lim_{\delta\to 0} \Omega_{M,\omega}^{r}(f,\delta) = 0$.

For formulations of our results, we need several lemmas.

Lemma A (see [36]). *For* $\alpha \in \mathbb{R}^+$, *we suppose that*

(i) $a_1 + a_2 + \cdots + a_n + \cdots$,

(ii) $a_1 + 2^{\alpha} a_2 + \cdots + n^{\alpha} a_n + \cdots$,

be two series in a Banach space $(B, \|\cdot\|)$. *Let*

$$\begin{aligned} R_n^{\langle\alpha\rangle} &:= \sum_{k=0}^{n} \left(1 - \left(\frac{k}{n+1}\right)^{\alpha}\right) a_k, \\ R_n^{\langle\alpha\rangle *} &:= \sum_{k=0}^{n} \left(1 - \left(\frac{k}{n+1}\right)^{\alpha}\right) k^{\alpha} a_k \end{aligned} \tag{2.27}$$

for $n = 1, 2, \ldots$. *Then,*

$$\left\| R_n^{\langle\alpha\rangle *} \right\| \le c, \quad n = 1, 2, \ldots \tag{2.28}$$

for some $c > 0$ *if and only if there exists a* $R \in B$ *such that*

$$\left\| R_n^{\langle\alpha\rangle} - R \right\| \le \frac{C}{n^{\alpha}}, \tag{2.29}$$

where c *and* C *are constants depending only on one another.*

If $M \in QC_2^{\theta}(0,1)$, $\omega \in A_{p(M)}$, and $f \in L_{M,\omega}(\mathsf{T})$, then from Theorem A(ii) and Abel's transformation we get

$$\left\| R_n^{\langle\alpha\rangle}(f,\cdot) \right\|_{M,\omega} \le c\|f\|_{M,\omega}, \quad n = 1,2,3,\ldots, \ x \in \mathsf{T} \tag{2.30}$$

and therefore from (2.14) and(2.30)

$$\left\| \Theta_n^{\langle\alpha\rangle}(f,\cdot) \right\|_{M,\omega} \le c\|f\|_{M,\omega}, \quad n = 1,2,3,\ldots, \ x \in \mathsf{T}. \tag{2.31}$$

From the property

$$\Theta_m^{\langle\alpha\rangle}(f)(x) = \frac{1}{\sum_{k=m+1}^{2m}\left[(k+1)^{\alpha} - k^{\alpha}\right]} \sum_{k=m+1}^{2m} \left[(k+1)^{\alpha} - k^{\alpha}\right] S_k(x,f), \quad x \in \mathsf{T}, \ f \in L^1(\mathsf{T}) \tag{2.32}$$

it is known that

$$\Theta_m^{\langle\alpha\rangle}(T_m) = T_m \tag{2.33}$$

for $T_m \in \mathcal{T}_m$, $m = 1,2,3,\ldots$.

Lemma 2.2. *Let $T_n \in \mathcal{T}_n$, $n = 1,2,3,\ldots, M \in QC_2^{\theta}(0,1)$, and $\omega \in A_{p(M)}$. If $\alpha \in \mathbb{R}^+$, then there exists a constant $c > 0$ independent of n such that*

$$\left\| T_n^{(\alpha)} \right\|_{M,\omega} \le cn^{\alpha}\|T_n\|_{M,\omega} \tag{2.34}$$

holds.

Proof. Without loss of generality one can assume that $\|T_n\|_{M,\omega} = 1$. Since

$$\begin{gathered} T_n = \sum_{k=0}^{n}(A_k(x) + A_{-k}(x)), \qquad \frac{\widetilde{T}_n}{n^{\alpha}} = \sum_{k=1}^{n}\left[\frac{(A_k(x) - A_{-k}(x))}{n^{\alpha}}\right], \\ \frac{T_n^{(\alpha)}}{(in)^{\alpha}} = \sum_{k=1}^{n} k^{\alpha}\left[\frac{(A_k(x) - A_{-k}(x))}{n^{\alpha}}\right] \end{gathered} \tag{2.35}$$

we have by (2.30) and Theorem A(iii) that

$$\left\| R_m^{\langle\alpha\rangle}\left(\frac{\widetilde{T}_n}{n^{\alpha}}\right) \right\|_{M,\omega} \le \frac{c}{n^{\alpha}}\left\|\widetilde{T}_n\right\|_{M,\omega} \le \frac{c}{n^{\alpha}}\|T_n\|_{M,\omega} = \frac{c}{n^{\alpha}} \tag{2.36}$$

and from Lemma A

$$\left\| R_m^{\langle\alpha\rangle}\left(\frac{T_n^{(\alpha)}}{(in)^\alpha}\right)\right\|_{M,\omega} \le c. \tag{2.37}$$

Hence from (2.33) and (2.31), we find

$$\left\|T_n^{(\alpha)}\right\|_{M,\omega} = n^\alpha \left\|\Theta_m^{\langle\alpha\rangle}\left(\frac{T_n^{(\alpha)}}{(in)^\alpha}\right)\right\|_{M,\omega} \le cn^\alpha \|T_n\|_{M,\omega}. \tag{2.38}$$

General case follows immediately from this. □

Let $M \in QC_2^\theta(0,1)$. We denote by $W_M^\alpha(\mathsf{T},\omega)$, $\alpha > 0$, $\omega \in A_{p(M)}$, the linear space of 2π-periodic real valued functions $f \in L_{M,\omega}(\mathsf{T})$ such that $f^{(\alpha)} \in L_{M,\omega}(\mathsf{T})$.

Lemma 2.3. *Let $M \in QC_2^\theta(0,1)$. If $f \in W_M^\alpha(\mathsf{T},\omega)$ with $\omega \in A_{p(M)}$ and $\alpha \ge 0$, then for $n = 0,1,2,\ldots$, there is a constant $c > 0$ dependent only on α and M such that*

$$\left\|f^{(\alpha)}(\cdot) - S_n^{(\alpha)}(\cdot,f)\right\|_{M,\omega} \le cE_n\left(f^{(\alpha)}\right)_{M,\omega} \tag{2.39}$$

holds.

Proof. If $\alpha = 0$, then from boundedness (see (2.21)) of the operator S_n we get that

$$\|f - S_n f\|_{M,\omega} \le cE_n(f)_{M,\omega}. \tag{2.40}$$

Let $W_n(f) := W_n(x,f) := (1/(n+1))\sum_{\nu=n}^{2n} S_\nu(x,f)$, $n = 0,1,2,\ldots$. Since

$$W_n\left(\cdot, f^{(\alpha)}\right) = W_n^{(\alpha)}(\cdot,f), \tag{2.41}$$

we have

$$\begin{aligned}\left\|f^{(\alpha)}(\cdot) - S_n^{(\alpha)}(\cdot,f)\right\|_{M,\omega} &\le \left\|f^{(\alpha)}(\cdot) - W_n\left(\cdot,f^{(\alpha)}\right)\right\|_{M,\omega} + \left\|S_n^{(\alpha)}(\cdot,W_n(f)) - S_n^{(\alpha)}(\cdot,f)\right\|_{M,\omega} \\ &\quad + \left\|W_n^{(\alpha)}(\cdot,f) - S_n^{(\alpha)}(\cdot,W_n(f))\right\|_{M,\omega} := I_1 + I_2 + I_3.\end{aligned} \tag{2.42}$$

From (2.21) we get the boundedness of W_n in $L_{M,\omega}(\mathsf{T})$ and we have

$$\begin{aligned}I_1 &\le \left\|f^{(\alpha)}(\cdot) - S_n\left(\cdot,f^{(\alpha)}\right)\right\|_{M,\omega} + \left\|S_n(\cdot,f^{(\alpha)}) - W_n(\cdot,f^{(\alpha)})\right\|_{M,\omega} \\ &\le cE_n\left(f^{(\alpha)}\right)_{M,\omega} + \left\|W_n\left(\cdot,S_n\left(f^{(\alpha)}\right) - f^{(\alpha)}\right)\right\|_{M,\omega} \le cE_n\left(f^{(\alpha)}\right)_{M,\omega}.\end{aligned} \tag{2.43}$$

From Lemma 2.2 we get

$$\begin{aligned} I_2 &\le cn^{\alpha}\left\|S_n(\cdot,W_n(f))-S_n(\cdot,f)\right\|_{M,\omega}, \\ I_3 &\le c(2n)^{\alpha}\left\|W_n(\cdot,f)-S_n(\cdot,W_n(f))\right\|_{M,\omega} \le c(2n)^{\alpha}E_n(W_n(f))_{M,\omega}. \end{aligned} \tag{2.44}$$

Now we have

$$\begin{aligned} \left\|S_n(\cdot,W_n(f))-S_n(\cdot,f)\right\|_{M,\omega} &\le \left\|S_n(\cdot,W_n(f))-W_n(\cdot,f)\right\|_{M,\omega} \\ &\quad + \left\|W_n(\cdot,f)-f(\cdot)\right\|_{M,\omega} + \left\|f(\cdot)-S_n(\cdot,f)\right\|_{M,\omega} \\ &\le cE_n(W_n(f))_{M,\omega} + cE_n(f)_{M,\omega}. \end{aligned} \tag{2.45}$$

Since

$$E_n(W_n(f))_{M,\omega} \le cE_n(f)_{M,\omega}, \tag{2.46}$$

we get

$$\begin{aligned} \left\|f^{(\alpha)}(\cdot)-T_n^{(\alpha)}(\cdot,f)\right\|_{M,\omega} &\le cE_n\left(f^{(\alpha)}\right)_{M,\omega} + cn^{\alpha}E_n(W_n(f))_{M,\omega} + cn^{\alpha}E_n(f)_{M,\omega} \\ &\quad + c(2n)^{\alpha}E_n(W_n(f))_{M,\omega} \le cE_n\left(f^{(\alpha)}\right)_{M,\omega} + Cn^{\alpha}E_n(f)_{M,\omega}. \end{aligned} \tag{2.47}$$

Now we show that

$$E_n(f)_{M,\omega} \le \frac{c}{(n+1)^{\alpha}}E_n\left(f^{(\alpha)}\right)_{M,\omega}. \tag{2.48}$$

For this we set

$$A_k(x,f) := a_k\cos kx + b_k\sin kx. \tag{2.49}$$

For given $f \in L_{M,\omega}(\mathrm{T})$ and $\varepsilon > 0$, by Lemma 3 of [37], there exists a trigonometric polynomial T such that

$$\int_{\mathrm{T}} M(|f(x)-T(x)|)\omega(x)dx < \varepsilon \tag{2.50}$$

which by (2.7) this implies that

$$\left\|f-T\right\|_{M,\omega} < \varepsilon, \tag{2.51}$$

and hence we obtain

$$E_n(f)_{M,\omega} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{2.52}$$

In this case from (2.40) we have

$$f(x) = \sum_{k=0}^{\infty} A_k(x, f) \tag{2.53}$$

in $\|\cdot\|_{M,\omega}$ norm. If $k = 1, 2, 3, \ldots$, then

$$\begin{aligned} A_k(x, f) &= A_k\left(x + \frac{\alpha\pi}{2k}, f\right)\cos\frac{\alpha\pi}{2} + A_k\left(x + \frac{\alpha\pi}{2k}, \tilde{f}\right)\sin\frac{\alpha\pi}{2}, \\ A_k\left(x, f^{(\alpha)}\right) &= k^{\alpha} A_k\left(x + \frac{\alpha\pi}{2k}, f\right). \end{aligned} \tag{2.54}$$

Hence,

$$\begin{aligned} \sum_{k=0}^{\infty} A_k(x, f) &= A_0(x, f) + \cos\frac{\alpha\pi}{2}\sum_{k=1}^{\infty} A_k\left(x + \frac{\alpha\pi}{2k}, f\right) + \sin\frac{\alpha\pi}{2}\sum_{k=1}^{\infty} A_k\left(x + \frac{\alpha\pi}{2k}, \tilde{f}\right) \\ &= A_0(x, f) + \cos\frac{\alpha\pi}{2}\sum_{k=1}^{\infty} k^{-\alpha} A_k\left(x, f^{(\alpha)}\right) + \sin\frac{\alpha\pi}{2}\sum_{k=1}^{\infty} k^{-\alpha} A_k\left(x, \tilde{f}^{(\alpha)}\right). \end{aligned} \tag{2.55}$$

Therefore,

$$f(x) - S_n(x, f) = \cos\frac{\alpha\pi}{2}\sum_{k=n+1}^{\infty}\frac{1}{k^{\alpha}} A_k\left(x, f^{(\alpha)}\right) + \sin\frac{\alpha\pi}{2}\sum_{k=n+1}^{\infty}\frac{1}{k^{\alpha}} A_k\left(x, \tilde{f}^{(\alpha)}\right). \tag{2.56}$$

Since

$$\begin{aligned} \sum_{k=n+1}^{\infty} k^{-\alpha} A_k\left(x, f^{(\alpha)}\right) &= \sum_{k=n+1}^{\infty} k^{-\alpha}\left[\left(S_k\left(\cdot, f^{(\alpha)}\right) - f^{(\alpha)}(\cdot)\right) - \left(S_{k-1}\left(\cdot, f^{(\alpha)}\right) - f^{(\alpha)}(\cdot)\right)\right] \\ &= \sum_{k=n+1}^{\infty}\left(k^{-\alpha} - (k+1)^{-\alpha}\right)\left(S_k\left(\cdot, f^{(\alpha)}\right) - f^{(\alpha)}(\cdot)\right) \\ &\quad - (n+1)^{-\alpha}\left(S_n\left(\cdot, f^{(\alpha)}\right) - f^{(\alpha)}(\cdot)\right), \\ \sum_{k=n+1}^{\infty} k^{-\alpha} A_k\left(x, \tilde{f}^{(\alpha)}\right) &= \sum_{k=n+1}^{\infty}\left(k^{-\alpha} - (k+1)^{-\alpha}\right)\left(S_k\left(\cdot, \tilde{f}^{(\alpha)}\right) - \tilde{f}^{(\alpha)}(\cdot)\right) \\ &\quad - (n+1)^{-\alpha}\left(S_n\left(\cdot, \tilde{f}^{(\alpha)}\right) - \tilde{f}^{(\alpha)}(\cdot)\right), \end{aligned} \tag{2.57}$$

we obtain

$$\begin{aligned}
\|f(\cdot)-S_n(\cdot,f)\|_{M,\omega} &\le \sum_{k=n+1}^{\infty}\left(k^{-\alpha}-(k+1)^{-\alpha}\right)\left\|S_k\left(\cdot,f^{(\alpha)}\right)-f^{(\alpha)}(\cdot)\right\|_{M,\omega} \\
&\quad+(n+1)^{-\alpha}\left\|S_n\left(\cdot,f^{(\alpha)}\right)-f^{(\alpha)}(\cdot)\right\|_{M,\omega} \\
&\quad+\sum_{k=n+1}^{\infty}\left(k^{-\alpha}-(k+1)^{-\alpha}\right)\left\|S_k\left(\cdot,\tilde{f}^{(\alpha)}\right)-\tilde{f}^{(\alpha)}(\cdot)\right\|_{M,\omega} \\
&\quad+(n+1)^{-\alpha}\left\|S_n\left(\cdot,\tilde{f}^{(\alpha)}\right)-\tilde{f}^{(\alpha)}(\cdot)\right\|_{M,\omega} \\
&\le c\left[\sum_{k=n+1}^{\infty}\left(k^{-\alpha}-(k+1)^{-\alpha}\right)E_k(f)_{M,\omega}+(n+1)^{-\alpha}E_n\left(f^{(\alpha)}\right)_{M,\omega}\right] \\
&\quad+C\left[\sum_{k=n+1}^{\infty}\left(k^{-\alpha}-(k+1)^{-\alpha}\right)E_k(f)_{M,\omega}+(n+1)^{-\alpha}E_n\left(\tilde{f}^{(\alpha)}\right)_{M,\omega}\right].
\end{aligned} \tag{2.58}$$

Consequently,

$$\begin{aligned}
\|f(x)-S_n(x,f)\|_{M,\omega} &\le cE_k\left(f^{(\alpha)}\right)_{M,\omega}\left[\sum_{k=n+1}^{\infty}\left(k^{-\alpha}-(k+1)^{-\alpha}\right)+(n+1)^{-\alpha}\right] \\
&\quad+cE_n\left(\tilde{f}^{(\alpha)}\right)_{M,\omega}\left[\sum_{k=n+1}^{\infty}\left(k^{-\alpha}-(k+1)^{-\alpha}\right)+(n+1)^{-\alpha}\right] \\
&\le cE_n\left(f^{(\alpha)}\right)_{M,\omega}\left[\sum_{k=n+1}^{\infty}\left(k^{-\alpha}-(k+1)^{-\alpha}\right)+(n+1)^{-\alpha}\right] \\
&\le \frac{c}{(n+1)^{\alpha}}E_n\left(f^{(\alpha)}\right)_{M,\omega},
\end{aligned} \tag{2.59}$$

and (2.48) holds. Now (2.47) and (2.48) imply the result. □

Lemma 2.4. *Let $T_n \in \mathcal{T}_n$, $n=0,1,2,\ldots$, $M \in QC_2^{\theta}(0,1)$, and $\omega \in A_{p(M)}$. If $\alpha \in \mathbb{R}^+$, then*

$$\Omega_{M,\omega}^{\alpha}\left(T_n,\frac{\pi}{n+1}\right) \le \frac{c}{(n+1)^{\alpha}}\left\|T_n^{(\alpha)}\right\|_{M,\omega} \tag{2.60}$$

hold, where the constant $c>0$ is dependent only on α and M.

Proof. First we prove that if $0<\alpha<\beta$, then

$$\Omega_{M,\omega}^{\beta}(T_n,\cdot) \le c\Omega_{M,\omega}^{\alpha}(T_n,\cdot). \tag{2.61}$$

It is easily seen that if $\alpha \le \beta$, $\alpha, \beta \in \mathbb{Z}^+$, then (2.61) holds. Now, we assume $0 < \alpha < \beta \le 1$. In this case, putting $K(x) := \sigma_t^{\alpha} T_n(x)$, we have

$$\begin{aligned}
\sigma_t^{\beta-\alpha} K(x) &= \sum_{j=0}^{\infty} (-1)^j \binom{\beta-\alpha}{j} \frac{1}{(2t)^j} \int_{-t}^{t} \cdots \int_{-t}^{t} K(x+u_1+\cdots u_j)\, du_1 \cdots du_j \\
&= \sum_{j=0}^{\infty} (-1)^j \binom{\beta-\alpha}{j} \frac{1}{(2t)^j} \\
&\quad \times \int_{-t}^{t} \cdots \int_{-t}^{t} \left[\sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} \frac{1}{(2t)^k} \right. \\
&\qquad \left. \times \int_{-t}^{t} \cdots \int_{-t}^{t} T_n(x+u_1+\cdots u_j+u_{j+1}+\cdots u_{j+k})\, du_{j+1} \cdots du_{j+k} \right] du_1 \cdots du_j \\
&= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (-1)^{j+k} \binom{\beta-\alpha}{j} \binom{\alpha}{k} \left[\frac{1}{(2t)^{j+k}} \int_{-t}^{t} \cdots \int_{-t}^{t} T_n(x+u_1+\cdots u_{j+k})\, du_1 \cdots du_{j+k} \right] \\
&= \sum_{\upsilon=0}^{\infty} \left\{ \sum_{\mu=0}^{\upsilon} (-1)^{\upsilon-\mu} \binom{\beta-\alpha}{\upsilon-\mu} \binom{\alpha}{\mu} \frac{1}{(2t)^{\upsilon}} \int_{-t}^{t} \cdots \int_{-t}^{t} T_n\{x+u_1+\cdots u_{\upsilon}\}\, du_1 \cdots du_{\upsilon} \right\} \\
&= \sum_{\upsilon=0}^{\infty} (-1)^{\upsilon} \frac{1}{(2t)^{\upsilon}} \int_{-t}^{t} \cdots \int_{-t}^{t} T_n(x+u_1+\ldots u_{\upsilon})\, du_1 \cdots du_{\upsilon} \sum_{\mu=0}^{\upsilon} \binom{\beta-\alpha}{\upsilon-\mu} \binom{\alpha}{\mu} \\
&= \sum_{\upsilon=0}^{\infty} (-1)^{\upsilon} \binom{\beta}{\upsilon} \frac{1}{(2t)^{\upsilon}} \int_{-t}^{t} \cdots \int_{-t}^{t} T_n(x+u_1+\ldots u_{\upsilon})\, du_1 \cdots du_{\upsilon} = \sigma_t^{\beta} T_n(x).
\end{aligned} \tag{2.62}$$

Then,

$$\left\| \sigma_t^{\beta} T_n \right\|_{M,\omega} = \left\| \sigma_t^{\beta-\alpha} K \right\|_{M,\omega} \le c \left\| \sigma_t^{\alpha} T_n \right\|_{M,\omega}, \tag{2.63}$$

and hence (2.61) holds. We note that if $r_1, r_2 \in \mathbb{Z}^+$, $\alpha_1, \beta_1 \in (0,1)$ taking $\alpha := r_1 + \alpha_1, \beta := r_2 + \beta_1$ for the remaining cases $r_1 = r_2$, $\alpha_1 < \beta_1$ or $r_1 < r_2$, $\alpha_1 < \beta_1$ or $r_1 < r_2$, $\alpha_1 > \beta_1$, it can easily be obtained from the last inequality that the required inequality (2.61) holds. Now we will show that if $\varrho \in W_M^{(2r)}(\mathrm{T}, \omega)$, $r = 1, 2, 3, \ldots$, then

$$\Omega_{M,\omega}^{r}(\varrho, \delta) \le c\delta^{2r} \left\| \varrho^{(2r)} \right\|_{M,\omega}. \tag{2.64}$$

Putting

$$g(x) := \prod_{i=2}^{r} (I - \sigma_{h_i}) \varrho(x), \tag{2.65}$$

we have

$$(I-\sigma_{h_1})g(x)=\prod_{i=1}^{r}(I-\sigma_{h_i})\varrho(x),$$
$$\prod_{i=1}^{r}(I-\sigma_{h_i})\varrho(x)=\frac{1}{2h_1}\int_{-h_1}^{h_1}\big(g(x)-g(x+t)\big)dt=-\frac{1}{8h_1}\int_0^{h_1}\int_0^t\int_{-u}^{u}g''(x+s)ds\,du\,dt. \tag{2.66}$$

Therefore,

$$\begin{aligned}
&\left\|\prod_{i=1}^{r}(I-\sigma_{h_i})\varrho(x)\right\|_{M,\omega}\\
&\quad=\frac{1}{8h_1}\sup\left\{\int_{\mathrm{T}}\left|\int_0^{h_1}\int_0^t\int_{-u}^{u}g''(x+s)ds\,du\,dt\right||v(x)|\omega(x)dx:\int_{\mathrm{T}}\widetilde{M}(|v(x)|)\omega(x)dx\le 1\right\}\\
&\quad\le\frac{1}{8h_1}\int_0^{h_1}\int_0^t 2u\left\|\frac{1}{2u}\int_{-u}^{u}g''(x+s)ds\right\|_{M,\omega}du\,dt\\
&\quad\le\frac{c}{8h_1}\int_0^{h_1}\int_0^t 2u\|g''\|_{M,\omega}\,du\,dt=ch_1^2\|g''\|_{M,\omega}.
\end{aligned}\tag{2.67}$$

Since

$$g''(x)=\prod_{i=2}^{r}(I-\sigma_{h_i})\varrho''(x), \tag{2.68}$$

we obtain that

$$\begin{aligned}
\Omega_{M,\omega}^{r}(\varrho,\delta)&\le\sup_{\substack{0<h_i\le\delta\\ i=1,2,\dots,r}}ch_1^2\|g''\|_{M,\omega}=c\delta^2\left\|\prod_{i=2}^{r}(I-\sigma_{h_i})\varrho''(x)\right\|_{M,\omega}\\
&=c\delta^2\sup_{\substack{0<h_i\le\delta\\ i=2,\dots,r}}\left\|\prod_{i=2}^{r}(I-\sigma_{h_i})\varrho''(x)\right\|_{M,\omega}=c\delta^2\Omega_{M,\omega}^{r-1}(\varrho'',\delta)\\
&\le c\delta^4\Omega_{M,\omega}^{r-2}\left(\varrho^{(4)},\delta\right)\le\cdots\le C\delta^{2r}\left\|\varrho^{(2r)}\right\|_{M,\omega}.
\end{aligned}\tag{2.69}$$

Using (2.61), (2.64), and Lemma 2.2, we get

$$\begin{aligned}
\Omega_{M,\omega}^{\alpha}\left(T_n,\frac{\pi}{n+1}\right)&\le c\Omega_{M,\omega}^{[\alpha]}\left(T_n,\frac{\pi}{n+1}\right)\le c\left(\frac{\pi}{n+1}\right)^{2[\alpha]}\left\|T_n^{(2[\alpha])}\right\|_{M,\omega}\\
&\le\frac{c}{(n+1)^{2[\alpha]}}(n+1)^{[\alpha]-(\alpha-[\alpha])}\left\|T_n^{(\alpha)}\right\|_{M,\omega}=\frac{c}{(n+1)^{\alpha}}\left\|T_n^{(\alpha)}\right\|_{M,\omega}
\end{aligned}\tag{2.70}$$

which is the required result (2.60) for $\alpha \geq 1$. On the other hand in case of $0 < \alpha < 1$ the inequality (2.60) can be obtained by Marcinkiewicz Multiplier Theorem for $L_{M,\omega}(\mathsf{T})$ where $M \in QC_2^{\theta}(0,1)$ and $\omega \in A_{p(M)}$. □

Definition 2.5. For $f \in L_{M,\omega}(\mathsf{T})$, $\delta > 0$, and $r = 1,2,3,\ldots$, the Peetre$K$-functional is defined as

$$K(\delta, f; L_{M,\omega}(\mathsf{T}), W_M^{\alpha}(\mathsf{T},\omega)) := \inf_{g \in W_M^{\alpha}(\mathsf{T},\omega)} \left\{ \|f-g\|_{M,\omega} + \delta \left\| g^{(r)} \right\|_{M,\omega} \right\}. \tag{2.71}$$

Proposition 2.6. *Let $M \in QC_2^{\theta}(0,1)$, $\omega \in A_{p(M)}$, and $f \in L_{M,\omega}(\mathsf{T})$. Then the K-functional $K(\delta^{2r}, f; L_{M,\omega}(\mathsf{T}), W_M^{2r}(\mathsf{T},\omega))$ in* (2.71) *and the modulus $\Omega_{M,\omega}^r(f,\delta)$, $r = 1,2,3,\ldots$, are equivalent.*

Proof. If $h \in W_M^{2r}(\mathsf{T},\omega)$, then we have

$$\Omega_{M,\omega}^r(f,\delta) \leq c\|f-h\|_{M,\omega} + c\delta^{2r}\left\|h^{(2r)}\right\|_{M,\omega} \leq cK\left(\delta^{2r}, f; L_{M,\omega}(\mathsf{T}), W_M^{2r}(\mathsf{T},\omega)\right). \tag{2.72}$$

Putting

$$(L_\delta f)(x) := 3\delta^{-3}\int_0^{\delta}\int_0^{u}\int_{-t}^{t} f(x+s)\,ds\,dt\,du, \quad x \in \mathsf{T}, \tag{2.73}$$

we have

$$\frac{d^2}{dx^2}L_\delta f = \frac{c}{\delta^2}(I-\sigma_\delta)f, \tag{2.74}$$

and hence

$$\frac{d^{2r}}{dx^{2r}}L_\delta^r f = \frac{c}{\delta^{2r}}(I-\sigma_\delta)^r, \quad r = 1,2,3,\ldots. \tag{2.75}$$

On the other hand, we find

$$\|L_\delta f\|_{M,\omega} \leq 3\delta^{-3}\int_0^{\delta}\int_0^{u} 2t\|\sigma_t f\|_{M,\omega}\,dt\,du \leq c\|f\|_{M,\omega}. \tag{2.76}$$

Now, let $A_\delta^r := I - (I - L_\delta^r)^r$. Then $A_\delta^r f \in W_M^{2r}(\mathsf{T},\omega)$ and

$$\left\|\frac{d^{2r}}{dx^{2r}}A_\delta^r f\right\|_{M,\omega} \leq c\left\|\frac{d^{2r}}{dx^{2r}}L_\delta^r f\right\|_{M,\omega} = \frac{c}{\delta^{2r}}\|(I-\sigma_\delta)^r\|_{M,\omega} \leq \frac{c}{\delta^{2r}}\Omega_{M,\omega}^r(f,\delta). \tag{2.77}$$

Since

$$I - L_\delta^r = (I - L_\delta)\sum_{j=0}^{r-1} L_\delta^j, \tag{2.78}$$

we get

$$\begin{aligned}\left\|(I-L_\delta^r)g\right\|_{M,\omega} &\le c\left\|(I-L_\delta)g\right\|_{M,\omega} \le 3c\delta^{-3}\int_0^\delta\int_0^u 2t\left\|(I-\sigma_t)g\right\|_{M,\omega}dt\,du \\ &\le c\sup_{0<t\le\delta}\left\|(I-\sigma_t)g\right\|_{M,\omega}.\end{aligned} \tag{2.79}$$

Taking into account

$$\left\|f-A_\delta^r f\right\|_{M,\omega} = \left\|(I-L_\delta^r)^r f\right\|_{M,\omega} \tag{2.80}$$

by a recursive procedure, we obtain

$$\begin{aligned}\left\|f-A_\delta^r f\right\|_{M,\omega} &\le c\sup_{0<t_1\le\delta}\left\|(I-\sigma_{t_1})(I-L_\delta^r)^{r-1}f\right\|_{M,\omega} \\ &\le c\sup_{0<t_1\le\delta}\sup_{0<t_2\le\delta}\left\|(I-\sigma_{t_1})(I-\sigma_{t_2})(I-L_\delta^r)^{r-2}f\right\|_{M,\omega} \\ &\le\cdots\le c\sup_{\substack{0<t_i\le\delta\\ i=1,2,\ldots,r}}\left\|\prod_{i=1}^r(I-\sigma_{t_i})f(x)\right\|_{M,\omega} = c\Omega_{M,\omega}^r(f,\delta).\end{aligned} \tag{2.81}$$

Now we can formulate the results. □

Theorem 2.7. *Let $M\in QC_2^\theta(0,1)$ and $r\in\mathbb{R}^+$. If $f\in L_{M,\omega}(\mathsf{T})$ with $\omega\in A_{p(M)}$, then there is a constant $c>0$ dependent only on r and M such that for $n=0,1,2,3,\ldots$*

$$E_n(f)_{M,\omega}\le c\Omega_{M,\omega}^r\left(f,\frac{1}{n+1}\right) \tag{2.82}$$

holds.

Proof. We put $k-1<r\le k$, $k\in\mathbb{Z}^+$. From Remark 2.1(i), (2.64), (2.71), Proposition 2.6, and (2.61), we get for every $g\in W_{M,\omega}^{2k}(\mathbb{T})$ and $n=0,1,2,3,\ldots$

$$\begin{aligned}E_n(f)_{M,\omega} &\le E_n(f-g)_{M,\omega}+E_n(g)_{M,\omega} \le c\left[\left\|f-g\right\|_{M,\omega}+(n+1)^{-2k}\left\|g^{(2k)}\right\|_{M,\omega}\right] \\ &\le cK\left((n+1)^{-2k},f;L_{M,\omega}(\mathsf{T}),W_M^{2k}(\mathsf{T},\omega)\right)\le c\Omega_{M,\omega}^k\left(f,\frac{1}{(n+1)}\right) \\ &\le c\Omega_{M,\omega}^r\left(f,\frac{1}{(n+1)}\right).\end{aligned} \tag{2.83}$$

□

Theorem 2.8. *Let $M \in QC_2^{\theta}(0,1)$ and $r \in \mathbb{R}^+$. If $f \in L_{M,\omega}(\mathsf{T})$ with $\omega \in A_{p(M)}$, then there is a constant $c > 0$ dependent only on r and M such that for $n = 0, 1, 2, 3, \ldots$*

$$\Omega_{M,\omega}^r\left(f, \frac{\pi}{n+1}\right) \leq \frac{c}{(n+1)^r} \sum_{\nu=0}^{n} (\nu+1)^{r-1} E_\nu(f)_{M,\omega} \tag{2.84}$$

holds.

Proof. Let $T_n \in \mathcal{T}_n$ be the best approximating polynomial of $f \in L_{M,\omega}(\mathsf{T})$ and let $m \in \mathbb{Z}^+$. Then,

$$\begin{aligned} \Omega_{M,\omega}^r\left(f, \frac{\pi}{n+1}\right) \leq \Omega_{M,\omega}^r\left(f - T_{2^m}, \frac{\pi}{(n+1)}\right) + \Omega_{M,\omega}^r\left(T_{2^m}, \frac{\pi}{(n+1)}\right) \leq cE_{2^m}(f)_{M,\omega} \\ + \Omega_{M,\omega}^r\left(T_{2^m}, \frac{\pi}{(n+1)}\right). \end{aligned} \tag{2.85}$$

By Lemma 2.4 we have

$$\Omega_{M,\omega}^r\left(T_{2^m}, \frac{\pi}{(n+1)}\right) \leq c\left(\frac{1}{n+1}\right)^r \left\|T_{2^m}^{(r)}\right\|_{M,\omega}. \tag{2.86}$$

Since

$$T_{2^m}^{(r)}(x) = T_1^{(r)}(x) + \sum_{\nu=0}^{m-1} \left\{T_{2^{\nu+1}}^{(r)}(x) - T_{2^\nu}^{(r)}(x)\right\}, \tag{2.87}$$

we get

$$\Omega_{M,\omega}^r\left(T_{2^m}, \frac{\pi}{(n+1)}\right) \leq \frac{c}{(n+1)^r} \left\{\left\|T_1^{(r)}\right\|_{M,\omega} + \sum_{\nu=0}^{m-1} \left\|T_{2^{\nu+1}}^{(r)} - T_{2^\nu}^{(r)}\right\|_{M,\omega}\right\}. \tag{2.88}$$

Fractional Bernstein inequality of Lemma 2.2 gives

$$\begin{aligned} \left\|T_{2^{\nu+1}}^{(r)} - T_{2^\nu}^{(r)}\right\|_{M,\omega} \leq c2^{\nu r}\|T_{2^{\nu+1}} - T_{2^\nu}\|_{M,\omega} \leq c2^{\nu r+1}E_{2^\nu}(f)_{M,\omega}, \\ \left\|T_1^{(r)}\right\|_{M,\omega} = \left\|T_1^{(r)} - T_0^{(r)}\right\|_{M,\omega} \leq cE_0(f)_{M,\omega}. \end{aligned} \tag{2.89}$$

Hence,

$$\Omega_{M,\omega}^r\left(T_{2^m}, \frac{\pi}{(n+1)}\right) \leq \frac{c}{(n+1)^r} \left\{E_0(f)_{M,\omega} + \sum_{\nu=0}^{m-1} 2^{(\nu+1)r} E_{2^\nu}(f)_{M,\omega}\right\}. \tag{2.90}$$

It is easily seen that

$$2^{(\nu+1)r}E_{2^\nu}(f)_{M,\omega} \le c^* \sum_{\mu=2^{\nu-1}+1}^{2^\nu} \mu^{r-1}E_\mu(f)_{M,\omega}, \quad \nu = 1,2,3,\ldots, \tag{2.91}$$

where

$$c^* = \begin{cases} 2^{r+1}, & 0 < r < 1, \\ 2^{2r}, & r \ge 1. \end{cases} \tag{2.92}$$

Therefore,

$$\begin{aligned} \Omega^r_{M,\omega}\left(T_{2^m}, \frac{\pi}{(n+1)}\right) &\le \frac{c}{(n+1)^r}\left\{E_0(f)_{M,\omega} + 2^r E_1(f)_{M,\omega} + C\sum_{\nu=1}^{m}\sum_{\mu=2^{\nu-1}+1}^{2^\nu}\mu^{r-1}E_\mu(f)_{M,\omega}\right\} \\ &\le \frac{c}{(n+1)^r}\left\{E_0(f)_{M,\omega} + \sum_{\mu=1}^{2^m}\mu^{r-1}E_\mu(f)_{M,\omega}\right\} \\ &\le \frac{c}{(n+1)^r}\sum_{\nu=0}^{2^m-1}(\nu+1)^{r-1}E_\nu(f)_{M,\omega}. \end{aligned} \tag{2.93}$$

If we choose $2^m \le n+1 \le 2^{m+1}$, then

$$\begin{aligned} \Omega^r_{M,\omega}\left(T_{2^m}, \frac{\pi}{(n+1)}\right) &\le \frac{c}{(n+1)^r}\sum_{\nu=0}^{n}(\nu+1)^{r-1}E_\nu(f)_{M,\omega}, \\ E_{2^m}(f)_{M,\omega} \le E_{2^{m-1}}(f)_{M,\omega} &\le \frac{c}{(n+1)^r}\sum_{\nu=0}^{n}(\nu+1)^{r-1}E_\nu(f)_{M,\omega}. \end{aligned} \tag{2.94}$$

Last two inequalities complete the proof. □

From Theorems 2.7 and 2.8 we have the following corollaries.

Corollary 2.9. *Let $M \in QC_2^\theta(0,1)$ and $r \in \mathbb{R}^+$. If $f \in L_{M,\omega}(\mathrm{T})$ with $\omega \in A_{p(M)}$ and*

$$E_n(f)_{M,\omega} = \mathcal{O}(n^{-\sigma}), \quad \sigma > 0,\ n = 1,2,\ldots, \tag{2.95}$$

then

$$\Omega^r_{M,\omega}(f,\delta) = \begin{cases} \mathcal{O}(\delta^\sigma); & r > \sigma, \\ \mathcal{O}\left(\delta^\sigma \left|\log\left(\frac{1}{\delta}\right)\right|\right); & r = \sigma, \\ \mathcal{O}(\delta^\alpha); & r < \sigma, \end{cases} \tag{2.96}$$

hold.

Definition 2.10. Let $M \in QC_2^\theta(0,1)$ and $r \in \mathbb{R}^+$. *If* $f \in L_{M,\omega}(\mathsf{T})$ and $\omega \in A_{p(M)}$ then for $0 < \sigma < r$ we set $\mathrm{Lip}\sigma(r,M,\omega) := \{f \in L_{M,\omega}(\mathsf{T}) : \Omega^r_{M,\omega}(f,\delta) = \mathcal{O}(\delta^\sigma), \delta > 0\}$.

Corollary 2.11. *Let* $M \in QC_2^\theta(0,1)$ *and* $r \in \mathbb{R}^+$. *If* $f \in L_{M,\omega}(\mathsf{T})$, $\omega \in A_{p(M)}$, $0 < \sigma < r$ *and* $E_n(f)_{M,\omega} = \mathcal{O}(n^{-\sigma}), n = 1,2,\ldots$, *then* $f \in \mathrm{Lip}\sigma(r,M,\omega)$.

Corollary 2.12. *Let* $0 < \sigma < r$ *and let* $f \in L_{M,\omega}(\mathsf{T})$, $\omega \in A_{p(M)}$, *where* $M \in QC_2^\theta(0,1)$. *Then the following conditions are equivalent:*

$$\begin{aligned} &\text{(a)} \quad f \in \mathrm{Lip}\sigma(r,M,\omega), \\ &\text{(b)} \quad E_n(f)_{M,\omega} = \mathcal{O}(n^{-\sigma}), \quad n = 1,2,\ldots. \end{aligned} \tag{2.97}$$

Theorem 2.13. *Let* $f \in L_{M,\omega}(\mathsf{T})$, $\omega \in A_{p(M)}$, *where* $M \in QC_2^\theta(0,1)$. *If* $\alpha \in (0,\infty)$ *and*

$$\sum_{\nu=1}^{\infty} \nu^{\alpha-1} E_\nu(f)_{M,\omega} < \infty, \tag{2.98}$$

then

$$E_n\left(f^{(\alpha)}\right)_{M,\omega} \le c\left((n+1)^\alpha E_n(f)_{M,\omega} + \sum_{\nu=n+1}^{\infty} \nu^{\alpha-1} E_\nu(f)_{M,\omega}\right) \tag{2.99}$$

hold where the constant $c > 0$ *is dependent only on* α *and* M.

Proof of Theorem 2.13. The condition (2.98) and Lemma 2.3 implies that $f^{(\alpha)}$ exist and $f^{(\alpha)} \in L_{M,\omega}(\mathsf{T})$. Since

$$\left\|f^{(\alpha)} - S_n\left(f^{(\alpha)}\right)\right\|_{M,\omega} \le \left\|S_{2^{m+2}}\left(f^{(\alpha)}\right) - S_n\left(f^{(\alpha)}\right)\right\|_{M,\omega} + \sum_{k=m+2}^{\infty} \left\|S_{2^{k+1}}\left(f^{(\alpha)}\right) - S_{2^k}\left(f^{(\alpha)}\right)\right\|_{M,\omega}, \tag{2.100}$$

we have for $2^m < n < 2^{m+1}$

$$\left\|S_{2^{m+2}}\left(f^{(\alpha)}\right) - S_n\left(f^{(\alpha)}\right)\right\|_{M,\omega} \le c2^{(m+2)\alpha} E_n(f)_{M,\omega} \le C(n+1)^\alpha E_n(f)_{M,\omega}. \tag{2.101}$$

On the other hand, we find

$$\begin{aligned}\sum_{k=m+2}^{\infty}\left\|S_{2^{k+1}}\left(f^{(\alpha)}\right)-S_{2^k}\left(f^{(\alpha)}\right)\right\|_{M,\omega} &\le c\sum_{k=m+2}^{\infty}2^{(k+1)\alpha}E_{2^k}(f)_{M,\omega}\\ &\le C\sum_{k=m+2}^{\infty}\sum_{\mu=2^{k-1}+1}^{2^k}\mu^{\alpha-1}E_{\mu}(f)_{M,\omega}\\ &= c\sum_{\nu=2^{m+1}+1}^{\infty}\nu^{\alpha-1}E_{\nu}(f)_{M,\omega}\le c\sum_{\nu=n+1}^{\infty}\nu^{\alpha-1}E_{\nu}(f)_{M,\omega},\end{aligned}\tag{2.102}$$

and Theorem 2.13 is proved. □

As a corollary of Theorems 2.7, 2.8, and 2.13 we have the following.

Corollary 2.14. *Let $f\in W_M^{\alpha}(\mathrm{T},\omega)$, $\omega\in A_{p(M)}$, $r\in(0,\infty)$, and*

$$\sum_{\nu=1}^{\infty}\nu^{\alpha-1}E_{\nu}(f)_{M,\omega}<\infty\tag{2.103}$$

for some $\alpha>0$. In this case for $n=0,1,2,\ldots$, there exists a constant $c>0$ dependent only on α, r, and M such that

$$\Omega_{M,\omega}^{r}\left(f^{(\alpha)},\frac{\pi}{n+1}\right)\le\frac{c}{(n+1)^r}\sum_{\nu=0}^{n}(\nu+1)^{\alpha+r-1}E_{\nu}(f)_{M,\omega}+c\sum_{\nu=n+1}^{\infty}\nu^{\alpha-1}E_{\nu}(f)_{M,\omega}\tag{2.104}$$

hold.

3. Near Best Approximants in Weighted Smirnov-Orlicz Space

Let $w=\varphi(z)$ and $w=\varphi_1(z)$ be the conformal mappings of G_∞ and G_0 onto the complement $\mathbb{D}_\infty$ of $\mathbb{D}$, normalized by the conditions

$$\begin{aligned}&\varphi(\infty)=\infty, &&\lim_{z\to\infty}\varphi(z)/z>0,\\ &\varphi_1(0)=\infty, &&\lim_{z\to 0}z\,\varphi_1(z)>0,\end{aligned}\tag{3.1}$$

respectively. We denote by ψ and ψ_1 the inverse mappings of φ and φ_1, respectively, and $\mathbb{T}:=\partial\mathbb{D}$. These mappings ψ and ψ_1 have in some deleted neighborhood of ∞ the representations

$$\psi(w)=\alpha w+\alpha_0+\sum_{k=1}^{\infty}\frac{\alpha_k}{w^k},\quad \alpha>0,\qquad \psi_1(w)=\sum_{l=1}^{\infty}\frac{\beta_l}{w^l},\quad \beta_1>0.\tag{3.2}$$

Therefore, the functions

$$\frac{\psi'(w)}{\psi(w)-z}, \quad z \in G_0, \qquad \frac{\psi_1'(w)}{\psi_1(w)-z}, \quad z \in G_\infty \tag{3.3}$$

are analytic in $\mathbb{D}_\infty$ and have, respectively, simple zero and zero of order 2 at ∞. Hence they have expansions

$$\begin{aligned} \frac{\psi'(w)}{\psi(w)-z} &= \sum_{k=0}^{\infty} \frac{F_k(z)}{w^{k+1}}, \quad z \in G_0,\ w \in \mathbb{D}_\infty, \\ \frac{\psi_1'(w)}{\psi_1(w)-z} &= \sum_{k=1}^{\infty} \frac{\widetilde{F}_k(1/z)}{w^{k+1}}, \quad z \in G_\infty,\ w \in \mathbb{D}_\infty, \end{aligned} \tag{3.4}$$

where $F_k(z)$ and $\widetilde{F}_k(1/z)$ are, respectively, Faber Polynomials of degree k for continuums $\overline{G_0}$ and $\overline{\mathbb{C}} \setminus G_0$, with the integral representations [38, pp. 35, 255]

$$\begin{aligned} F_k(z) &= \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{w^k \psi'(w)}{\psi(w)-z} dw, \quad z \in G_0, \\ \widetilde{F}_k\left(\frac{1}{z}\right) &= \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{w^k \psi_1'(w)}{\psi_1(w)-z} dw, \quad z \in G_\infty, \end{aligned} \tag{3.5}$$

$$F_k(z) = \varphi^k(z) + \frac{1}{2\pi i} \int_{\Gamma} \frac{\varphi^k(\varsigma)}{\varsigma - z} d\varsigma, \quad z \in G_\infty,\ k = 0, 1, 2, \ldots, \tag{3.6}$$

$$\widetilde{F}_k\left(\frac{1}{z}\right) = \varphi_1^k(z) - \frac{1}{2\pi i} \int_{\Gamma} \frac{\varphi_1^k(\varsigma)}{\varsigma - z} d\varsigma, \quad z \in G_0 \setminus \{0\}. \tag{3.7}$$

We put

$$\begin{aligned} a_k :&= a_k(f) := \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f_0(w)}{w^{k+1}} dw, \quad k = 0, 1, 2, \ldots, \\ \widetilde{a}_k :&= \widetilde{a}_k(f) := \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f_1(w)}{w^{k+1}} dw, \quad k = 1, 2, \ldots \end{aligned} \tag{3.8}$$

and correspond the series

$$\sum_{k=0}^{\infty} a_k F_k(z) + \sum_{k=1}^{\infty} \widetilde{a}_k \widetilde{F}_k\left(\frac{1}{z}\right) \tag{3.9}$$

with the function $f \in L^1(\Gamma)$, that is,

$$f(z) \sim \sum_{k=0}^{\infty} a_k F_k(z) + \sum_{k=1}^{\infty} \widetilde{a}_k \widetilde{F}_k\left(\frac{1}{z}\right). \tag{3.10}$$

This series is called the *Faber-Laurent* series of the function f and the coefficients a_k and $\tilde{a}_k$ are said to be the *Faber-Laurent coefficients* of f. For further information about the Faber polynomials and Faber Laurent series, we refer to monographs [39, Chapter I, Section 6], [40, Chapter II], and [38].

It is well known that, using the Faber polynomials, approximating polynomials can be constructed [3]. The interpolating polynomials can also be used for this aim. In their work [41] under the assumption $\Gamma \in C(2,\alpha)$, $0 < \alpha < 1$, Shen and Zhong obtain a series of interpolation nodes in G_0 and show that interpolating polynomials and best approximating polynomial in $E^p(G_0)$, $1 < p < \infty$, have the same order of convergence. In [42] considering $\Gamma \in C(1,\alpha)$ and choosing the interpolation nodes as the zeros of the Faber polynomials, Zhu obtain similar result.

In the above-cited works, Γ does not admit corners, whereas many domains in the complex plain may have corners. When Γ is a piecewise Vanishing Rotation curve [43] Zhong and Zhu show that the interpolating polynomials based on the zeros of the Faber polynomials converge to f in the $E^p(G_0)$, $1 < p < \infty$ norm.

A function $\omega : \Gamma \to [0,\infty]$ is called a *weight* on Γ, if ω is measurable and $\omega^{-1}(\{0,\infty\})$ has measure zero. We denote by $L_{M,\omega}(\Gamma)$ the linear space of Lebesgue measurable functions $f : \Gamma \to \mathbb{C}$ satisfying the condition

$$\int_\Gamma M\left[\alpha\left|f(z)\right|\right]\omega(z)|dz| < \infty \tag{3.11}$$

for some $\alpha > 0$.

The space $L_{M,\omega}(\Gamma)$ becomes a Banach space with the *Orlicz norm*

$$\|f\|_{M,\Gamma,\omega} := \sup\left\{\int_\Gamma \left|f(z)g(z)\right|\omega(z)|dz| : g \in L_{N,\omega}(\Gamma);\ \rho(g;N) \le 1\right\}, \tag{3.12}$$

where N is the complementary function of M and

$$\rho(g;N) := \int_\Gamma N\left[\left|g(z)\right|\right]\omega(z)|dz|. \tag{3.13}$$

The Banach space $L_{M,\omega}(\Gamma)$ is called weighted Orlicz space on Γ.

For $z \in \Gamma$ and $\epsilon > 0$ let $\Gamma(z,\epsilon) := \{t \in \Gamma : |t - z| < \epsilon\}$. For fixed $p \in [1,\infty)$, the set of all weights $\omega : \Gamma \to [0,\infty]$ satisfying the relation

$$\begin{gathered}\sup_{z\in\Gamma}\sup_{\epsilon>0}\left(\frac{1}{\epsilon}\int_{\Gamma(z,\epsilon)}\omega(\tau)|d\tau|\right)\left(\frac{1}{\epsilon}\int_{\Gamma(z,\epsilon)}\omega(\tau)^{-1/(p-1)}|d\tau|\right)^{p-1} < \infty, \quad \text{if } p > 1,\\ \sup_{\epsilon>0}\frac{1}{\epsilon}\int_{\Gamma(z,\epsilon)}\omega(\tau)|d\tau| \le c\omega(z), \quad \forall z \in \Gamma,\ \text{if } p = 1\end{gathered} \tag{3.14}$$

is denoted by $A_p(\Gamma)$.

We denote by $L^p(\Gamma,\omega)$ the set of all measurable functions $f : \Gamma \to \mathbb{C}$ such that $|f|\omega^{1/p}$ belongs to Lebesgue space $L^p(\Gamma)$, $1 \le p < \infty$, on Γ.

Definition 3.1. Let ω be a weight on Γ and let $E_{M,\omega}(G_0) := \{f \in E^1(G_0) : f \in L_{M,\omega}(\Gamma)\}$, $E_{M,\omega}(G_\infty) := \{f \in E^1(G_\infty) : f \in L_{M,\omega}(\Gamma)\}$, $\tilde{E}_{M,\omega}(G_\infty) := \{f \in E_{M,\omega}(G_\infty) : f(\infty) = 0\}$. The classes of functions $E_{M,\omega}(G_0)$ and $E_{M,\omega}$ (G_∞) will be called weighted Smirnov-Orlicz classes with respect to domains G_0 and G_∞, respectively.

In this chapter, we prove that the convergence rate of the interpolating polynomials based on the zeros of the F_n is the same with the best approximating algebraic polynomials in the weighted Smirnov-Orlicz class $E_{M,\omega}(G_0)$ under the assumption that Γ is a closed Radon curve. This means that interpolating polynomials based on the zeros of the Faber polynomials are near best approximant of f belonging to weighted Smirnov-Orlicz class $E_{M,\omega}(G_0)$.

In the case that all of the zeros of the nth Faber polynomial F_n are in G_0, we denote by $L_n(f,\cdot)$ the $(n-1)$th interpolating polynomial for $f \in E_{M,\omega}(G_0)$ based on the zeros of F_n.

Let $f \in L^1(\Gamma)$. Then the functions f^+ and f^- defined by

$$f^+(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\varsigma)}{\varsigma - z}d\varsigma, \quad z \in G_0, \qquad f^-(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\varsigma)}{\varsigma - z}d\varsigma, \quad z \in G_\infty \tag{3.15}$$

are analytic in G_0 and G_∞, respectively, and $f^-(\infty) = 0$.

We denote by

$$E_n(f)_{M,\Gamma,\omega} := \inf\left\{\|f - p\|_{M,\Gamma,\omega} : p \in \mathcal{P}_n\right\} \tag{3.16}$$

the minimal error of approximation by polynomials of f, where $\mathcal{P}_n$ is the set of algebraic polynomials of degree not greater than n.

Let Γ be a rectifiable Jordan curve, $f \in L^1(\Gamma)$, and let

$$(S_\Gamma f)(t) := \lim_{\varepsilon \to 0}\frac{1}{2\pi i}\int_{\Gamma\setminus\Gamma(t,\varepsilon)} \frac{f(\varsigma)}{\varsigma - t}d\varsigma, \quad t \in \Gamma \tag{3.17}$$

be *Cauchy's singular integral* of f at the point t. The linear operator $S_\Gamma : f \to S_\Gamma f$ is called the *Cauchy singular operator*.

If one of the functions f^+ or f^- has the nontangential limits a.e. on Γ, then $S_\Gamma f(z)$ exists a.e. on Γ and also the other one has the nontangential limits a.e. on Γ. Conversely, if $S_\Gamma f(z)$ exists a.e. on Γ, then both functions f^+ and f^- have the nontangential limits a.e. on Γ. In both cases, the formulae

$$f^+(z) = (S_\Gamma f)(z) + \frac{f(z)}{2}, \qquad f^-(z) = (S_\Gamma f)(z) - \frac{f(z)}{2} \tag{3.18}$$

hold, and hence

$$f = f^+ - f^- \tag{3.19}$$

holds a.e. on Γ (see, e.g., [1, page 431]).

Lemma 3.2. *$f\Gamma$ is a regular curve, $M \in QC_2^{\theta}(0,1)$ and $\omega \in A_{p(M)}(\Gamma)$, then for every $f \in E_{M,\omega}(G_0)$ one has*

$$\|S_\Gamma(f,\cdot)\|_{M,\Gamma,\omega} \le c\|f\|_{M,\Gamma,\omega}, \tag{3.20}$$

where the constant c depends only on Γ and M.

Proof. Assertion (3.20) immediately follows from modular inequality

$$\int_\Gamma M(|S_\Gamma(f,t)|)\omega(t)dt \le c\int_\Gamma M(|f(t)|)\omega(t)dt \tag{3.21}$$

given in (7.5.13) of [23]. □

Theorem 3.3. *If Γ is a closed Radon curve, $M \in QC_2^{\theta}(0,1)$ and $\omega \in A_{p(M)}(\Gamma)$, then for every $f \in E_{M,\omega}(G_0)$ one has*

$$\|f - L_n(f,\cdot)\|_{M,\Gamma,\omega} \le cE_n(f)_{M,\Gamma,\omega}, \tag{3.22}$$

where the constant c depends only on Γ and M.

Proof. First of all we know [16] that all zeros of the Faber polynomials are in G_0. Since interpolating operator $L_n(f,\cdot)$ is linear and corresponds f by a polynomial of degree not more than $n-1$, we need only to show that, for large values of n, $L_n(f,\cdot)$ is uniformly bounded in weighted Smirnov-Orlicz class $E_{M,\omega}(G_0)$. We suppose that P_{n-1} is the $(n-1)$th best approximating algebraic polynomial for f in $E_{M,\omega}(G_0)$. In this case we have

$$\|f - L_n(f,\cdot)\|_{M,\Gamma,\omega} = \|f - P_{n-1} - L_n(f - P_{n-1},\cdot)\|_{M,\Gamma,\omega} \le (1 + \|L_n\|)\|f - P_{n-1}\|_{M,\Gamma,\omega}. \tag{3.23}$$

Since we assumed the interpolation nodes as the zeros of the Faber polynomials F_n, using [39, page 59], we have

$$f(z) - L_n(f,z) = \frac{F_n(z)}{2\pi i}\int_\Gamma \frac{f(\varsigma)}{F_n(\varsigma)(\varsigma - z)}d\varsigma = F_n(z)\left(S_\Gamma\left(\frac{f}{F_n}\right)\right)(z), \quad z \in G_0 \tag{3.24}$$

and consequently

$$\|f(z) - L_n(f,z)\|_{M,\Gamma,\omega} = \left\|F_n(\cdot)\left(S_\Gamma\left(\frac{f}{F_n}\right)\right)(\cdot)\right\|_{M,\Gamma,\omega} \le \left\{\max_{z\in\Gamma}|F_n(z)|\right\}\left\|S_\Gamma\left(\frac{f}{F_n}\right)\right\|_{M,\Gamma,\omega}. \tag{3.25}$$

By Lemma 3.2, we get

$$\|f - L_n(f,\cdot)\|_{M,\Gamma,\omega} \le c\left\{\max_{z\in\Gamma}|F_n(z)|\right\}\left\|\frac{f}{F_n}\right\|_{M,\Gamma,\omega} \le c\left\{\max_{z,\varsigma\in\Gamma}\left|\frac{F_n(z)}{F_n(\varsigma)}\right|\right\}\|f\|_{M,\Gamma,\omega}. \tag{3.26}$$

We set $\kappa := \max_{z\in\Gamma}|\vartheta_z - 1|$, where $\vartheta_z\pi$ is the exterior angle of the point $z \in \Gamma$. By the Radon assumption on Γ we get $0 \le \kappa < 1$. Then one can find for $z \in \Gamma$

$$0,5 - 0,5 \cdot \kappa < |F_n(z)| < 1,5 + 0,5 \cdot \kappa, \tag{3.27}$$

and therefore

$$\max_{z,\varsigma\in\Gamma}\left|\frac{F_n(z)}{F_n(\varsigma)}\right| \le \frac{3+\kappa}{1-\kappa}. \tag{3.28}$$

From the last inequality we obtain

$$\|f - L_n(f,\cdot)\|_{M,\Gamma,\omega} \le c\frac{3+\kappa}{1-\kappa}\|f\|_{M,\Gamma,\omega}. \tag{3.29}$$

Since

$$\|L_n(f,\cdot)\|_{M,\Gamma,\omega} \le \|f\|_{M,\Gamma,\omega} + \|f - L_n(f,\cdot)\|_{M,\Gamma,\omega} \le \left(1 + c\frac{3+\kappa}{1-\kappa}\right)\|f\|_{M,\Gamma,\omega}, \tag{3.30}$$

we obtain that $L_n(f,\cdot)$ is uniformly bounded in $E_{M,\omega}(G_0)$, namely,

$$\|L_n\| \le c. \tag{3.31}$$

Therefore, we conclude that

$$\|f - L_n(f,\cdot)\|_{M,\Gamma,\omega} \le c\|f - P_{n-1}\|_{M,\Gamma,\omega} = cE_n(f)_{M,\Gamma,\omega} \tag{3.32}$$

and interpolating polynomial $L_n(f,\cdot)$ is near best approximant for f. □

If Γ is Dini-smooth, then [44] there exist constants c and C such that

$$0 < c < |\psi'(w)| < C < \infty, \quad |w| \ge 1. \tag{3.33}$$

Similar inequalities hold also for ψ_1' and φ_1', in case of $|w| = 1$ and $z \in \Gamma$, respectively.

We define Poisson polynomial for function $f \in E_{M,\omega}\ (G_0)$

$$V_n(f,z) := \sum_{k=0}^{n} c_k F_k(z) + \sum_{k=n+1}^{2n-1}\left(2 - \frac{k}{n}\right)c_k F_k(z), \quad z \in G_0. \tag{3.34}$$

Theorem 3.4. *If Γ is a Dini-smooth curve, $M \in QC_2^{\theta}(0,1)$ and $\omega \in A_{p(M)}(\Gamma)$, then for every $f \in E_{M,\omega}(G_0)$ one has*

$$\left\| f - V_n(f,\cdot) \right\|_{M,\Gamma,\omega} \le c E_n(f)_{M,\Gamma,\omega}, \tag{3.35}$$

where the constant c depends only on Γ and M.

Proof. From (3.8) and (3.5), we have

$$V_n(f,z) = \frac{1}{2\pi}\int_0^{2\pi} f\left(\psi\left(e^{it}\right)\right)\frac{dt}{2\pi i}\int_{\Gamma}\frac{\sum_{k=-(2n-1)}^{2n-1}\lambda_{|k|}\varphi^k(\varsigma)e^{-ikt}}{\varsigma - z}d\varsigma, \tag{3.36}$$

where $z \in G_0$ and

$$\lambda_{|k|} := \begin{cases} 1, & 0 \le k \le n, \\ 2 - \dfrac{k}{n}, & n+1 \le k \le 2n-1. \end{cases} \tag{3.37}$$

If $P_n \in \mathcal{P}_n$ is near best approximant for $f \in E_{M,\omega}(G_0)$, we get

$$\left\| f - V_n(f,\cdot) \right\|_{M,\Gamma,\omega} \le E_n(f)_{M,\Gamma,\omega} + \left\| P_n - V_n(f,\cdot) \right\|_{M,\Gamma,\omega}. \tag{3.38}$$

Using

$$P_n(z) = \frac{1}{2\pi}\int_0^{2\pi} P_n\left(\psi\left(e^{it}\right)\right)\frac{dt}{2\pi i}\int_{\Gamma}\frac{\sum_{k=-(2n-1)}^{2n-1}\lambda_{|k|}^{k}\varphi(\varsigma)e^{-ikt}}{\varsigma - z}d\varsigma, \quad z \in G_0, \tag{3.39}$$

we find

$$\begin{aligned} &P_n(z) - V_n(f,z) \\ &\quad = \frac{1}{2\pi}\int_0^{2\pi}\left\{P_n\left(\psi\left(e^{it}\right)\right) - f\left(\psi\left(e^{it}\right)\right)\right\}\frac{dt}{2\pi i}\int_{\Gamma}\frac{\sum_{k=-(2n-1)}^{2n-1}\lambda_{|k|}^{k}\varphi(\varsigma)e^{-ikt}}{\varsigma - z}d\varsigma, \quad z \in G_0. \end{aligned} \tag{3.40}$$

Taking in the last inequality, the nontangential boundary values from inside of Γ, $z \to z_0 \in \Gamma$ and using (3.18), we have

$$\begin{aligned} P_n(z_0) - V_n(f,z_0) = {} & \frac{1}{2\pi}\int_0^{2\pi}\left\{P_n\left(\psi\left(e^{it}\right)\right) - f\left(\psi\left(e^{it}\right)\right)\right\}dt \\ & \times\left[\frac{1}{2}\sum_{k=-(2n-1)}^{2n-1}\lambda_{|k|}^{k}\varphi(z_0)e^{-ikt} + \frac{1}{2\pi i}\int_{\Gamma}\frac{\sum_{k=-(2n-1)}^{2n-1}\lambda_{|k|}^{k}\varphi(\varsigma)e^{-ikt}}{\varsigma - z_0}d\varsigma\right]. \end{aligned} \tag{3.41}$$

Since $\varphi^{-2n}(\varsigma)\sum_{k=-(2n-1)}^{2n-1}\lambda_{|k|}\varphi^k(\varsigma)e^{-ikt}$ is analytic in G_∞, we have

$$\frac{1}{2\pi i}\int_\Gamma \frac{\sum_{k=-(2n-1)}^{2n-1}\lambda_{|k|}\varphi^k(\varsigma)e^{-ikt}}{(\varsigma-z)\varphi^{2n}(\varsigma)}d\varsigma = 0, \quad z\in G_0, \tag{3.42}$$

and taking nontangential limit in (3.42) we get

$$\frac{1}{2}\sum_{k=-(2n-1)}^{2n-1}\lambda_{|k|}\varphi^k(z_0)e^{-ikt} = -\frac{1}{2\pi i}\int_\Gamma \frac{\varphi^{2n}(z_0)}{\varphi^{2n}(\varsigma)}\frac{\sum_{k=-(2n-1)}^{2n-1}\lambda_{|k|}\varphi^k(\varsigma)e^{-ikt}}{\varsigma-z_0}d\varsigma, \tag{3.43}$$

and hence by transformation $z_0=\psi(w_0)$ we obtain

$$\begin{aligned}
&P_n(z_0)-V_n(f,z_0)\\
&\quad=\frac{1}{2\pi}\int_0^{2\pi}\left\{P_n\left(\psi\left(e^{it}\right)\right)-f\left(\psi\left(e^{it}\right)\right)\right\}\frac{dt}{2\pi i}\int_{\mathbb{T}}\left(1-\frac{w_0^{2n}}{w^{2n}}\right)\frac{\sum_{k=-(2n-1)}^{2n-1}\lambda_{|k|}w^k e^{-ikt}}{\psi(w)-\psi(w_0)}\psi'(w)\,dw.
\end{aligned} \tag{3.44}$$

Since one has

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{\mathbb{T}}\left(1-\frac{w_0^{2n}}{w^{2n}}\right)\frac{\sum_{k=-(2n-1)}^{2n-1}\lambda_{|k|}w^k e^{-ikt}}{\psi(w)-\psi(w_0)}\psi'(w)dw\\
&\quad=\frac{1}{2\pi i}\int_{\mathbb{T}}\left(1-\frac{w_0^{2n}}{w^{2n}}\right)\sum_{k=-(2n-1)}^{2n-1}\lambda_{|k|}w^k e^{-ikt}\left[\frac{\psi'(w)}{\psi(w)-\psi(w_0)}-\frac{1}{w-w_0}\right]dw\\
&\qquad+\frac{1}{2\pi i}\int_{\mathbb{T}}\left(1-\frac{w_0^{2n}}{w^{2n}}\right)\frac{1}{w-w_0}\sum_{k=-(2n-1)}^{2n-1}\lambda_{|k|}w^k e^{-ikt},
\end{aligned} \tag{3.45}$$

we can write

$$\begin{aligned}
&P_n(z_0)-V_n(f,z_0)\\
&\quad=\frac{1}{2\pi}\int_0^{2\pi}\left\{P_n\left(\psi\left(e^{it}\right)\right)-f\left(\psi\left(e^{it}\right)\right)\right\}dt\\
&\qquad\times\frac{1}{2\pi i}\int_{\mathbb{T}}\left(1-\frac{w_0^{2n}}{w^{2n}}\right)\sum_{k=-(2n-1)}^{2n-1}\lambda_{|k|}w^k e^{-ikt}\left[\frac{\psi'(w)}{\psi(w)-\psi(w_0)}-\frac{1}{w-w_0}\right]dw\\
&\qquad+\frac{1}{2\pi}\int_0^{2\pi}\left\{P_n\left(\psi\left(e^{it}\right)\right)-f\left(\psi\left(e^{it}\right)\right)\right\}\frac{dt}{2\pi i}\int_{\mathbb{T}}\left(1-\frac{w_0^{2n}}{w^{2n}}\right)\frac{1}{w-w_0}\\
&\qquad\times\sum_{k=-(2n-1)}^{2n-1}\lambda_{|k|}w^k e^{-ikt} := I_1+I_2.
\end{aligned} \tag{3.46}$$

From equality

$$\frac{1}{2\pi i}\int_{\mathbb{T}}\left(1-\frac{w_0^{2n}}{w^{2n}}\right)\frac{1}{w-w_0}\sum_{k=-(2n-1)}^{2n-1}\lambda_{|k|}w^k e^{-ikt}dw=\sum_{k=-(2n-1)}^{2n-1}\lambda_{|k|}w_0^k e^{-ikt}, \tag{3.47}$$

we have

$$\|I_2\|_{M,\Gamma,\omega}\le E_n(f)_{M,\Gamma,\omega}\frac{1}{2\pi}\int_0^{2\pi}\left|\sum_{k=-(2n-1)}^{2n-1}\lambda_{|k|}w_0^k e^{-ikt}\right|dt. \tag{3.48}$$

On the other hand,

$$\|I_1\|_{M,\Gamma,\omega}\le E_n(f)_{M,\Gamma,\omega}\frac{1}{4\pi^2}\int_0^{2\pi}\left|\sum_{k=-(2n-1)}^{2n-1}\lambda_{|k|}w^k e^{-ikt}\right|\int_{\mathbb{T}}\left|1-\frac{w_0^{2n}}{w^{2n}}\right|\left|\frac{\psi'(w)}{\psi(w)-\psi(w_0)}-\frac{1}{w-w_0}\right||dw|. \tag{3.49}$$

We denote by A a subarc of $\mathbb{T}$ with the center w_0 such that it has arc lenght $\mathcal{O}(1/n)$. In this case

$$\begin{aligned}&\int_A\left|1-\frac{w_0^{2n}}{w^{2n}}\right|\left|\frac{\psi'(w)}{\psi(w)-\psi(w_0)}-\frac{1}{w-w_0}\right||dw|\\&\quad\le\int_A\left|w^{2n}-w_0^{2n}\right|\left|\frac{|\psi'(w)|}{|\psi(w)-\psi(w_0)|}-\frac{1}{|w-w_0|}\right||dw|\le c\end{aligned} \tag{3.50}$$

and, by (1.3),

$$\begin{aligned}&\int_{\mathbb{T}\setminus A}\left|1-\frac{w_0^{2n}}{w^{2n}}\right|\left|\frac{\psi'(w)}{\psi(w)-\psi(w_0)}-\frac{1}{w-w_0}\right||dw|\\&\quad\le 2\int_{\mathbb{T}\setminus A}\frac{\left|\psi'(w)(w-w_0)-[\psi(w)-\psi(w_0)]\right|}{|\psi(w)-\psi(w_0)||w-w_0|}|dw|\\&\quad\le c\int_{1/n}^1\frac{\omega(\psi',t)}{t}dt\le c.\end{aligned} \tag{3.51}$$

Hence,

$$\|I_1\|_{M,\Gamma,\omega}\le E_n(f)_{M,\Gamma,\omega}\int_0^{2\pi}\left|\sum_{k=-(2n-1)}^{2n-1}\lambda_{|k|}w^k e^{-ikt}\right|dt. \tag{3.52}$$

Inequalities (3.46), (3.48), and (3.52) imply that

$$\|P_n - V_n(f,\cdot)\|_{M,\Gamma,\omega} \le E_n(f)_{M,\Gamma,\omega}\left\{\int_0^{2\pi}\left|\sum_{k=-(2n-1)}^{2n-1}\lambda_{|k|}w_0^k e^{-ikt}\right| + \int_0^{2\pi}\left|\sum_{k=-(2n-1)}^{2n-1}\lambda_{|k|}w^k e^{-ikt}\right|\right\}. \tag{3.53}$$

For every $w \in \mathbb{T}$, one has

$$\int_0^{2\pi}\left|\sum_{k=-(2n-1)}^{2n-1}\lambda_{|k|}w^k e^{-ikt}\right| \le c, \tag{3.54}$$

and therefore we get the required inequality of Theorem 3.4. □

Theorem 3.4 signifies that Poisson polynomial is near best approximant for f.

For $g \in L_{M,\omega}(\mathbb{T})$, we set

$$\sigma_h(g)(w) := \frac{1}{2h}\int_{-h}^{h} g\left(we^{it}\right)dt, \quad 0 < h < \pi, \quad w \in \mathbb{T}. \tag{3.55}$$

If $M \in QC_2^{\theta}(0,1)$ and $\omega \in A_{p(M)}(\mathbb{T})$, then by Theorem A(ii) we have

$$\|\sigma_h(g)\|_{M,\mathbb{T},\omega} \le c\|g\|_{M,\mathbb{T},\omega}, \tag{3.56}$$

and consequently $\sigma_h(g) \in L_{M,\omega}(\mathbb{T})$ for any $g \in L_{M,\omega}(\mathbb{T})$.

Definition 3.5. Let $M \in QC_2^{\theta}(0,1)$, $\omega \in A_{p(M)}(\mathbb{T})$, and $r > 0$. The function

$$\Omega^r_{M,\mathbb{T},\omega}(g,\delta) := \sup_{0<h_i,t\le\delta}\left\|\prod_{i=1}^{[r]}(I-\sigma_{h_i})(I-\sigma_t)^{r-[r]}g\right\|_{M,\mathbb{T},\omega}, \quad \delta > 0 \tag{3.57}$$

is called rth modulus of smoothness of $g \in L_{M,\omega}(\mathbb{T})$.

It can easily be verified that the function $\Omega^r_{M,\mathbb{T},\omega}(g,\cdot)$ is continuous, nonnegative, subadditive and satisfy $\lim_{\delta\to 0}\Omega^r_{M,\mathbb{T},\omega}(g,\delta) = 0$ for $g \in L_{M,\omega}(\mathbb{T})$.

Let Γ be a Dini-smooth curve and ω be a weight on Γ. We associate with ω the following two weights defined on $\mathbb{T}$ by

$$\omega_0 := \omega \circ \varphi, \qquad \omega_1 := \omega \circ \varphi_1 \tag{3.58}$$

and let $f_0 := f \circ \psi$, $f_1 := f \circ \psi_1$ for $f \in L_{M,\omega}(\Gamma)$. Then from (3.33), we have $f_0 \in L_{M,\omega_0}(\mathbb{T})$ and $f_1 \in L_{M,\omega_1}(\mathbb{T})$ for $f \in L_{M,\omega}(\Gamma)$. Using the nontangential boundary values of f_0^+ and f_1^+ on $\mathbb{T}$, we define

$$\begin{aligned} \Omega^r_{M,\Gamma,\omega}(f,\delta) &:= \Omega^r_{M,\mathbb{T},\omega_0}(f_0^+,\delta), \\ \widetilde{\Omega}^r_{M,\Gamma,\omega}(f,\delta) &:= \Omega^r_{M,\mathbb{T},\omega_1}(f_1^+,\delta) \end{aligned} \tag{3.59}$$

for r, $\delta > 0$.

We set

$$E_n(f)_{M,\mathbb{T},\omega} := \inf_{P\in\mathcal{P}_n(\mathbb{D})} \|f - P\|_{M,\mathbb{T},\omega}, \qquad \widetilde{E}_n(g)_{M,\Gamma,\omega} := \inf_{R\in\mathcal{R}_n} \|g - R\|_{M,\Gamma,\omega}, \tag{3.60}$$

where $f \in E_{M,\omega}(\mathbb{D})$, $g \in E_{M,\omega}(G_\infty)$, and $\mathcal{R}_n$ is the set of rational functions of the form $\sum_{k=0}^{n} a_k z^{-k}$.

Now we can give several applications of approximation theorems of Section 2.

Theorem 3.6. *Let Γ be a Dini-smooth curve, $M \in QC_2^\theta(0,1)$ and $f \in L_{M,\omega}(\Gamma)$ with $\omega \in A_1(\Gamma)$. Then there is a constant $c > 0$ such that for any natural number n*

$$\|f - R_n(\cdot,f)\|_{M,\Gamma,\omega} \le c\left\{\Omega^r_{M,\Gamma,\omega}\left(f,\frac{1}{(n+1)}\right) + \widetilde{\Omega}^r_{M,\Gamma,\omega}\left(f,\frac{1}{(n+1)}\right)\right\}, \tag{3.61}$$

where $r > 0$ and $R_n(\cdot,f)$ is the nth partial sum of the Faber-Laurent series of f.

Corollary 3.7. *Let Γ be a Dini-smooth curve, $M \in QC_2^\theta(0,1)$ and $f \in E_{M,\omega}(G_0)$ with $\omega \in A_1(\Gamma)$. Then there is a constant $c > 0$ such that for every natural number n*

$$\|f - P_n(\cdot,f)\|_{M,\Gamma,\omega} \le c\,\Omega^r_{M,\Gamma,\omega}\left(f,\frac{1}{(n+1)}\right), \quad r > 0, \tag{3.62}$$

where $P_n(\cdot,f)$ is the nth partial sum of the Faber series of f.

Corollary 3.8. *Let Γ be a Dini-smooth curve, $M \in QC_2^\theta(0,1)$ and $f \in \widetilde{E}_{M,\omega}(G_\infty)$ with $\omega \in A_1$ (Γ). Then there is a constant $c > 0$ such that for every natural number n*

$$\|f - R_n(\cdot,f)\|_{M,\Gamma,\omega} \le c_8\,\widetilde{\Omega}^r_{M,\Gamma,\omega}\left(f,\frac{1}{(n+1)}\right), \quad r > 0, \tag{3.63}$$

where $R_n(\cdot,f)$ is as in Theorem 3.6.

Theorem 3.9. *Let Γ be a Dini-smooth curve, $M \in QC_2^\theta(0,1)$ and $f \in E_{M,\omega}(G_0)$ with $\omega \in A_1(\Gamma)$. Then for $r > 0$ there exists a constant $c > 0$ such that*

$$\Omega^r_{M,\Gamma,\omega}\left(f,\frac{1}{n}\right) \le \frac{c}{n^r}\left\{E_0(f)_{M,\Gamma,\omega} + \sum_{k=1}^{n} k^{r-1} E_k(f)_{M,\Gamma,\omega}\right\} \tag{3.64}$$

hold.

Corollary 3.10. *Under the conditions of Corollary 3.7, if*

$$E_n(f)_{M,\Gamma,\omega} = \mathcal{O}(n^{-\alpha}), \quad \alpha > 0,\ n = 1,2,3,\ldots, \tag{3.65}$$

then for $f \in E_{M,\omega}(G_0)$ *and* $r > 0$

$$\Omega^r_{M,\Gamma,\omega}\ (f,\delta) = \begin{cases} \mathcal{O}(\delta^\alpha), & r > \alpha, \\ \mathcal{O}\left(\delta^\alpha \left|\log \dfrac{1}{\delta}\right|\right), & r = \alpha, \\ \mathcal{O}(\delta^r), & r < \alpha. \end{cases} \tag{3.66}$$

Definition 3.11. Let $M \in QC_2^\theta\ (0,1)$ and $\alpha \in \mathbb{R}^+$. If $f \in E_{M,\omega}\ (G_0)$, then for $0 < \sigma < \alpha$ we set

$$\begin{aligned} \widetilde{\mathrm{Lip}}\sigma(\alpha, M, \Gamma, \omega) &: = \left\{ f \in \tilde{E}_{M,\omega}(G_\infty) : \tilde{\Omega}^\alpha_{M,\Gamma,\omega}(f,\delta) = \mathcal{O}(\delta^\alpha) \right\}, \\ \mathrm{Lip}\sigma(\alpha, M, \Gamma, \omega) &: = \left\{ f \in E_{M,\omega}(G_0) : \Omega^\alpha_{M,\Gamma,\omega}(f,\delta) = \mathcal{O}(\delta^\sigma),\ \delta > 0 \right\}. \end{aligned} \tag{3.67}$$

Corollary 3.12. *Let* $M \in QC_2^\theta(0,1)$ *and* $\alpha \in \mathbb{R}^+$. *If* $f \in E_{M,\omega}(G_0)$, $\omega \in A_1(\Gamma)$, $0 < \sigma < \alpha$ *and* $E_n(f)_{M,\Gamma,\omega} = \mathcal{O}(n^{-\sigma})$, $n = 1,2,\ldots$, *then* $f \in \mathrm{Lip}\sigma(\alpha, M, \Gamma, \omega)$.

By Corollaries 3.7 and 3.10 we have the constructive characterization of the class $\mathrm{Lip}\sigma(\alpha, M, \Gamma, \omega)$.

Corollary 3.13. *Let* $0 < \sigma < \alpha$ *and* $f \in E_{M,\omega}(G_0)$, $\omega \in A_1(\Gamma)$, *where* $M \in QC_2^\theta(0,1)$, *be fulfilled. Then the following conditions are equivalent:*

(a) $f \in \mathrm{Lip}\sigma(\alpha, M, \Gamma, \omega)$.

(b) $E_n(f)_{M,\Gamma,\omega} = \mathcal{O}(n^{-\sigma}),\ n = 1,2,\ldots$.

The inverse theorem for unbounded domains has the following form.

Theorem 3.14. *Let* Γ *be a Dini-smooth curve,* $M \in QC_2^\theta(0,1)$ *and* $f \in \tilde{E}_{M,\omega}(G_\infty)$ *with* $\omega \in A_1(\Gamma)$. *Then there is a constant* $c > 0$ *such that for every natural number* n

$$\tilde{\Omega}^r_{M,\Gamma,\omega}\left(f, \frac{1}{n}\right) \le \frac{c}{n^r}\left\{ \tilde{E}_0(f)_{M,\Gamma,\omega} + \sum_{k=1}^{n} k^{r-1} \tilde{E}_k(f)_{M,\Gamma,\omega} \right\}, \quad r > 0 \tag{3.68}$$

holds.

By the similar way to that of $E_{M,\omega}(G_0)$, we obtain the following corollaries.

Corollary 3.15. *Under the conditions of Corollary 3.8, if*

$$\tilde{E}_n(f)_{M,\Gamma,\omega} = \mathcal{O}(n^{-\alpha}), \quad \alpha > 0,\ n = 1,2,3,\ldots, \tag{3.69}$$

then for $f \in \widetilde{E}_{M,\omega}(G_\infty)$ *and* $r > 0$

$$\widetilde{\Omega}^r_{M,\Gamma,\omega}(f,\delta) = \begin{cases} \mathcal{O}\ (\delta^\alpha), & r > \alpha, \\ \mathcal{O}\left(\delta^\alpha \left|\log \dfrac{1}{\delta}\right|\right), & r = \alpha, \\ \mathcal{O}\ (\delta^r), & r < \alpha. \end{cases} \tag{3.70}$$

Corollary 3.16. *Under the conditions of Theorem 3.14, if*

$$\widetilde{E}_n(f)_{M,\Gamma,\omega} = \mathcal{O}(n^{-\alpha}), \quad \alpha > 0,\ n = 1,2,3,\ldots, \tag{3.71}$$

then $f \in \widetilde{\mathrm{Lip}}\sigma(\alpha, M, \Gamma, \omega)$.

By Corollaries 3.8 and 3.15, we have the following.

Corollary 3.17. *Let* $\alpha > 0$ *and the conditions of Theorem 3.14 be fulfilled. Then the following conditions are equivalent,*

(a) $f \in \widetilde{\mathrm{Lip}}\sigma(\alpha, M, \Gamma, \omega)$,

(b) $\widetilde{E}_n(f)_{M,\Gamma,\omega} = \mathcal{O}(n^{-\alpha}), \quad n = 1,2,3,\ldots.$

Before the proofs, we need some auxiliary lemmas.

Lemma 3.18. *Let* Γ *be a Dini-smooth curve,* $M \in QC_2^\theta(0,1)$ *and* $f \in L_{M,\omega}(\Gamma)$ *with* $\omega \in A_1(\Gamma)$. *Then,* $f^+ \in E_{M,\omega}(G_0)$ *and* $f^- \in \widetilde{E}_{M,\omega}(G_\infty)$ *for every* $f \in L_{M,\omega}\ (\Gamma)$.

Proof. Using $M \in \Delta_2$, we can find a $p \in (1,\infty)$ such that $L_{M,\omega}(\Gamma) \subset L^p(\Gamma,\omega)$, where the inclusion maps being continuous (see, e.g., Lemma 2.13 of [20]). Since $\omega \in A_p(\Gamma)$ by [9], we get $f^+ \in E^1(G_0)$ and $f^- \in E^1(G_\infty)$. Using $\omega \in A_{p(M)}(\Gamma)$ and boundedness of operator S_Γ in $L_{M,\omega}(\Gamma)$, we obtain from (3.18) that

$$f^+ \in L_{M,\omega}(\Gamma), \qquad f^- \in L_{M,\omega}(\Gamma). \tag{3.72}$$

□

Lemma 3.19. *Let* $M \in QC_2^\theta(0,1)$ *and* $\omega \in A_{p(M)}(\mathbb{T})$. *Then there exists a constant* $c > 0$ *such that for every natural number* n

$$\|g - T_n g\|_{M,\mathbb{T},\omega} \le c\,\Omega^r_{M,\mathbb{T},\omega}\left(g, \frac{1}{(n+1)}\right), \quad g \in E_{M,\omega}(\mathbb{D}), \tag{3.73}$$

where $r > 0$ *and* $T_n g$ *is* n*th partial sum of the Taylor series of* g *at the origin.*

Proof. Using Theorem 2.7 this lemma can be proved by the same method of Theorem 3 of [45]. □

Let $\mathcal{P}$ be the set of all polynomials (with no restrictions on the degree), and let $\mathcal{P}(\mathbb{D})$ be the set of traces of members of $\mathcal{P}$ on $\mathbb{D}$. We define the operators $T : \mathcal{P}(\mathbb{D}) \to E_{M,\omega}(G_0)$ and $\widetilde{T} : \mathcal{P}(\mathbb{D}) \to \widetilde{E}_{M,\omega}(G_\infty)$ defined on $\mathcal{P}(\mathbb{D})$ as

$$\begin{aligned} T(P)(z) &:= \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{P(w)\psi'(w)}{\psi(w)-z}dw, \quad z \in G_0, \\ \widetilde{T}(P)(z) &:= \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{P(w)\psi_1'(w)}{\psi_1(w)-z}dw, \quad z \in G_\infty. \end{aligned} \tag{3.74}$$

Then it is readily seen that

$$T\left(\sum_{k=0}^{n} b_k w^k\right) = \sum_{k=0}^{n} b_k F_k(z), \qquad \widetilde{T}\left(\sum_{k=0}^{n} d_k w^k\right) = \sum_{k=0}^{n} d_k \widetilde{F}_k\left(\frac{1}{z}\right). \tag{3.75}$$

If $z' \in G_0$, then

$$T(P)(z') = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{P(w)\psi'(w)}{\psi(w)-z'}dw = \frac{1}{2\pi i}\int_{\Gamma} \frac{(P\circ\varphi)(\varsigma)}{\varsigma - z'}d\varsigma = (P\circ\varphi)^{+}(z'), \tag{3.76}$$

which, by (3.18), implies that

$$T(P)(z) = S_\Gamma(P\circ\varphi)(z) + \left(\frac{1}{2}\right)(P\circ\varphi)(z) \tag{3.77}$$

a.e. on Γ.

Similarly taking from outside of Γ the nontangential limit $z'' \to z \in \Gamma$ in the relation

$$\widetilde{T}(P)(z'') = \frac{1}{2\pi i}\int_{\Gamma} \frac{P(\varphi_1(\varsigma))}{\varsigma - z''}d\varsigma = \left[(P\circ\varphi_1)\right]^{-}(z''), \quad z'' \in G_\infty, \tag{3.78}$$

we get

$$\widetilde{T}(P)(z) = -\left(\frac{1}{2}\right)(P\circ\varphi_1)(z) + S_\Gamma(P\circ\varphi_1)(z) \tag{3.79}$$

a.e. on Γ.

Since S_Γ is bounded in $L_{M,\omega}(\Gamma)$, we have the following result.

Lemma 3.20. *Let Γ be a Dini-smooth curve, $M \in QC_2^{\theta}(0,1)$ and $f \in L_{M,\omega}(\Gamma)$ with $\omega \in A_{p(M)}(\Gamma)$. Then the linear operators*

$$T : \mathcal{P}(\mathbb{D}) \longrightarrow E_{M,\omega}(G_0), \qquad \widetilde{T} : \mathcal{P}(\mathbb{D}) \longrightarrow \widetilde{E}_{M,\omega}(G_\infty) \tag{3.80}$$

are bounded.

The set of trigonometric polynomials is dense in $L_{M,\omega}(\mathsf{T})$, which implies density of the algebraic polynomials in $E_{M,\omega}(\mathbb{D})$. Consequently, from Lemma 3.20, we can extend the operators T and $\widetilde{T}$ from $\mathcal{P}(\mathbb{D})$ to the spaces $E_{M,\omega_0}(\mathbb{D})$ and $E_{M,\omega_1}(\mathbb{D})$ as linear and bounded operators, respectively, and for the extensions $T : E_{M,\omega_0}(\mathbb{D}) \to E_{M,\omega}(G_0)$ and $\widetilde{T} : E_{M,\omega_1}(\mathbb{D}) \to \widetilde{E}_{M,\omega}(G_\infty)$, we have the representations

$$\begin{aligned} T(g)(z) &= \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{g(w)\psi'(w)}{\psi(w)-z}dw, \quad z \in G_0,\ g \in E_{M,\omega_0}(\mathbb{D}), \\ \widetilde{T}(g)(z) &= \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{g(w)\psi_1'(w)}{\psi_1(w)-z}dw, \quad z \in G_\infty,\ g \in E_{M,\omega_1}(\mathbb{D}). \end{aligned} \tag{3.81}$$

Lemma 3.21. *Let $M \in QC_2^\theta(0,1)$ and $f \in L_{M,\omega}(\mathbb{T})$ with $\omega \in A_{p(M)}(\mathbb{T})$. Then,*

$$\|P_r(f) - f\|_{M,\mathbb{T},\omega} \longrightarrow 0, \quad \text{as } r \longrightarrow 1^-, \tag{3.82}$$

where

$$P_r(f)\left(e^{i\theta}\right) := \frac{1}{2\pi}\int_0^{2\pi} P(r,\theta-t) f\left(e^{it}\right)dt, \quad 0 < r < 1 \tag{3.83}$$

and $P(r,\theta-t)$ is the Poisson kernel.

Proof. There are numbers p and q such that

$$1 < p < p(M) < q < \infty, \qquad \omega \in A_p(\mathbb{T}). \tag{3.84}$$

Since [46, Theorem 10] P_r is a bounded operator in $L^p(\mathbb{T},\omega)$ for every $1 < p < \infty$, we have by Marcinkiewicz Interpolation Theorem

$$\|P_r(f)\|_{M,\mathbb{T},\omega} \le c\|f\|_{M,\mathbb{T},\omega}. \tag{3.85}$$

From density of trigonometric polynomials in $L_{M,\omega}(\mathsf{T})$, we have density of the set of continuous functions on $\mathbb{T}$ in $L_{M,\omega}(\mathbb{T})$. Consequently, there is a continuous function f^* on $\mathbb{T}$ such that, for given $\epsilon > 0$ and $f \in L_{M,\omega}(\mathbb{T})$,

$$\|f - f^*\|_{M,\mathbb{T},\omega} < \epsilon. \tag{3.86}$$

On the other hand, since the Poisson integral of a continuous function converges to it uniformly on $\mathbb{T}$ [47, page 239], we have by (2.7) and $\omega \in A_p(\mathbb{T})$

$$\begin{aligned} \|P_r(f^*) - f^*\|_{M,\mathbb{T},\omega} &= \sup_{\rho(g,\widetilde{M})\le 1} \int_{\mathbb{T}} \left|P_r(f^*)(w) - f^*(w)\right|\left|g(w)\right|\omega(w)|dw| \\ &< \epsilon\left(M(1)\int_{\mathbb{T}} \omega(w)|dw| + 1\right) \le C\epsilon \end{aligned} \tag{3.87}$$

for $0 < 1 - r < \delta(\epsilon)$. Then, from (3.85), (3.86), and (3.87), we conclude that

$$\begin{aligned}\|P_r(f) - f\|_{M,\mathbb{T},\omega} &\le \|P_r(f) - P_r(f^*)\|_{M,\mathbb{T},\omega} + \|P_r(f^*) - f^*\|_{M,\mathbb{T},\omega} + \|f^* - f\|_{M,\mathbb{T},\omega} \\ &= \|P_r(f - f^*)\|_{M,\mathbb{T},\omega} + \|P_r(f^*) - f^*\|_{M,\mathbb{T},\omega} + \|f^* - f\|_{M,\mathbb{T},\omega} \\ &\le c\|f^* - f\|_{M,\mathbb{T},\omega} + \|P_r(f^*) - f^*\|_{M,\mathbb{T},\omega} < \{c + C\}\epsilon.\end{aligned} \tag{3.88}$$

This completes the proof. □

Theorem 3.22. *Let Γ be a Dini-smooth curve, $M \in QC_2^{\theta}(0,1)$ and $f \in L_{M,\omega}(\Gamma)$ with $\omega \in A_1(\Gamma)$. Then the linear operators*

$$T : E_{M,\omega_0}(\mathbb{D}) \longrightarrow E_{M,\omega}(G_0), \qquad \widetilde{T} : E_{M,\omega_1}(\mathbb{D}) \longrightarrow \widetilde{E}_{M,\omega}(G_\infty) \tag{3.89}$$

are one-to-one and onto.

Proof. The proof we give, only for the operator T. For the operator $\widetilde{T}$ is the proof goes similarly. Let $g \in E_{M,\omega_0}(\mathbb{D})$ with the Taylor expansion

$$g(w) := \sum_{k=0}^{\infty} \alpha_k w^k, \quad w \in \mathbb{D}. \tag{3.90}$$

Since Γ is a Dini-smooth curve, the conditions $\omega \in A_1(\Gamma)$, $\omega_0 \in A_1(\mathbb{T})$, and $\omega_1 \in A_1(\mathbb{T})$ are equivalent.

Let $g_r(w) := g(rw)$, $0 < r < 1$. Since $g \in E^1(\mathbb{D})$ is the Poisson integral of its boundary function [48, page 41], we have

$$\|g_r - g\|_{M,\mathbb{T},\omega_0} = \|P_r(g) - g\|_{M,\mathbb{T},\omega_0}, \tag{3.91}$$

and using Lemma 3.21, we get $\|g_r - g\|_{M,\mathbb{T},\omega_0} \to 0$, as $r \to 1^-$.

Therefore, the boundedness of the operator T implies that

$$\|T(g_r) - T(g)\|_{M,\Gamma,\omega} \longrightarrow 0, \quad \text{as } r \longrightarrow 1^-. \tag{3.92}$$

Since $\sum_{k=0}^{\infty} \alpha_k r^k w^k$ is uniformly convergent on $\mathbb{T}$, one has

$$\begin{aligned}T(g_r)(z') &= \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{g_r(w)\psi'(w)}{\psi(w) - z'}dw = \sum_{m=0}^{\infty} \alpha_m r^m \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{w^m \psi'(w)}{\psi(w) - z'}dw \\ &= \sum_{m=0}^{\infty} \alpha_m r^m F_m(z'), \quad z' \in G_0.\end{aligned} \tag{3.93}$$

From the last equality and Lemma 3 of [39, page 43] we have

$$\begin{aligned} a_k(T(g_r)) &= \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{T(g_r)(\psi(w))}{w^{k+1}}dw = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{\sum_{m=0}^{\infty}\alpha_m r^m F_m(\psi(w))}{w^{k+1}}dw \\ &= \sum_{m=0}^{\infty}\alpha_m r^m \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{F_m(\psi(w))}{w^{k+1}}dw = \alpha_k r^k, \end{aligned} \tag{3.94}$$

and therefore

$$a_k(T(g_r)) \longrightarrow \alpha_k, \quad \text{as } r \longrightarrow 1^-. \tag{3.95}$$

On the other hand, applying (3.33), (2.7), and weighted version of Hölder's inequality

$$\int_{\Gamma} |u(z)v(z)|\omega(z)|dz| \le c\|u\|_{M,\Gamma,\omega}\|v\|_{\widetilde{M},\Gamma,\omega}, \quad u \in L_{M,\omega}(\Gamma),\ v \in L_{\widetilde{M},\omega}(\Gamma), \tag{3.96}$$

we obtain

$$\begin{aligned} &|a_k(T(g_r)) - a_k(T(g))| \\ &\quad = \left|\frac{1}{2\pi i}\int_{\mathbb{T}} \frac{[T(g_r) - T(g)](\psi(w))}{w^{k+1}}dw\right| \\ &\quad \le \frac{1}{2\pi}\int_{\mathbb{T}} |[T(g_r) - T(g)](\psi(w))||dw| = \frac{1}{2\pi}\int_{\Gamma} |[T(g_r) - T(g)](z)||\varphi'(z)||dz| \\ &\quad \le \frac{c}{2\pi}\int_{\Gamma} |[T(g_r) - T(g)](z)||dz| = \frac{c}{2\pi}\int_{\Gamma} |[T(g_r) - T(g)](z)|\omega^{-1}(z)\omega(z)|dz| \\ &\quad \le \frac{c}{2\pi}\|T(g_r) - T(g)\|_{M,\Gamma,\omega}\left\|\omega^{-1}\right\|_{\widetilde{M},\Gamma,\omega} \le \frac{c}{2\pi}\|T(g_r) - T(g)\|_{M,\Gamma,\omega}, \end{aligned} \tag{3.97}$$

because $\|\omega^{-1}\|_{\widetilde{M},\Gamma,\omega} \le \widetilde{M}(1)\ \text{mes}\ (\Gamma) + 1 \le c < \infty$.

Using here the relation (3.92), we get

$$a_k(T(g_r)) \longrightarrow a_k(T(g)), \quad \text{as } r \longrightarrow 1^-, \tag{3.98}$$

and then by (3.95), $a_k(T(g)) = \alpha_k$ for $k = 0, 1, 2, \ldots$. If $T(g) = 0$, then $\alpha_k = a_k(T(g)) = 0,\ k = 0, 1, 2, \ldots$, and therefore $g = 0$. This means that the operator T is one-to-one.

Now we take a function $f \in E_{M,\omega}(G_0)$ and consider the function $f_0 = f \circ \psi \in L_{M,\omega_0}(\mathbb{T})$. The Cauchy type integral

$$\frac{1}{2\pi i}\int_{\mathbb{T}} \frac{f_0(\tau)}{\tau - w}d\tau \tag{3.99}$$

represents analytic functions f_0^+ and f_0^- in $\mathbb{D}$ and $\mathbb{D}_\infty$, respectively. Since $\omega_0 \in A_1(\mathbb{T})$, by Lemma 3.18, we have

$$f_0^+ \in E_{M,\omega_0}(\mathbb{D}), \qquad f_0^- \in \widetilde{E}_{M,\omega_0}(\mathbb{D}_\infty), \tag{3.100}$$

and moreover

$$f_0(w) = f_0^+(w) - f_0^-(w) \tag{3.101}$$

a.e. on $\mathbb{T}$. Since $f_0^- \in E^1(\mathbb{D}_\infty)$ and $f_0^-(\infty) = 0$, we have

$$a_k = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{f_0(w)}{w^{k+1}}dw = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{f_0^+(w)}{w^{k+1}}dw - \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{f_0^-(w)}{w^{k+1}}dw = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{f_0^+(w)}{w^{k+1}}dw, \tag{3.102}$$

which proves that the coefficients a_k, $k = 0, 1, 2, \ldots$, also become the Taylor coefficients of the function f_0^+ at the origin, that is,

$$f_0^+(w) = \sum_{k=0}^{\infty} a_k w^k, \quad w \in \mathbb{D}, \tag{3.103}$$

and also

$$T(f_0^+) \sim \sum_{k=0}^{\infty} a_k F_k. \tag{3.104}$$

Hence the functions $T(f_0^+)$ and f have the same Faber coefficients a_k, $k = 0, 1, 2, \ldots$, and therefore $T(f_0^+) = f$. This proves that the operator T is onto. □

Proof of Theorem 3.6. We prove that the rational function

$$R_n(z, f) := \sum_{k=0}^{n} a_k F_k(z) + \sum_{k=1}^{n} \widetilde{a}_k \widetilde{F}_k\left(\frac{1}{z}\right) \tag{3.105}$$

satisfies the required inequality of Theorem 3.6. This inequality is true if we can show that

$$\left\| f^-(z) + \sum_{k=1}^{n} \widetilde{a}_k \widetilde{F}_k\left(\frac{1}{z}\right) \right\|_{M,\Gamma,\omega} \le c\, \widetilde{\Omega}^r_{M,\Gamma,\omega}\left(f, \frac{1}{(n+1)}\right), \tag{3.106}$$

$$\left\| f^+(z) - \sum_{k=0}^{n} a_k F_k(z) \right\|_{M,\Gamma,\omega} \le c\, \Omega^r_{M,\Gamma,\omega}\left(f, \frac{1}{(n+1)}\right), \tag{3.107}$$

because $f(z) = f^+(z) - f^-(z)$ a.e. on Γ.

First we prove (3.106). Let $f \in L_{M,\omega}(\Gamma)$. Then $f_1 \in L_{M,\omega_1}(\mathbb{T})$ and $f_0 \in L_{M,\omega_0}(\mathbb{T})$. According to (3.101),

$$f(\varsigma) = f_0^+(\varphi(\varsigma)) - f_0^-(\varphi(\varsigma)) \tag{3.108}$$

a.e. on Γ. On the other hand,

$$f_1(w) = f_1^+(w) - f_1^-(w), \tag{3.109}$$

which implies the inequality

$$f(\varsigma) = f_1^+(\varphi_1(\varsigma)) - f_1^-(\varphi_1(\varsigma)) \tag{3.110}$$

a.e. on Γ.

Let $z' \in G_0 \setminus \{0\}$. Using (3.7) and (3.110), we have

$$\begin{aligned}
\sum_{k=1}^{n} \tilde{a}_k \tilde{F}_k\left(\frac{1}{z'}\right) &= \sum_{k=1}^{n} \tilde{a}_k \varphi_1^k(z') - \frac{1}{2\pi i}\int_\Gamma \frac{\sum_{k=1}^{n} \tilde{a}_k \varphi_1^k(\varsigma)}{\varsigma - z'} d\varsigma \\
&= \sum_{k=1}^{n} \tilde{a}_k \varphi_1^k(z') - \frac{1}{2\pi i}\int_\Gamma \frac{\left(\sum_{k=1}^{n} \tilde{a}_k \varphi_1^k(\varsigma) - f_1^+(\varphi_1(\varsigma))\right)}{\varsigma - z'} d\varsigma - \frac{1}{2\pi i}\int_\Gamma \frac{f_1^-(\varphi_1(\varsigma))}{\varsigma - z'} d\varsigma \\
&\quad - \frac{1}{2\pi i}\int_\Gamma \frac{f(\varsigma)}{\varsigma - z'} d\varsigma \\
&= \sum_{k=1}^{n} \tilde{a}_k \varphi_1^k(z') - \frac{1}{2\pi i}\int_\Gamma \frac{\left(\sum_{k=1}^{n} \tilde{a}_k \varphi_1^k(\varsigma) - f_1^+(\varphi_1(\varsigma))\right)}{\varsigma - z'} d\varsigma - f_1^-(\varphi_1(z')) - f^-(z').
\end{aligned} \tag{3.111}$$

Hence, taking the nontangential limit $z' \to z \in \Gamma$, inside of Γ, we obtain

$$\begin{aligned}
\sum_{k=1}^{n} \tilde{a}_k \tilde{F}_k\left(\frac{1}{z}\right) &= \sum_{k=1}^{n} \tilde{a}_k \varphi_1^k(z) - \frac{1}{2}\left(\sum_{k=1}^{n} \tilde{a}_k \varphi_1^k(z) - f_1^+(\varphi_1(z))\right) \\
&\quad - S_\Gamma\left[\sum_{k=1}^{n} \tilde{a}_k \varphi_1^k - (f_1^+ \circ \varphi_1)\right] - f_1^-(\varphi_1(z)) - f^+(z)
\end{aligned} \tag{3.112}$$

a.e. on Γ.

Using (3.19), (3.110), Minkowski's inequality, and the boundedness of S_Γ, we get

$$\begin{aligned}
&\left\| f^-(z) + \sum_{k=1}^{n} \tilde{a}_k \tilde{F}_k\left(\frac{1}{z}\right) \right\|_{M,\Gamma,\omega} \\
&\quad = \left\| \frac{1}{2}\left(\sum_{k=1}^{n} \tilde{a}_k \varphi_1^k(z) - f_1^+(\varphi_1(z)) \right) - S_\Gamma\left[\sum_{k=1}^{n} \tilde{a}_k \varphi_1^k - (f_1^+ \circ \varphi_1) \right](z) \right\|_{M,\Gamma,\omega} \\
&\quad \le c \left\| \sum_{k=1}^{n} \tilde{a}_k \varphi_1^k(z) - f_1^+(\varphi_1(z)) \right\|_{M,\Gamma,\omega} \le c \left\| f_1^+(w) - \sum_{k=1}^{n} \tilde{a}_k w^k \right\|_{M,\mathbb{T},\omega_1}.
\end{aligned} \tag{3.113}$$

On the other hand, from the proof of Theorem 3.22 we know that the Faber-Laurent coefficients $\tilde{a}_k$ of the function f and the Taylor coefficients of the function f_1^+ at the origin are the same. Then taking Lemma 3.19 into account, we conclude that

$$\left\| f^- + \sum_{k=1}^{n} \tilde{a}_k \tilde{F}_k\left(\frac{1}{z}\right) \right\|_{M,\Gamma,\omega} \le c\, \Omega^r_{M,\mathbb{T},\omega_1}\left(f_1^+, \frac{1}{(n+1)}\right) = c\, \tilde{\Omega}^r_{M,\Gamma,\omega}\left(f, \frac{1}{(n+1)}\right), \tag{3.114}$$

and (3.106) is proved.

The proof of relation (3.107) goes similarly; we use the relations (3.6) and (3.108) instead of (3.7) and (3.110), respectively. Hence (3.19), (3.106), and (3.107) complete the proof. □

Proof of Theorem 3.9. Let $f \in E_{M,\omega}(G_0)$. Then we have $T(f_0^+) = f$. Since by Theorem 3.22 the operator $T : E_{M,\omega_0}(\mathbb{D}) \to E_{M,\omega}(G_0)$ is linear, bounded, one-to-one and onto, the operator $T^{-1} : E_{M,\omega}(G_0) \to E_{M,\omega_0}(\mathbb{D})$ is also linear and bounded. We take a $p_n^* \in \mathcal{P}_n$ as the best approximating algebraic polynomial to f in $E_{M,\omega}(G_0)$, that is,

$$E_n(f)_{M,\Gamma,\omega} = \|f - p_n^*\|_{M,\Gamma,\omega}. \tag{3.115}$$

Then, $T^{-1}(p_n^*) \in \mathcal{P}_n(\mathbb{D})$, and therefore

$$\begin{aligned}
E_n(f_0^+)_{M,\mathbb{T},\omega_0} &\le \left\| f_0^+ - T^{-1}(p_n^*) \right\|_{M,\mathbb{T},\omega_0} = \left\| T^{-1}(f) - T^{-1}(p_n^*) \right\|_{M,\mathbb{T},\omega_0} \\
&= \left\| T^{-1}(f - p_n^*) \right\|_{M,\mathbb{T},\omega_0} \le \left\| T^{-1} \right\| \|f - p_n^*\|_{M,\Gamma,\omega} = \left\| T^{-1} \right\| E_n(f)_{M,\Gamma,\omega},
\end{aligned} \tag{3.116}$$

because the operator T^{-1} is bounded.

Theorem 2.8 and (3.116) imply that

$$\begin{aligned}
\Omega^r_{M,\Gamma,\omega}\left(f, \frac{1}{n}\right) = \Omega^r_{M,\mathbb{T},\omega_0}\left(f_0^+, \frac{1}{n}\right) &\le \frac{c}{n^r}\left\{ E_0(f_0^+)_{M,\mathbb{T},\omega_0} + \sum_{k=1}^{n} k^{r-1} E_k(f_0^+)_{M,\mathbb{T},\omega_0} \right\} \\
&\le \frac{c\|T^{-1}\|}{n^r}\left\{ E_0(f)_{M,\Gamma,\omega} + \sum_{k=1}^{n} k^{r-1} E_k(f)_{M,\Gamma,\omega} \right\}, \quad r > 0.
\end{aligned} \tag{3.117}$$

□

References

[1] G. M. Goluzin, *Geometric Theory of Functions of a Complex Variable*, vol. 26 of *Translations of Mathematical Monographs*, American Mathematical Society, Providence, RI, USA, 1969.

[2] J. L. Walsh and H. G. Russell, "Integrated continuity conditions and degree of approximation by polynomials or by bounded analytic functions," *Transactions of the American Mathematical Society*, vol. 92, pp. 355–370, 1959.

[3] S. Ya. Al'per, "Approximation in the mean of analytic functions of class E^p," in *Investigations on the Modern Problems of the Function Theory of a Complex Variable*, pp. 273–286, Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, Russia, 1960.

[4] V. M. Kokilashvili, "Approximation in the mean of analytic functions of class E_p," *Soviet Mathematics. Doklady*, vol. 8, pp. 1393–1397, 1967.

[5] W. Kokilashvili, "A direct theorem on mean approximation of analytic functions by polynomials," *Soviet Mathematics. Doklady*, vol. 10, pp. 411–414, 1969.

[6] J.-E. Andersson, "On the degree of polynomial approximation in $E^p(D)$," *Journal of Approximation Theory*, vol. 19, no. 1, pp. 61–68, 1977.

[7] D. M. Israfilov, "Approximate properties of the generalized Faber series in an integral metric," *Izvestiya Akademii Nauk Azerbaidzhanskoi SSR*, vol. 2, pp. 10–14, 1987 (Russian).

[8] A. Çavuş and D. M. Israfilov, "Approximation by Faber-Laurent rational functions in the mean of functions of class $L_p(\Gamma)$ with $1 < p < \infty$," *Approximation Theory and its Applications*, vol. 11, no. 1, pp. 105–118, 1995.

[9] D. M. Israfilov, "Approximation by p-Faber polynomials in the weighted Smirnov class $E^p(G,\omega)$ and the Bieberbach polynomials," *Constructive Approximation*, vol. 17, no. 3, pp. 335–351, 2001.

[10] D. M. Israfilov, "Approximation by p-Faber-Laurent rational functions in the weighted Lebesgue spaces," *Czechoslovak Mathematical Journal*, vol. 54, no. 3, pp. 751–765, 2004.

[11] D. M. Israfilov and A. Guven, "Approximation in weighted Smirnov classes," *East Journal on Approximations*, vol. 11, no. 1, pp. 91–102, 2005.

[12] A. Guven and D. M. Israfilov, "Improved inverse theorems in weighted Lebesgue and Smirnov spaces," *Bulletin of the Belgian Mathematical Society. Simon Stevin*, vol. 14, no. 4, pp. 681–692, 2007.

[13] W. Kokilashvili, "On analytic functions of Smirnov-Orlicz classes," *Studia Mathematica*, vol. 31, pp. 43–59, 1968.

[14] A. Guven and D. M. Israfilov, "Polynomial approximation in Smirnov-Orlicz classes," *Computational Methods and Function Theory*, vol. 2, no. 2, pp. 509–517, 2002.

[15] D. M. Israfilov, B. Oktay, and R. Akgun, "Approximation in Smirnov-Orlicz classes," *Glasnik Matematički III*, vol. 40, no. 1, pp. 87–102, 2005.

[16] R. Akgün and D. M. Israfilov, "Approximation by interpolating polynomials in Smirnov-Orlicz class," *Journal of the Korean Mathematical Society*, vol. 43, no. 2, pp. 413–424, 2006.

[17] D. M. Israfilov and R. Akgün, "Approximation in weighted Smirnov-Orlicz classes," *Journal of Mathematics of Kyoto University*, vol. 46, no. 4, pp. 755–770, 2006.

[18] R. Akgün and D. M. Israfilov, "Polynomial approximation in weighted Smirnov-Orlicz space," *Proceedings of A. Razmadze Mathematical Institute*, vol. 139, pp. 89–92, 2005.

[19] D. M. Israfilov and R. Akgün, "Approximation by polynomials and rational functions in weighted rearrangement invariant spaces," *Journal of Mathematical Analysis and Applications*, vol. 346, no. 2, pp. 489–500, 2008.

[20] R. Akgün and D. M. Israfilov, "Approximation and moduli of smoothness of fractional order in Smirov-Orlicz spaces," *Glasnik Matematicki III*, vol. 42, no. 1, pp. 121–136, 2008.

[21] A. I. Shvai, "Approximation on analytic functions by Poisson polynomials," *Ukrainian Mathematical Journal*, vol. 25, no. 6, pp. 710–713, 1973.

[22] A. Gogatishvili and V. Kokilashvili, "Criteria of weighted inequalities in Orlicz classes for maximal functions defined on homogeneous type spaces," *Georgian Mathematical Journal*, vol. 1, no. 6, pp. 641–673, 1994.

[23] I. Genebashvili, A. Gogatishvili, V. Kokilashvili, and M. Krbec, *Weight Theory for Integral Transforms on Spaces of Homogeneous Type*, vol. 92 of *Pitman Monographs and Surveys in Pure and Applied Mathematics*, Longman, Harlow, UK, 1998.

[24] R. Akgün and D. M. Israfilov, "Simultaneous and converse approximation theorems in weighted Orlicz spaces," *Bulletin of the Belgian Mathematical Society. Simon Stevin*, vol. 17, no. 1, pp. 13–28, 2010.

[25] E. A. Gadjieva, *Investigation of the Properties of Functions with Quasimonotone Fourier Coeffcients in Generalized Nikolskii-Besov Spaces*, Author's Summary of Dissertation, Tbilisi, Georgia, 1986.

[26] N. X. Ky, "On approximation by trigonometric polynomials in L_u^p-spaces," *Studia Scientiarum Mathematicarum Hungarica*, vol. 28, no. 1-2, pp. 183–188, 1993.
[27] N. X. Ky, "Moduli of mean smoothness and approximation with A_p-weights," *Annales Universitatis Scientiarum Budapestinensis de Rolando Eötvös Nominatae. Sectio Mathematica*, vol. 40, pp. 37–48, 1997.
[28] L. Ephremidze, V. Kokilashvili, and Y. E. Yildirir, "On the inverse inequalities for trigonometric polynomial approximations in weighted Lorentz spaces," *Proceedings of A. Razmadze Mathematical Institute*, vol. 144, pp. 132–136, 2007.
[29] V. Kokilashvili and Y. E. Yildirir, "On the approximation in weighted Lebesgue spaces," *Proceedings of A. Razmadze Mathematical Institute*, vol. 143, pp. 103–113, 2007.
[30] G. Mastroianni and V. Totik, "Best approximation and moduli of smoothness for doubling weights," *Journal of Approximation Theory*, vol. 110, no. 2, pp. 180–199, 2001.
[31] Z. Ditzian and D. S. Lubinsky, "Jackson and smoothness theorems for Freud weights in $L_p(0 < p \leq \infty)$," *Constructive Approximation*, vol. 13, no. 1, pp. 99–152, 1997.
[32] Z. Ditzian and V. Totik, *Moduli of Smoothness*, vol. 9 of *Springer Series in Computational Mathematics*, Springer, New York, NY, USA, 1987.
[33] H. N. Mhaskar, *Introduction to the Theory of Weighted Polynomial Approximation*, vol. 7 of *Series in Approximations and Decompositions*, World Scientific, River Edge, NJ, USA, 1996.
[34] A. Zygmund, *Trigonometric Series*, vol. 1-2, Cambridge University Press, New York, NY, USA, 2nd edition, 1959.
[35] S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional Integrals and Derivatives*, Gordon and Breach Science Publishers, Yverdon, Switzerland, 1993.
[36] I. Joó, "Saturation theorems for Hermite-Fourier series," *Acta Mathematica Hungarica*, vol. 57, no. 1-2, pp. 169–179, 1991.
[37] M. Khabazi, "The mean convergence of trigonometric Fourier series in weighted Orlicz classes," *Proceedings of A. Razmadze Mathematical Institute*, vol. 129, pp. 65–75, 2002.
[38] P. K. Suetin, *Series of Faber Polynomials*, vol. 1 of *Analytical Methods and Special Functions*, Gordon and Breach Science Publishers, Amsterdam, The Netherlands, 1998.
[39] D. Gaier, *Lectures on Complex Approximation*, Birkhäuser, Boston, Mass, USA, 1987.
[40] V. I. Smirnov and N. A. Lebedev, *Functions of a Complex Variable: Constructive Theory*, Translated from the Russian by Scripta Technica, The M.I.T. Press, Cambridge, Mass, USA, 1968.
[41] X. C. Shen and L. Zhong, "On Lagrange Interpolation in $E^p(D)$ for $1 < p < \infty$," *Advanced Mathematics*, vol. 18, pp. 342–345, 1989 (Chinese).
[42] L. Y. Zhu, "A new class of interpolation nodes," *Advances in Mathematics*, vol. 24, no. 4, pp. 327–334, 1995 (Chinese).
[43] L. Zhong and L. Zhu, "Convergence of interpolants based on the roots of Faber polynomials," *Acta Mathematica Hungarica*, vol. 65, no. 3, pp. 273–283, 1994.
[44] S. Warschawski, "Über das randverhalten der ableitung der abbildungsfunktion bei konformer abbildung," *Mathematische Zeitschrift*, vol. 35, no. 1, pp. 321–456, 1932.
[45] D. M. Israfilov and A. Guven, "Approximation by trigonometric polynomials in weighted Orlicz spaces," *Studia Mathematica*, vol. 174, no. 2, pp. 147–168, 2006.
[46] B. Muckenhoupt, "Weighted norm inequalities for the Hardy maximal function," *Transactions of the American Mathematical Society*, vol. 165, pp. 207–226, 1972.
[47] W. Rudin, *Real and Complex Analysis*, McGraw-Hill, New York, NY, USA, 3rd edition, 1987.
[48] P. L. Duren, *Theory of H^p Spaces*, vol. 38 of *Pure and Applied Mathematics*, Academic Press, New York, NY, USA, 1970.