Research Article

Boundedness of n -Multiple Discrete Hardy Operators with Weights for $1 < q < p < \infty$

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We find necessary and sufficient conditions on weighted sequences ω_i , $i = 1, 2, \dots, n-1$, u , and v , for which the operator $(S_{n-1}f)_i = \sum_{k_1=1}^i \omega_{1,k_1} \cdots \sum_{k_{n-1}=1}^{k_{n-2}} \omega_{n-1,k_{n-1}} \sum_{j=1}^{k_{n-1}} f_j$, $i \geq 1$, is bounded from $l_{p,v}$ to $l_{q,u}$ for $1 < q < p < \infty$.

1. Introduction

Let $1 < p, q < \infty$ and $1/p + 1/p' = 1$. Let $v = \{v_k\}_{k=1}^\infty$ be positive, $\omega_i = \{\omega_{i,k}\}_{k=1}^\infty$, $i = 1, 2, \dots, n-1$, and $u = \{u_k\}_{k=1}^\infty$ nonnegative number sequences, hereinafter referred to as weights.

Let $l_{p,v}$ be the space of sequences $f = \{f_k\}_{k=1}^\infty$, for which the following norm is finite:

$$\|f\|_{p,v} = \left(\sum_{i=1}^\infty |v_i f_i|^p \right)^{1/p}. \quad (1)$$

Consider the operator S_{n-1} that acts on the sequence $f = \{f_k\}_{k=1}^\infty$ as follows:

$$(S_{n-1}f)_i = \sum_{k_1=1}^i \omega_{1,k_1} \sum_{k_2=1}^{k_1} \omega_{2,k_2} \cdots \sum_{k_{n-1}=1}^{k_{n-2}} \omega_{n-1,k_{n-1}} \sum_{j=1}^{k_{n-1}} f_j, \quad i \geq 1, \quad (2)$$

and call it n -multiple discrete Hardy operator with weights.

By changing the order of summation in (2) we have

$$(S_{n-1}f)_i = \sum_{j=1}^i A_{n-1,1}(i, j) f_j, \quad i \geq 1, \quad (3)$$

where $A_{n-1,1}(i, j) \equiv 1$ for $n = 1$ and

$$A_{n-1,1}(i, j) = \sum_{k_{n-1}=j}^i \omega_{n-1,k_{n-1}} \sum_{k_{n-2}=k_{n-1}}^i \omega_{n-2,k_{n-2}} \cdots \sum_{k_1=k_2}^i \omega_{1,k_1}, \quad i \geq j \geq 1, \quad (4)$$

for $n > 1$.

Together with operator (3) we consider its dual operator

$$(S_{n-1}^*f)_j = \sum_{i=j}^\infty A_{n-1,1}(i, j) f_i, \quad j \geq 1. \quad (5)$$

When $n = 1$, the operators S_{n-1} and S_{n-1}^* are simple discrete Hardy operators $(S_0f)_i = \sum_{j=1}^i f_j$ and $(S_0^*f)_i = \sum_{j=i}^\infty f_j$; the problem of boundedness from $l_{p,v}$ to $l_{q,u}$ is investigated in detail in [1–3] for all values of the parameters p and q .

In paper [4] necessary and sufficient conditions of boundedness of operators (3) and (5) from $l_{p,v}$ to $l_{q,u}$ are obtained for the case $1 < p \leq q < \infty$. However, the problem of boundedness of these operators has not been studied for the case $1 < q < p < \infty$. In this paper we consider this problem.

Let us notice that the boundedness problem of continuous Hardy-type operators has been studied and developed

in an extraordinary depth (see, e.g., [3]). The corresponding results for n -multiple integral Hardy operators were considered by Baiarystanov in [5]. Namely, he investigated the continuous analogue of operators (3) and (5) and found necessary and sufficient conditions of their boundedness from L_p to L_q for the case $1 < p \leq q < \infty$. Moreover, there he obtained sufficient conditions of the same problem for the case $1 < q < p < \infty$. Later in [6] Sagindykov proved that sufficient conditions found by Baiarystanov for the case $1 < q < p < \infty$ are also necessary. However, the method for the continuous case does not work in the discrete situation as it was mentioned in [4]. Thus, here we present a completely different method.

In the sequel we suppose that the sum $\sum_{i=t}^k$ is equal to zero for $t > k$; the symbol $F \ll E$ means $F \leq cE$, where a positive constant c does not depend on arguments of the expressions F and E but can depend on the parameters p and q . The relationship $F \approx E$ means $F \ll E \ll F$.

2. Auxiliary Statements

For all $i \geq j \geq 1$ we assume that $A_{l,m}(i, j) \equiv 1$ if $l < m$ and $A_{l,m}(i, j) = \sum_{k_l=j}^i \omega_{l,k_l} \sum_{k_{l-1}=k_l}^i \omega_{l-1,k_{l-1}} \cdots \sum_{k_m=k_{m+1}}^i \omega_{m,k_m}$ if $n-1 \geq l \geq m \geq 1$. Moreover, for all $i < j$ we suppose that $A_{l,m}(i, j) = 0$ if $l, m \geq 1$.

In paper [4] the following lemma is proved.

Lemma 1 (see [4]). *For all $i, j, \tau : 1 \leq j \leq \tau \leq i$ one has*

$$\begin{aligned} \max_{m-1 \leq r \leq l} A_{r,m}(i, \tau) A_{l,r+1}(\tau, j) &\leq A_{l,m}(i, j) \\ &\leq \sum_{r=m-1}^l A_{r,m}(i, \tau) A_{l,r+1}(\tau, j), \end{aligned} \quad (6)$$

when $n-1 \geq l \geq m \geq 1$.

Lemma 2. *Let $(B_{i,j})_{i,j=1}^\infty$ be a nonnegative matrix whose elements do not decrease in the first index and do not increase in the second index. Let $\gamma > 0$. Then for $1 \leq t, k \leq \infty$ one has*

$$\Delta_j^+ \left(\sum_{i=j}^k B_{i,j} \right)^\gamma \approx \left(\sum_{i=j}^k B_{i,j} \right)^{\gamma-1} \Delta_j^+ \left(\sum_{i=j}^k B_{i,j} \right), \quad (7)$$

$$\Delta_j^- \left(\sum_{i=t}^j B_{i,j} \right)^\gamma \approx \left(\sum_{i=t}^j B_{i,j} \right)^{\gamma-1} \Delta_j^- \left(\sum_{i=t}^j B_{i,j} \right), \quad (8)$$

where $\Delta_j^+ E_{i,j} = E_{i,j} - E_{i,j+1}$ and $\Delta_i^- E_{i,j} = E_{i,j} - E_{i-1,j}$.

Generally speaking the statement of Lemma 2 is known, but for more complete presentation let us give its proof. We prove only relation (7). Relation (8) can be proved similarly.

Proof. Let $\gamma \geq 1$. Then

$$\begin{aligned} \Delta_j^+ \left(\sum_{i=j}^k B_{i,j} \right)^\gamma &= \left(\sum_{i=j}^k B_{i,j} \right)^\gamma - \left(\sum_{i=j+1}^k B_{i,j+1} \right)^\gamma \\ &\leq \gamma \left(\sum_{i=j}^k B_{i,j} \right)^{\gamma-1} \Delta_j^+ \left(\sum_{i=j}^k B_{i,j} \right), \\ \Delta_j^+ \left(\sum_{i=j}^k B_{i,j} \right)^\gamma &\geq \left(\sum_{i=j}^k B_{i,j} \right)^\gamma - \left(\sum_{i=j}^k B_{i,j} \right)^{\gamma-1} \sum_{i=j+1}^k B_{i,j+1} \\ &= \left(\sum_{i=j}^k B_{i,j} \right)^{\gamma-1} \Delta_j^+ \left(\sum_{i=j}^k B_{i,j} \right). \end{aligned} \quad (9)$$

If $0 < \gamma < 1$, then these inequalities hold in opposite direction. Therefore (7) holds for all $\gamma > 0$. The proof of Lemma 2 is complete. $\square$

Moreover, we need the following obvious relations:

$$\begin{aligned} \left(\sum_{i=t}^k B_{i,t} \right)^\gamma &= \sum_{j=t}^k \Delta_j^+ \left(\sum_{i=j}^k B_{i,j} \right)^\gamma, \\ \left(\sum_{j=t}^k B_{k,j} \right)^\gamma &= \sum_{i=t}^k \Delta_i^- \left(\sum_{j=t}^i B_{i,j} \right)^\gamma. \end{aligned} \quad (10)$$

We also use the following lemma.

Lemma 3 (see [1]). *Let $1 < p < q < \infty$. Then the operator S_0^* is bounded from $l_{p,v}$ to $l_{q,u}$ if and only if*

$$\begin{aligned} B_0(1) &= \left(\sum_{i=1}^\infty \left(\sum_{j=1}^i u_j^q \right)^{p/(p-q)} \left(\sum_{k=i}^\infty v_k^{-p'} \right)^{p(q-1)/(p-q)} v_i^{-p'} \right)^{(p-q)/pq} \\ &< \infty; \end{aligned} \quad (11)$$

moreover, $B_0(1) \approx \|S_0^*\|$.

3. Main and Associated Results

For $r = 0, 1, \dots, n-1$, we assume that

$$A_r(n) = \left(\sum_{i=1}^\infty \left(\sum_{j=i}^\infty A_{r,1}^q(j, i) u_j^q \right) \right)^{q/(p-q)}$$

$$\begin{aligned}
& \times \left(\sum_{k=1}^i A_{n-1,r+1}^{p'}(i, k) v_k^{-p'} \right)^{q(p-1)/(p-q)} \\
& \times \Delta_i^+ \left(\sum_{j=i}^{\infty} A_{r,1}^q(j, i) u_j^q \right)^{(p-q)/pq}, \\
B_r(n) &= \left(\sum_{i=1}^{\infty} \left(\sum_{j=i}^{\infty} A_{r,1}^{p'}(j, i) v_j^{-p'} \right)^{p(q-1)/(p-q)} \right. \\
& \times \left(\sum_{k=1}^i A_{n-1,r+1}^q(i, k) u_k^q \right)^{p/(p-q)} \\
& \left. \times \Delta_i^+ \left(\sum_{j=i}^{\infty} A_{r,1}^{p'}(j, i) v_j^{-p'} \right)^{(p-q)/pq} \right). \quad (12)
\end{aligned}$$

Theorem 4. Let $1 < q < p < \infty$ and $n \geq 1$. Then operator (3) is bounded from $l_{p,v}$ to $l_{q,u}$ if and only if $A(n) = \max_{0 \leq r \leq n-1} A_r(n) < \infty$. Moreover, for the norm of operator (3) from $l_{p,v}$ to $l_{q,u}$ the following relation $\|S_{n-1}\| \approx A(n)$ holds.

Theorem 5. Let $1 < q < p < \infty$ and $n \geq 1$. Then operator (5) is bounded from $l_{p,v}$ to $l_{q,u}$ if and only if $B(n) = \max_{0 \leq r \leq n-1} B_r(n) < \infty$. Moreover, for the norm of operator (5) from $l_{p,v}$ to $l_{q,u}$ the following relation $\|S_{n-1}^*\| \approx B(n)$ holds.

For the proofs of Theorems 4 and 5 we need to establish several statements.

Assume that

$$\begin{aligned}
\tilde{u}_i^q &= \Delta_i^- \left(\sum_{j=1}^i A_{n-1,r+1}^q(i, j) u_j^q \right), \\
(S_r f)_i &= \sum_{j=1}^i A_{r,1}(i, j) f_j, \quad (13) \\
(S_r^* g)_j &= \sum_{i=j}^{\infty} A_{r,1}(i, j) g_i, \quad i, j \geq 1, \quad 0 \leq r \leq n-1.
\end{aligned}$$

Consider the inequality

$$\|S_r^* g\|_{q,\tilde{u}} \leq C \|g\|_{p,v}. \quad (14)$$

Lemma 6. Let $1 < p, q < \infty$. Then the inequality

$$\|S_r f \tilde{u}^q\|_{p',v^{-1}} \leq C \|f\|_{q',\tilde{u}^{q-1}}, \quad f \geq 0, \quad (15)$$

is dual to (14) with respect to the linear form $\sum_{i=1}^{\infty} f_i g_i \tilde{u}_i^q$.

Proof of Lemma 6. If $C > 0$ is the best constant in (14), then on the basis of principle of duality in l_q we have

$$\begin{aligned}
C &= \sup_{g \geq 0} \frac{\|S_r^* g\|_{q,\tilde{u}}}{\|g\|_{p,v}} = \sup_{g \geq 0} \sup_{f \geq 0} \frac{\sum_{k=1}^{\infty} f_k (S_r^* g)_k \tilde{u}_k^q}{\|f\|_{q',\tilde{u}^{q-1}} \|g\|_{p,v}} \\
&= \sup_{g \geq 0} \sup_{f \geq 0} \frac{\sum_{i=1}^{\infty} g_i (S_r f \tilde{u}^q)_i}{\|g\|_{p,v} \|f\|_{q',\tilde{u}^{q-1}}} = \sup_{f \geq 0} \frac{\|S_r f \tilde{u}^q\|_{p',v^{-1}}}{\|f\|_{q',\tilde{u}^{q-1}}}. \quad (16)
\end{aligned}$$

□

Lemma 7. Let $1 < q < p < \infty$ and $0 \leq r \leq n-1$. If inequality (14) holds with the least constant $C > 0$, then

$$B_r(n) \leq C. \quad (17)$$

Proof of Lemma 7. Suppose that inequality (14) holds. Then due to Lemma 6 inequality (15) also holds with the same best constant as in (14).

Let $t \geq x \geq 1$. Introduce the test sequence $f = \{f_j\}_{j=1}^{\infty}$ as follows: $f_j = 0$ for $j > x$ and $f_j = (\sum_{i=1}^j \tilde{u}_i^q)^{(q-1)/(p-q)} (\sum_{k=j}^t A_{r,1}^{p'}(k, j) v_k^{-p'})^{(p-1)(q-1)/(p-q)}$ for $1 \leq j \leq x$.

If we substitute this sequence in the right side of (15), we have

$$\begin{aligned}
& \|f\|_{q',\tilde{u}^{q-1}} \\
&= \left(\sum_{j=1}^x \left(\sum_{i=1}^j \tilde{u}_i^q \right)^{q/(p-q)} \right. \\
& \quad \left. \times \left(\sum_{k=j}^t A_{r,1}^{p'}(k, j) v_k^{-p'} \right)^{q(p-1)/(p-q)} \tilde{u}_j^q \right)^{1/q'}. \quad (18)
\end{aligned}$$

Now we will work with the left side of (15) and the test sequence:

$$\begin{aligned}
& \|S_r f \tilde{u}^q\|_{p',v^{-1}}^{p'} \\
&= \sum_{i=1}^{\infty} v_i^{-p'} \left(\sum_{j=1}^i A_{r,1}(i, j) f_j \tilde{u}_j^q \right) \left(\sum_{k=1}^i A_{r,1}(i, k) f_k \tilde{u}_k^q \right)^{p'-1} \quad (19)
\end{aligned}$$

(reduce the sum in the second brackets)

$$\geq \sum_{i=1}^{\infty} v_i^{-p'} \sum_{j=1}^i A_{r,1}(i, j) f_j \tilde{u}_j^q \left(\sum_{k=1}^j A_{r,1}(i, k) f_k \tilde{u}_k^q \right)^{p'-1} \quad (20)$$

(change order of summation)

$$= \sum_{j=1}^{\infty} f_j \sum_{i=j}^{\infty} A_{r,1}(i, j) v_i^{-p'} \left(\sum_{k=1}^j A_{r,1}(i, k) f_k \tilde{u}_k^q \right)^{p'-1} \tilde{u}_j^q \quad (21)$$

(use nonincreasing of $A_{r,1}(i, k)$ in the second argument)

$$\geq \sum_{j=1}^{\infty} f_j \sum_{i=j}^{\infty} A_{r,1}^{p'}(i, j) v_i^{-p'} \left(\sum_{k=1}^j f_k \tilde{u}_k^q \right)^{p'-1} \tilde{u}_j^q \quad (22)$$

(substitute the test sequence)

$$\begin{aligned} &\geq \sum_{j=1}^x \left(\sum_{i=1}^j \tilde{u}_i^q \right)^{(q-1)/(p-q)} \left(\sum_{k=j}^t A_{r,1}^{p'}(k, j) v_k^{-p'} \right)^{(p-1)(q-1)/(p-q)} \\ &\quad \times \sum_{i=j}^t A_{r,1}^{p'}(i, j) v_i^{-p'} \\ &\quad \times \left(\sum_{k=1}^j \left(\sum_{i=1}^k \tilde{u}_i^q \right)^{(q-1)/(p-q)} \right. \\ &\quad \left. \times \left(\sum_{i=k}^t A_{r,1}^{p'}(i, k) v_i^{-p'} \right)^{(p-1)(q-1)/(p-q)} \tilde{u}_k^q \right)^{p'-1} \tilde{u}_j^q \end{aligned} \quad (23)$$

(join together the similar cofactors)

$$\begin{aligned} &\geq \sum_{j=1}^x \left(\sum_{i=1}^j \tilde{u}_i^q \right)^{(q-1)/(p-q)} \left(\sum_{k=j}^t A_{r,1}^{p'}(k, j) v_k^{-p'} \right)^{q(p-1)/(p-q)} \\ &\quad \times \left(\sum_{k=1}^j \left(\sum_{i=1}^k \tilde{u}_i^q \right)^{(q-1)/(p-q)} \tilde{u}_k^q \right)^{p'-1} \tilde{u}_j^q \end{aligned} \quad (24)$$

(use (8))

$$\begin{aligned} &\gg \sum_{j=1}^x \left(\sum_{i=1}^j \tilde{u}_i^q \right)^{(q-1)/(p-q)} \left(\sum_{k=j}^t A_{r,1}^{p'}(k, j) v_k^{-p'} \right)^{q(p-1)/(p-q)} \\ &\quad \times \left(\sum_{k=1}^j \Delta_k^- \left(\sum_{i=1}^k \tilde{u}_i^q \right)^{(p-1)/(p-q)} \right)^{p'-1} \tilde{u}_j^q \end{aligned} \quad (25)$$

(simplify)

$$\begin{aligned} &= \sum_{j=1}^x \left(\sum_{k=j}^t A_{r,1}^{p'}(k, j) v_k^{-p'} \right)^{q(p-1)/(p-q)} \\ &\quad \times \left(\sum_{i=1}^j \tilde{u}_i^q \right)^{q/(p-q)} \tilde{u}_j^q. \end{aligned} \quad (26)$$

Therefore,

$$\begin{aligned} \|S_r f \tilde{u}^q\|_{p', v^{-1}}^{p'} &\gg \left(\sum_{j=1}^x \left(\sum_{k=j}^t A_{r,1}^{p'}(k, j) v_k^{-p'} \right)^{q(p-1)/(p-q)} \right. \\ &\quad \left. \times \left(\sum_{i=1}^j \tilde{u}_i^q \right)^{q/(p-q)} \tilde{u}_j^q \right)^{1/p'} \end{aligned} \quad (27)$$

which, together with (15) and (18), gives

$$\begin{aligned} C &\gg \left(\sum_{j=1}^x \left(\sum_{k=j}^t A_{r,1}^{p'}(k, j) v_k^{-p'} \right)^{q(p-1)/(p-q)} \right. \\ &\quad \left. \times \left(\sum_{i=1}^j \tilde{u}_i^q \right)^{q/(p-q)} \tilde{u}_j^q \right)^{(p-q)/pq} \end{aligned} \quad (28)$$

for all $t \geq x \geq 1$.

Now in (28) we approach $t \rightarrow \infty$ and use (8)

$$\begin{aligned} C &\gg \left(\sum_{j=1}^x \left(\sum_{k=j}^{\infty} A_{r,1}^{p'}(k, j) v_k^{-p'} \right)^{q(p-1)/(p-q)} \right. \\ &\quad \left. \times \Delta_j^- \left(\sum_{i=1}^j \tilde{u}_i^q \right)^{p/(p-q)} \right)^{(p-q)/pq} \end{aligned} \quad (29)$$

(then we use the Abelian transformation)

$$\begin{aligned} &= \left(\sum_{j=1}^x \left(\sum_{i=1}^j \tilde{u}_i^q \right)^{p/(p-q)} \Delta_j^+ \left(\sum_{k=j}^{\infty} A_{r,1}^{p'}(k, j) v_k^{-p'} \right)^{q(p-1)/(p-q)} \right. \\ &\quad \left. + \left(\sum_{i=1}^x \tilde{u}_i^q \right)^{p/(p-q)} \right. \\ &\quad \left. \times \left(\sum_{k=x+1}^{\infty} A_{r,1}^{p'}(k, j) v_k^{-p'} \right)^{q(p-1)/(p-q)} \right)^{(p-q)/pq} \end{aligned} \quad (30)$$

(and use (7))

$$\begin{aligned} &\gg \left(\sum_{j=1}^x \left(\sum_{i=1}^j \tilde{u}_i^q \right)^{p/(p-q)} \left(\sum_{k=j}^{\infty} A_{r,1}^{p'}(k, j) v_k^{-p'} \right)^{p(q-1)/(p-q)} \right. \\ &\quad \left. \times \Delta_j^+ \left(\sum_{k=j}^{\infty} A_{r,1}^{p'}(k, j) v_k^{-p'} \right) \right)^{(p-q)/pq}. \end{aligned} \quad (31)$$

Substituting $\tilde{u}_i^q$ and approaching $x \rightarrow \infty$, we obtain (17). $\square$

Lemma 8. Let $1 < q < \infty$ and $0 \leq r \leq n-1$. Then

$$\|S_r^* f\|_{q, \tilde{u}} \ll \|S_{n-1}^* f\|_{q, u}, \quad f \geq 0. \quad (32)$$

Proof of Lemma 8. The expression $A_{n-1,1}(i, j)$ can be transformed as follows:

$$\begin{aligned} A_{n-1,1}(i, j) &= \sum_{k_{n-1}=j}^i \omega_{n-1,k_{n-1}} \cdots \sum_{k_{r+1}=k_{r+2}}^i \omega_{r+1,k_{r+1}} \\ &\quad \times \sum_{k_r=k_{r+1}}^i \omega_{r,k_r} \cdots \sum_{k_1=k_2}^i \omega_{1,k_1} \\ &= \sum_{k_{n-1}=j}^i \omega_{n-1,k_{n-1}} \cdots \sum_{k_{r+2}=k_{r+3}}^i \omega_{r+2,k_{r+2}} \\ &\quad \times \sum_{k_{r+1}=k_{r+2}}^i \omega_{r+1,k_{r+1}} A_{r,1}(i, k_{r+1}) \end{aligned} \quad (33)$$

(change of order of summation)

$$\begin{aligned} &= \sum_{k_{r+1}=j}^i A_{r,1}(i, k_{r+1}) \omega_{r+1,k_{r+1}} \\ &\quad \times \sum_{k_{n-1}=j}^{k_{r+1}} \omega_{n-1,k_{n-1}} \cdots \sum_{k_{r+2}=k_{r+3}}^{k_{r+1}} \omega_{r+2,k_{r+2}} \\ &= \sum_{t=j}^i A_{r,1}(i, t) \omega_{r+1,t} A_{n-1,r+2}(t, j). \end{aligned} \quad (34)$$

Since $\Delta_t^- A_{n-1,r+1}(t, j) = \omega_{r+1,t} A_{n-1,r+2}(t, j)$, then

$$A_{n-1}(i, j) = \sum_{t=j}^i A_{r,1}(i, t) \Delta_t^- A_{n-1,r+1}(t, j). \quad (35)$$

Let $f \geq 0$; then

$$\begin{aligned} \|S_{n-1}^* f\|_{q,u}^q &= \sum_{j=1}^{\infty} u_j^q \left(\sum_{i=j}^{\infty} A_{n-1,1}(i, j) f_i \right)^{q-1} \\ &\quad \times \sum_{k=j}^{\infty} A_{n-1,1}(k, j) f_k \\ &\geq \sum_{j=1}^{\infty} u_j^q \sum_{k=j}^{\infty} A_{n-1,1}(k, j) f_k \\ &\quad \times \left(\sum_{i=k}^{\infty} A_{n-1,1}(i, j) f_i \right)^{q-1} \end{aligned} \quad (36)$$

(use (35))

$$\begin{aligned} &= \sum_{j=1}^{\infty} u_j^q \sum_{k=j}^{\infty} f_k \sum_{t=j}^k A_{r,1}(k, t) \Delta_t^- A_{n-1,r+1}(t, j) \\ &\quad \times \left(\sum_{i=k}^{\infty} A_{n-1,1}(i, j) f_i \right)^{q-1} \end{aligned} \quad (37)$$

(change order of summation)

$$= \sum_{j=1}^{\infty} u_j^q \sum_{t=j}^{\infty} \Delta_t^- A_{n-1,r+1}(t, j) \left(\sum_{k=t}^{\infty} A_{r,1}(k, t) f_k \left(\sum_{i=k}^{\infty} A_{n-1,1}(i, j) f_i \right)^{q-1} \right) \quad (38)$$

(use $A_{n-1,1}(i, j) \geq A_{r,1}(i, t) A_{n-1,r+1}(t, j)$, which follows from (6))

$$\begin{aligned} &\geq \sum_{j=1}^{\infty} u_j^q \sum_{t=j}^{\infty} (\Delta_t^- A_{n-1,r+1}(t, j)) A_{n-1,r+1}^{q-1}(t, j) \\ &\quad \times \sum_{k=t}^{\infty} A_{r,1}(k, t) f_k \left(\sum_{i=k}^{\infty} A_{r,1}(i, t) f_i \right)^{q-1} \end{aligned} \quad (39)$$

(use (7) and (8))

$$\gg \sum_{j=1}^{\infty} u_j^q \sum_{t=j}^{\infty} \Delta_t^- A_{n-1,r+1}^q(t, j) \sum_{k=t}^{\infty} \Delta_k^+ \left(\sum_{i=k}^{\infty} A_{r,1}(i, t) f_i \right)^q \quad (40)$$

(use the relation $\Delta_k^+ (\sum_{i=k}^{\infty} A_{r,1}(i, t) f_i) = A_{r,1}(k, t) f_k$)

$$= \sum_{j=1}^{\infty} u_j^q \sum_{t=j}^{\infty} \Delta_t^- A_{n-1,r+1}^q(t, j) \left(\sum_{i=t}^{\infty} A_{r,1}(i, t) f_i \right)^q \quad (41)$$

(rearrange the cofactors)

$$= \sum_{t=1}^{\infty} \left(\sum_{i=t}^{\infty} A_{r,1}(i, t) f_i \right)^q \sum_{j=1}^t \Delta_t^- A_{n-1,r+1}^q(t, j) u_j^q = \quad (42)$$

(take into account that $A_{n-1,r+1}(t-1, t) = 0$)

$$\begin{aligned} &= \sum_{t=1}^{\infty} \left(\sum_{i=t}^{\infty} A_{r,1}(i, t) f_i \right)^q \Delta_t^- \left(\sum_{j=1}^t A_{n-1,r+1}^q(t, j) u_j^q \right) \\ &= \|S_r^* f\|_{q,\bar{u}}^q; \end{aligned} \quad (43)$$

that is, we have (32). $\square$

4. Proofs of Theorems 4 and 5

First we prove Theorem 5.

Proof of Theorem 5

Necessity. Suppose that operator (5) is bounded from $l_{p,v}$ to $l_{q,u}$ that equivalently means the validity of the following inequality:

$$\|S_{n-1}^* f\|_{q,u} \leq C \|f\|_{p,v}, \quad f \geq 0. \quad (44)$$

Then due to (32) inequality (14) holds for any $r = 0, 1, \dots, n-1$. Therefore, by Lemma 7 we have $B_r(n) \ll C$, $0 \leq r \leq n-1$, that means $B(n) \ll C$, where C is the best constant in (44); that is, $C = \|S_{n-1}^*\|$.

Sufficiency. will be proved by the induction method.

For $n = 1$ the operator $S_{n-1}^* = S_0^*$ is the Hardy operator. Thus by Lemma 3 if $B_0(1) < \infty$, then the operator S_0^* is bounded from $l_{p,v}$ to $l_{q,u}$ with the estimate $\|S_0^*\| \ll B_0(1)$.

Next we assume that for $n = 1, 2, \dots, l$, $1 \leq l < n-1$, if $B(n) < \infty$, then the operator S_{n-1}^* , $n = 1, 2, \dots, l$, is bounded from $l_{p,v}$ to $l_{q,u}$ with the estimate $\|S_{n-1}^*\| \ll B(n)$.

Now we need to prove that for $n = l+1$ if $B(l+1) = \max_{0 \leq r \leq l} B_r(l+1) < \infty$, then the operator S_l^* is bounded from $l_{p,v}$ to $l_{q,u}$ with the estimate $\|S_l^*\| \ll B(l+1)$.

Let $f \geq 0$ and $T_j = \{k \in \mathbb{Z} : 2^{-k} \leq (S_l^* f)_j\}$, $j \in \mathbb{N}$. Suppose that $k_j = \inf T_j$ for $T_j \neq \emptyset$ and $k_j = \infty$ for $T_j = \emptyset$. When $k_j < \infty$, then $2^{-k_j} \leq (S_l^* f)_j < 2^{-k_j+1}$. Further, for exception of the trivial case we suppose that $(S_l^* f)_1 > 0$.

Let $m_1 = 1$, $M_1 = \{j \in \mathbb{N} : k_j = k_{m_1}\}$. Assume that $\sup M_1 + 1 = m_2$. If $\sup M_1 < \infty$, then $m_2 < \infty$ and $\sup M_1 = \max M_1 = m_2 - 1$. It is obvious that $m_1 < m_2$. Suppose that we have found $m_1 < m_2 < \dots < m_s < \infty$, $s > 1$. Then we determine m_{s+1} as $m_{s+1} = \sup M_s + 1$, where $M_s = \{j \in \mathbb{N} : k_j = k_{m_s}\}$. By the definition $2^{-k_{m_s}} \leq (S_l^* f)_j < 2^{-k_{m_s}+1}$, $m_s \leq j \leq m_{s+1} - 1$, and $k_{m_s} < k_{m_{s+1}}$, $s \geq 1$.

Further, for convenience let $k_{m_s} = n_s$ and $m_{s+1} - 1 = m'_{s+1}$; then

$$2^{-n_s} \leq (S_l^* f)_j < 2^{-n_s+1}, \quad m_s \leq j \leq m'_{s+1}, \quad (45)$$

and $n_s < n_{s+1}$, $s \geq 1$.

Let $\mathbb{N}_0 = \{s \in \mathbb{N} : m_s < \infty\}$. Then from (45) it follows

$$\mathbb{N} = \bigcup_{s \in \mathbb{N}_0} [m_s, m_{s+1}). \quad (46)$$

There are two possible cases: (1) $\mathbb{N}_0 = \mathbb{N}$ and (2) $\mathbb{N}_0 = \{1, 2, \dots, s_0\}$, $s_0 > 1$.

Case (1) $\mathbb{N}_0 = \mathbb{N}$. Since $n_s < n_{s+1}$, $s \geq 1$, then $-n_s - 1 \leq -n_{s+2} + 1$. Therefore, using (45) and (6), we have

$$\begin{aligned} 2^{-n_s-1} &= 2^{-n_s} - 2^{-n_s-1} \leq 2^{-n_s} - 2^{-n_{s+2}+1} \\ &\leq (S_l^* f)_{m'_{s+1}} - (S_l^* f)_{m_{s+2}} = \sum_{i=m'_{s+1}}^{m'_{s+2}} A_{l,1}(i, m'_{s+1}) f_i \\ &\quad + \sum_{i=m_{s+2}}^{\infty} [A_{l,1}(i, m'_{s+1}) - A_{l,1}(i, m_{s+2})] f_i \\ &\leq \sum_{i=m'_{s+1}}^{m'_{s+2}} A_{l,1}(i, m'_{s+1}) f_i + \sum_{i=0}^{l-1} A_{l,r+1}(m_{s+2}, m'_{s+1}) \\ &\quad \times \sum_{i=m_{s+2}}^{\infty} A_{r,1}(i, m_{s+2}) f_i. \end{aligned} \quad (47)$$

Using (45), (46), and (47), we get

$$\begin{aligned} \|S_l^* f\|_{q,u}^q &= \sum_{s \geq 1} \sum_{i=m_s}^{m'_{s+1}} u_i^q (S_l^* f)_i^q < \sum_s (2^{-n_s+1})^q \sum_{i=m_s}^{m'_{s+1}} u_i^q \\ &= 2^{2q} \sum_s (2^{-n_s-1})^q \sum_{i=m_s}^{m'_{s+1}} u_i \\ &\ll \sum_s \left(\sum_{i=m'_{s+1}}^{m'_{s+2}} A_{l,1}(i, m'_{s+1}) f_i \right)^q \sum_{i=m_s}^{m'_{s+1}} u_i^q \\ &\quad + \sum_{r=0}^{l-1} \sum_s A_{l,r+1}^q(m_{s+2}, m'_{s+1}) \\ &\quad \times \left(\sum_{i=m_{s+2}}^{\infty} A_{r,1}(i, m_{s+2}) f_i \right)^q \sum_{i=m_s}^{m'_{s+1}} u_i^q = I_l + \sum_{i=0}^{l-1} I_r. \end{aligned} \quad (48)$$

First we estimate I_l . Using Hölder's inequality twice, we have

$$\begin{aligned} I_l &\leq \sum_s \left(\sum_{i=m'_{s+1}}^{m'_{s+2}} A_{l,1}^{p'}(i, m'_{s+1}) v_i^{-p'} \right)^{q/p'} \\ &\quad \times \sum_{i=m_s}^{m'_{s+1}} u_i^q \left(\sum_{i=m'_{s+1}}^{m'_{s+2}} (v_i f_i)^p \right)^{q/p} \\ &\leq \left(\sum_s \left(\sum_{i=m'_{s+1}}^{m'_{s+2}} A_{l,1}^{p'}(i, m'_{s+1}) v_i^{-p'} \right)^{q(p-1)/(p-q)} \right. \\ &\quad \times \left. \left(\sum_{i=m_s}^{m'_{s+1}} u_i^q \right)^{p/(p-q)} \right)^{(p-q)/p} \\ &\quad \times \left(\sum_s \sum_{i=m'_{s+1}}^{m'_{s+2}} (v_i f_i)^p \right)^{q/p} \ll \tilde{B}_l^{(p-q)/p} \|f\|_{p,v}^q, \end{aligned} \quad (49)$$

where

$$\begin{aligned} \tilde{B}_l &= \sum_s \left(\sum_{i=m'_{s+1}}^{m'_{s+2}} A_{l,1}^{p'}(i, m'_{s+1}) v_i^{-p'} \right)^{q(p-1)/(p-q)} \\ &\quad \times \left(\sum_{i=m_s}^{m'_{s+1}} u_i^q \right)^{p/(p-q)}. \end{aligned} \quad (50)$$

Next we need the following obvious inequality:

$$\begin{aligned}
 \Delta_i^+ \left(\sum_{j=i}^{\infty} A_{l,1}^{p'}(j,i) v_j^{-p'} \right) &= A_{l,1}^{p'}(i,i) + \sum_{j=i+1}^{\infty} \Delta_i^+ A_{l,1}^{p'}(j,i) v_j^{-p'} \\
 &\geq A_{l,1}^{p'}(i,i) + \sum_{j=i+1}^{m'_{s+2}} \Delta_i^+ A_{l,1}^{p'}(j,i) v_j^{-p'} \\
 &= \Delta_i^+ \left(\sum_{j=i}^{m'_{s+2}} A_{l,1}^{p'}(j,i) v_j^{-p'} \right), \quad m'_{s+1} \leq i \leq m'_{s+2}.
 \end{aligned} \tag{51}$$

Now, taking into account the above inequality (51) and using (10) and (7), we estimate $\tilde{B}_l$:

$$\begin{aligned}
 \tilde{B}_l &\ll \sum_s \sum_{i=m'_{s+1}}^{m'_{s+2}} \left(\sum_{j=i}^{m'_{s+2}} A_{l,1}^{p'}(j,i) v_j^{-p'} \right)^{p(q-1)/(p-q)} \\
 &\quad \times \left(\sum_{k=m_s}^{m'_{s+1}} u_k^q \right)^{p/(p-q)} \Delta_i^+ \left(\sum_{j=i}^{m'_{s+2}} A_{l,1}^{p'}(j,i) v_j^{-p'} \right) \\
 &\leq \sum_s \sum_{i=m'_{s+1}}^{m'_{s+2}} \left(\sum_{j=i}^{\infty} A_{l,1}^{p'}(j,i) v_j^{-p'} \right)^{p(q-1)/(p-q)} \\
 &\quad \times \left(\sum_{k=1}^i u_k^q \right)^{p/(p-q)} \Delta_i^+ \left(\sum_{j=i}^{\infty} A_{l,1}^{p'}(j,i) v_j^{-p'} \right) \\
 &\ll \sum_{i=1}^{\infty} \left(\sum_{j=i}^{\infty} A_{l,1}^{p'}(j,i) v_j^{-p'} \right)^{p(q-1)/(p-q)} \\
 &\quad \times \left(\sum_{k=1}^i u_k^q \right)^{p/(p-q)} \Delta_i^+ \left(\sum_{j=i}^{\infty} A_{l,1}^{p'}(j,i) v_j^{-p'} \right) \\
 &= (B_l(l+1))^{pq/(p-q)}.
 \end{aligned} \tag{52}$$

From (49) and (52) we have

$$I_l \ll B_l^q(l+1) \|f\|_{p,v}^q \leq B^q(l+1) \|f\|_{p,v}^q. \tag{53}$$

Now we estimate I_r , $r = 0, 1, \dots, l-1$. Let $\mathbb{N}_1 = \{k = m_{s+2} : s \in \mathbb{N}\}$.

Assume that $\Delta_k^q = A_{l,r+1}^q(m_{s+2}, m'_{s+1}) \sum_{i=m_s}^{m'_{s+1}} u_i^q$ if $k = m_{s+2} \in \mathbb{N}_1$ and $\Delta_k^q = 0$ if $k \notin \mathbb{N}_1$. Then

$$\begin{aligned}
 I_r &= \sum_{k=1}^{\infty} \Delta_k^q \left(\sum_{i=k}^{\infty} A_{r,1}(i,k) f_i \right)^q = \|S_r^* f\|_{q,\Delta}^q, \\
 &\quad r = 0, 1, \dots, l-1.
 \end{aligned} \tag{54}$$

The operator S_r^* is the operator S_{n-1}^* for $n = r+1$ and $1 \leq r+1 \leq l$. Therefore, by our assumption we have $\|S_r^*\| \ll \tilde{B}(r+1) = \max_{0 \leq t \leq r} \tilde{B}_t(r+1)$. Hence,

$$I_r \ll \max_{0 \leq t \leq r} \tilde{B}_t(r+1), \tag{55}$$

where

$$\begin{aligned}
 (\tilde{B}_t(r+1))^{pq/(p-q)} &= \sum_{i=1}^{\infty} \left(\sum_{j=i}^{\infty} A_{t,1}^{p'}(j,i) v_j^{-p'} \right)^{p(q-1)/(p-q)} \\
 &\quad \times \left(\sum_{k=1}^i A_{r,t+1}^q(i,k) \Delta_k^q \right)^{p/(p-q)} \\
 &\quad \times \Delta_i^+ \left(\sum_{j=i}^{\infty} A_{t,1}^{p'}(j,i) v_j^{-p'} \right).
 \end{aligned} \tag{56}$$

We estimate the expression $\sum_{k=1}^i A_{r,t+1}^q(i,k) \Delta_k^q$:

$$\sum_{k=1}^i A_{r,t+1}^q(i,k) \Delta_k^q \tag{57}$$

$$= \sum_{m_{s+2} \leq i} A_{r,t+1}^q(i, m_{s+2}) A_{l,r+1}^q(m_{s+2}, m'_{s+1}) \sum_{j=m_s}^{m'_{s+1}} u_j^q$$

(use the left side of (6) and nonincreasing of $A_{l,t+1}^q(i, j)$ in the second argument)

$$\begin{aligned}
 &\leq \sum_{m_{s+2} \leq i} A_{l,t+1}^q(i, m'_{s+1}) \sum_{j=m_s}^{m'_{s+1}} u_j^q \leq \sum_{m_{s+2} \leq i} \sum_{j=m_s}^{m'_{s+1}} A_{l,t+1}^q(i, j) u_j^q \\
 &\leq \sum_{j=1}^i A_{l,t+1}^q(i, j) u_j^q.
 \end{aligned} \tag{58}$$

Substituting the obtained estimate in (56), we have $\tilde{B}_t(r+1) \leq B_t(l+1)$. Therefore, from (55) we get

$$\begin{aligned}
 I_r &\ll \max_{0 \leq t \leq r} B_t^q(l+1) \|f\|_{p,v}^q \leq B^q(l+1) \|f\|_{p,v}^q, \\
 &\quad r = 0, 1, \dots, l-1.
 \end{aligned} \tag{59}$$

Then from (48), (53), and (59) we obtain

$$\|S_l^* f\|_{q,u} \ll B(l+1) \|f\|_{p,v}, \tag{60}$$

$$\|S_l^*\| \ll B(l+1). \tag{61}$$

Now we turn to case (2) $\mathbb{N}_0 = \{1, 2, \dots, s_0\}$, $1 \leq s_0 < \infty$. In this case we have $m_{s_0} < \infty$ and $m_{s_0+1} = \infty$. Here there are two possible cases: $n_{s_0} < \infty$ and $n_{s_0} = \infty$.

Below we suppose that $\sum_{s=k}^t = \sum_{s=1}^t$ if $k < 0$.

Let $n_{s_0} < \infty$. Then

$$\begin{aligned} \|S_l^* f\|_{q,u}^q &= \sum_{s=1}^{s_0} \sum_{j=m_s}^{m_{s+1}} (S_l^* f)_j^q u_j \\ &= \sum_{s=1}^{s_0-2} \sum_{j=m_s}^{m_{s+1}} (S_l^* f)_j^q u_j + \sum_{j=m_{s_0-1}}^{m_{s_0}'} (S_l^* f)_j^q u_j^q \quad (62) \\ &\quad + \sum_{j=m_{s_0}}^{\infty} (S_l^* f)_j^q u_j^q = J_1 + J_2 + J_3. \end{aligned}$$

If $J_1 \neq 0$, then $s_0 > 2$, and as in the case $\mathbb{N}_0 = \mathbb{N}$ using (47), we estimate J_1 and as a result we get

$$J_1 \ll B^q(l+1) \|f\|_{p,v}^q. \quad (63)$$

Using (45) and Hölder's inequality, we estimate the value J_2 :

$$\begin{aligned} J_2 &< 2^q (2^{-n_{s_0-1}})^q \sum_{j=m_{s_0-1}}^{m_{s_0}'} u_j^q \\ &\ll \left(\sum_{i=m_{s_0}'}^{\infty} A_{l,1}(i, m_{s_0}') f_i \right)^q \sum_{j=m_{s_0-1}}^{m_{s_0}'} u_j^q \\ &\leq \left(\sum_{i=m_{s_0}'}^{\infty} A_{l,1}^{p'}(i, m_{s_0}') v_i^{-p'} \right)^{q/p'} \\ &\quad \times \sum_{j=m_{s_0-1}}^{m_{s_0}'} u_j^q \left(\sum_{i=m_{s_0}'}^{\infty} |v_i f_i|^p \right)^{q/p} \quad (64) \\ &\leq \left[\left(\sum_{i=m_{s_0}'}^{\infty} A_{l,1}^{p'}(i, m_{s_0}') v_i^{-p'} \right)^{q(p-1)/(p-q)} \right. \\ &\quad \times \left. \left(\sum_{j=m_{s_0-1}}^{m_{s_0}'} u_j^q \right)^{p/(p-q)} \right]^{(p-q)/p} \|f\|_{p,v}^q. \end{aligned}$$

Using (7) and (10), we estimate the expression in the square brackets:

$$\left(\sum_{i=m_{s_0}'}^{\infty} A_{l,1}^{p'}(i, m_{s_0}') v_i^{-p'} \right)^{q(p-1)/(p-q)} \left(\sum_{j=m_{s_0-1}}^{m_{s_0}'} u_j^q \right)^{p/(p-q)}$$

$$\begin{aligned} &\ll \sum_{i=m_{s_0}'}^{\infty} \left(\sum_{j=i}^{\infty} A_{l,1}(j, i) v_j^{-p'} \right)^{p(q-1)/(p-q)} \\ &\quad \times \Delta_i^+ \left(\sum_{j=i}^{\infty} A_{l,1}(j, i) v_j^{-p'} \right) \left(\sum_{j=m_{s_0-1}}^{m_{s_0}'} u_j^q \right)^{p/(p-q)} \\ &\leq \sum_{i=m_{s_0}'}^{\infty} \left(\sum_{j=i}^{\infty} A_{l,1}(j, i) v_j^{-p'} \right)^{p(q-1)/(p-q)} \\ &\quad \times \left(\sum_{j=1}^i u_j^q \right)^{p/(p-q)} \Delta_i^+ \left(\sum_{j=i}^{\infty} A_{l,1}(j, i) v_j^{-p'} \right) \\ &\leq (B_l(l+1))^{pq/(p-q)}. \quad (65) \end{aligned}$$

Therefore,

$$J_2 \ll (B_l(l+1))^q \|f\|_{p,v}^q \leq B^q(l+1) \|f\|_{p,v}^q. \quad (66)$$

Now we estimate J_3 . Since $2^{-n_{s_0}} \leq (S_l^* f)_j < 2^{-n_{s_0}+1}$ for all $j \geq m_{s_0}$, then $(S_l^* f)_j \leq 2(S_l^* f)_t$ for all $j, t \geq m_{s_0}$. Hence,

$$\begin{aligned} J_3 &\leq \sup_{t \geq m_{s_0}} \sum_{j=m_{s_0}}^t (S_l^* f)_j^q u_j^q \\ &\leq 2^q \sup_{t \geq m_{s_0}} \left(\sum_{i=t}^{\infty} A_{n-1,1}(i, t) f_i \right)^q \sum_{j=m_{s_0}}^t u_j^q. \quad (67) \end{aligned}$$

Further, as for the estimate of J_2 by Hölder's inequality, we have

$$\begin{aligned} J_3 &\ll \sup_{t \geq m_{s_0}} \left[\left(\sum_{i=t}^{\infty} A_{l,1}^q(i, t) v_i^{-p'} \right)^{q(p-1)/(p-q)} \right. \\ &\quad \times \left. \left(\sum_{j=m_{s_0}}^t u_j^q \right)^{p/(p-q)} \right]^{(p-q)/p} \|f\|_{p,v}^q \quad (68) \\ &\ll B_l^q(l+1) \|f\|_{p,v}^q \leq B^q(l+1) \|f\|_{p,v}^q. \end{aligned}$$

From (62), (63), (66), and (68) we have (60) and (61). If $n_{s_0} = \infty$ that means $k_{m_s} = \infty$, then $T_j = \emptyset$ for $j \geq m_{s_0}$; that is, $(S_l^* f)_j = 0$ for $j \geq m_{s_0}$. By the assumption $(S_l^* f)_1 > 0$, therefore, $s_0 > 1$. Then $m_2 < \infty$ and $s_0 \geq 2$. Hence,

$$\|S_l^* f\|_{q,u}^q = \sum_{s=1}^{s_0-2} \sum_{j=m_s}^{m_{s+1}} (S_l^* f)_j^q u_j^q + \sum_{j=m_{s_0-1}}^{m_{s_0}'} (S_l^* f)_j^q u_j^q = J_1 + J_2. \quad (69)$$

This, together with estimates (63) and (66), gives (60) and (61). Thus, if $B(l+1) < \infty$, then the operator S_l^* is bounded from $l_{p,v}$ to $l_{q,u}$ and estimate (61) holds. Consequently, for any

$n \geq 1$ from $B(n) < \infty$ we have that the operator S_{n-1}^* is bounded from $l_{p,v}$ to $l_{q,u}$ and the estimate $\|S_{n-1}^*\| \ll B(n)$ holds. The last estimate and the estimate $\|S_{n-1}^*\| \gg B(n)$ obtained in the necessity part yield $\|S_{n-1}^*\| \approx B(n)$. $\square$

The proof of Theorem 4 follows from Theorem 5, because boundedness of the operator S_{n-1} from $l_{p,v}$ to $l_{q,u}$ is equivalent to boundedness of the operator S_{n-1}^* from $l_{q',u^{-1}}$ to $l_{p',v^{-1}}$, and the respective norms of the operators S_{n-1}^* and S_{n-1} coincide.

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