Research Article

Construction of Frames for Shift-Invariant Spaces

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We construct a sequence $\{\phi_i(\cdot - j) \mid j \in \mathbb{Z}, i = 1, \dots, r\}$ which constitutes a p -frame for the weighted shift-invariant space $V_\mu^p(\Phi) = \{\sum_{i=1}^r \sum_{j \in \mathbb{Z}} c_i(j) \phi_i(\cdot - j) \mid \{c_i(j)\}_{j \in \mathbb{Z}} \in \ell_\mu^p, i = 1, \dots, r, p \in [1, \infty]\}$, and generates a closed shift-invariant subspace of $L_\mu^p(\mathbb{R})$. The first construction is obtained by choosing functions $\phi_i, i = 1, \dots, r$, with compactly supported Fourier transforms $\widehat{\phi}_i, i = 1, \dots, r$. The second construction, with compactly supported $\phi_i, i = 1, \dots, r$, gives the Riesz basis.

1. Introduction and Preliminaries

The shift-invariant spaces $V_\mu^p(\Phi)$, $p \in [1, \infty]$, quoted in the abstract, are used in the wavelet analysis, approximation theory, sampling theory, and so forth. They have been extensively studied by many authors [1–18]. The aim of this paper is to construct $V_\mu^p(\Phi)$, $p \in [1, \infty]$, spaces with specially chosen functions $\phi_i, i = 1, \dots, r$, which generate its p -frame. These results extend and correct the construction obtained in [19]. For the first construction, we take functions $\phi_i, i = 1, \dots, r$, so that the Fourier transforms are compactly supported smooth functions. Also, we derive conditions for the collection $\{\phi_i(\cdot - j) \mid j \in \mathbb{Z}, i = 1, \dots, r\}$ to form a Riesz basis for $V_\mu^p(\Phi)$. We note that the properties of the constructed frame guarantee the feasibility of a stable and continuous reconstruction algorithm in $V_\mu^p(\Phi)$ [20]. We generalize these results for a shift-invariant subspace of $L_\mu^p(\mathbb{R}^d)$. The second construction is obtained by choosing compactly supported functions $\phi_i, i = 1, \dots, r$. In this way, we obtain the Riesz basis.

This paper is organized as follows. In Section 2 we quote some basic properties of certain subspaces of the weighted L^p and ℓ^p spaces. In Section 3 we derive conditions for functions

of the form $\widehat{\phi}_i(\cdot) = \theta(\cdot + k_i\pi)$, $k_i \in \mathbb{Z}, i = 1, 2, \dots, r, r \in \mathbb{N}$, to form a Riesz basis for $V_\mu^p(\Phi)$. We also show that using functions of the form $\widehat{\phi}_i(\cdot) = \theta(\cdot + i\pi)$, $i = 1, \dots, r$, where θ is compactly supported smooth function whose length of support is less than or equal to 2π , we cannot construct a p -frame for the shift-invariant space $V_\mu^p(\Phi)$. In Section 4 we construct a sequence $\{\phi_i(\cdot - j) \mid j \in \mathbb{Z}, i = 0, \dots, r\}$, where $r \in 2\mathbb{N}$ or $r \in 3\mathbb{N}$, which constitutes a p -frame for the weighted shift-invariant space $V_\mu^p(\Phi)$. Our construction shows that the sampling and reconstruction problem in the shift-invariant spaces is robust in the sense of [1]. In Section 5 we construct p -Riesz basis by using compactly supported functions $\phi_i, i = 1, \dots, r$.

2. Basic Spaces

Let a function ω be nonnegative, continuous, symmetric, and submultiplicative; that is, $\omega(x+y) \leq \omega(x)\omega(y)$, $x, y \in \mathbb{R}^d$; let a function μ be ω -moderate; that is, $\mu(x+y) \leq C\omega(x)\mu(y)$, $x, y \in \mathbb{R}^d$. Functions μ and ω are called weights. We consider the weighted function spaces L_μ^p and the weighted sequence

spaces $\ell_\mu^p(\mathbb{Z}^d)$ with ω -moderate weights μ (see [19]). Let $p \in [1, \infty)$. Then (with obvious modification for $p = \infty$)

$$\begin{aligned} \mathcal{L}_\omega^p &= \left\{ f \mid \|f\|_{\mathcal{L}_\omega^p} \right. \\ &= \left(\int_{[0,1]^d} \left(\sum_{j \in \mathbb{Z}^d} |f(x+j)| \omega(x+j) \right)^p dx \right)^{1/p} \\ &\left. < +\infty \right\}, \\ W_\omega^p &:= \left\{ f \mid \|f\|_{W_\omega^p} \right. \\ &= \left(\sum_{j \in \mathbb{Z}^d} \sup_{x \in [0,1]^d} |f(x+j)|^p \omega(j)^p \right)^{1/p} < +\infty \left. \right\}. \end{aligned} \quad (1)$$

In what follows, we use the notation $\Phi = (\phi_1, \dots, \phi_r)^T$. Define $\|\Phi\|_{\mathcal{H}} = \sum_{i=1}^r \|\phi_i\|_{\mathcal{H}}$, where $\mathcal{H} = L_\omega^p, \mathcal{L}_\omega^p$ or W_ω^p , $p \in [1, \infty]$. With $\mathcal{F}\phi = \widehat{\phi}$ we denote the Fourier transform of the function ϕ ; that is, $\widehat{\phi}(\xi) = \int_{\mathbb{R}^d} \phi(x) e^{-i\pi x \cdot \xi} dx$, $\xi \in \mathbb{R}^d$.

Let $c = \{c_i\}_{i \in \mathbb{N}} \in \ell_\mu^p$ and $f, g \in L_\omega^p$, $p \in [1, \infty]$. We define, as in [1], the semiconvolution $f *' c$ as $(f *' c)(x) = \sum_{j \in \mathbb{Z}^d} c_j f(x-j)$, $x \in \mathbb{R}^d$, and $\langle f, g \rangle = \int_{\mathbb{R}^d} f(x) \overline{g(x)} dx$.

The concept of a p -frame is introduced in [1].

It is said that a collection $\{\phi_i(\cdot - j) \mid j \in \mathbb{Z}^d, i = 1, \dots, r\}$ is a p -frame for $V_\mu^p(\Phi)$ if there exists a positive constant C (dependent upon Φ , p , and ω) such that

$$\begin{aligned} C^{-1} \|f\|_{L_\mu^p} &\leq \sum_{i=1}^r \left\| \left\{ \int_{\mathbb{R}^d} f(x) \overline{\phi_i(x-j)} dx \right\}_{j \in \mathbb{Z}^d} \right\|_{\ell_\mu^p} \\ &\leq C \|f\|_{L_\mu^p}, \quad f \in V_\mu^p(\Phi). \end{aligned} \quad (2)$$

Recall [21] that the shift-invariant spaces are defined by

$$\begin{aligned} V_\mu^p(\Phi) &:= \left\{ f \in L_\mu^p \mid f(\cdot) = \sum_{i=1}^r \sum_{j \in \mathbb{Z}^d} c_j^i \phi_i(\cdot - j), \right. \\ &\left. \{c_j^i\}_{j \in \mathbb{Z}^d} \in \ell_\mu^p, i = 1, \dots, r \right\}. \end{aligned} \quad (3)$$

Remark 1 (see [22]). Let $\Phi \in W_\omega^1$ and let μ be ω -moderate. Then $V_\mu^p(\Phi)$ is a subspace (not necessarily closed) of L_μ^p and W_μ^p for any $p \in [1, \infty]$. Clearly (2) implies that ℓ_μ^p and $V_\mu^p(\Phi)$ are isomorphic Banach spaces.

Let $\Phi = (\phi_1, \dots, \phi_r)^T$. Let

$$[\widehat{\Phi}, \widehat{\Phi}](\xi) = \left[\sum_{k \in \mathbb{Z}^d} \widehat{\phi}_i(\xi + 2k\pi) \overline{\widehat{\phi}_j(\xi + 2k\pi)} \right]_{1 \leq i \leq r, 1 \leq j \leq r}, \quad (4)$$

where we assume that $\widehat{\phi}_i(\xi) \overline{\widehat{\phi}_j(\xi)}$ is integrable for any $1 \leq i, j \leq r$. Let $A = [a(j)]_{j \in \mathbb{Z}^d}$ be an $r \times \infty$ matrix and $AA^T = [\sum_{j \in \mathbb{Z}^d} a_i(j) \overline{a_{i'}(j)}]_{1 \leq i, i' \leq r}$. Then $\text{rank } A = \text{rank } AA^T$.

We recall results from [1, 19] which are needed in the sequel.

Lemma 2 (see [1]). *The following statements are equivalent.*

- (1) $\text{rank}[\widehat{\Phi}(\xi + 2j\pi)]_{j \in \mathbb{Z}^d}$ is a constant function on $\mathbb{R}^d$.
- (2) $\text{rank}[\widehat{\Phi}, \widehat{\Phi}](\xi)$ is a constant function on $\mathbb{R}^d$.
- (3) There exists a positive constant C independent of ξ such that

$$\begin{aligned} C^{-1} [\widehat{\Phi}, \widehat{\Phi}](\xi) &\leq [\widehat{\Phi}, \widehat{\Phi}](\xi) \overline{[\widehat{\Phi}, \widehat{\Phi}](\xi)}^T \\ &\leq C [\widehat{\Phi}, \widehat{\Phi}](\xi), \quad \xi \in [-\pi, \pi]^d. \end{aligned} \quad (5)$$

The next theorem [19] derives necessary and sufficient conditions for an indexed family $\{\phi_i(\cdot - j) \mid j \in \mathbb{Z}^d, i = 1, \dots, r\}$ to constitute a p -frame for $V_\mu^p(\Phi)$, which is equivalent with the closedness of this space in L_μ^p . Thus, it is shown that under appropriate conditions on the frame vectors, there is an equivalence between the concept of p -frames, Banach frames, and the closedness of the space they generate.

Theorem 3 (see [19]). *Let $\Phi = (\phi_1, \dots, \phi_r)^T \in (W_\omega^1)^r$, $p_0 \in [1, \infty]$, and let μ be ω -moderate. The following statements are equivalent.*

- (i) $V_\mu^{p_0}(\Phi)$ is closed in $L_\mu^{p_0}$.
- (ii) $\{\phi_i(\cdot - j) \mid j \in \mathbb{Z}^d, i = 1, \dots, r\}$ is a p_0 -frame for $V_\mu^{p_0}(\Phi)$.
- (iii) There exists a positive constant C such that

$$\begin{aligned} C^{-1} [\widehat{\Phi}, \widehat{\Phi}](\xi) &\leq [\widehat{\Phi}, \widehat{\Phi}](\xi) \overline{[\widehat{\Phi}, \widehat{\Phi}](\xi)}^T \\ &\leq C [\widehat{\Phi}, \widehat{\Phi}](\xi), \quad \xi \in [-\pi, \pi]^d. \end{aligned} \quad (6)$$

- (iv) There exist positive constants C_1 and C_2 (depending on Φ and ω) such that

$$\begin{aligned} C_1 \|f\|_{L_\mu^{p_0}} &\leq \inf_{f = \sum_{i=1}^r \phi_i *' c^i} \sum_{i=1}^r \left\| \{c_j^i\}_{j \in \mathbb{Z}^d} \right\|_{\ell_\mu^{p_0}} \\ &\leq C_2 \|f\|_{L_\mu^{p_0}}, \quad f \in V_\mu^{p_0}(\Phi). \end{aligned} \quad (7)$$

(v) There exists $\Psi = (\psi_1, \dots, \psi_r)^T \in (W_\omega^1)^r$, such that

$$\begin{aligned} f &= \sum_{i=1}^r \sum_{j \in \mathbb{Z}^d} \langle f, \psi_i(\cdot - j) \rangle \phi_i(\cdot - j) \\ &= \sum_{i=1}^r \sum_{j \in \mathbb{Z}^d} \langle f, \phi_i(\cdot - j) \rangle \psi_i(\cdot - j), \quad f \in V_\mu^{p_0}(\Phi). \end{aligned} \quad (8)$$

Corollary 4 (see [19]). Let $\Phi = (\phi_1, \dots, \phi_r)^T \in (W_\omega^1)^r$, $p_0 \in [1, \infty]$, and let μ be ω -moderate.

- (i) If $\{\phi_i(\cdot - j) \mid j \in \mathbb{Z}^d, i = 1, \dots, r\}$ is a p_0 -frame for $V_\mu^{p_0}(\Phi)$, then the collection $\{\phi_i(\cdot - j) \mid j \in \mathbb{Z}^d, i = 1, \dots, r\}$ is a p -frame for $V_\mu^p(\Phi)$ for any $p \in [1, \infty]$.
- (ii) If $V_\mu^{p_0}(\Phi)$ is closed in $L_\mu^{p_0}$ and $W_\mu^{p_0}$, then $V_\mu^p(\Phi)$ is closed in L_μ^p and W_μ^p for any $p \in [1, \infty]$.
- (iii) If (7) holds for p_0 , then it holds for any $p \in [1, \infty]$.

3. Construction of Frames Using a Band-Limited Function

Considering the length of the support of a function θ , we have different cases for the rank of matrix $[\widehat{\Phi}(\xi + 2j\pi)]_{j \in \mathbb{Z}}$.

First, we consider the next claim.

Let $\theta \in C_0^\infty(\mathbb{R})$ be a nonnegative function such that $\theta(x) > 0$, $x \in (-\pi, \pi)$, and $\text{supp } \theta \subseteq [-\pi, \pi]$. Moreover, let

$$\widehat{\phi}_k(\xi) = \theta(\xi + k\pi), \quad k \in \mathbb{Z}, \quad (9)$$

and $\Phi = (\phi_i, \phi_{i+1}, \dots, \phi_{i+r})^T$, $i \in \mathbb{Z}$, $r \in \mathbb{N}$.

Then the rank of matrix $[\widehat{\Phi}(\xi + 2j\pi)]_{j \in \mathbb{Z}}$ is not a constant function on $\mathbb{R}$ and it depends on $\xi \in \mathbb{R}$.

As a matter of fact, by the Paley-Wiener theorem, $\phi_i \in \mathcal{S}(\mathbb{R}) \subset W_\mu^1(\mathbb{R})$, $i \in \mathbb{Z}$. For any $i \in \mathbb{Z}$, matrix $[\widehat{\phi}_i, \widehat{\phi}_i](\xi) = \sum_{j \in \mathbb{Z}} |\theta(\xi + i\pi + 2j\pi)|^2$, $\xi \in \mathbb{R}$, has the rank 0 or 1, depending on ξ . Moreover, we have $[\widehat{\phi}_{2i}, \widehat{\phi}_{2i}](\pi) = 0$ and $[\widehat{\phi}_{2i}, \widehat{\phi}_{2i}](0) > 0$. Because of that, the rank of the matrix $[\widehat{\Phi}(\xi + 2j\pi)]_{j \in \mathbb{Z}}$ is not a constant function on $\mathbb{R}$ and it depends on $\xi \in \mathbb{R}$.

Theorem 5. Let $\theta \in C_0^\infty(\mathbb{R})$ be a non-negative function such that $\theta(x) > 0$, $x \in (-\pi - \varepsilon, \pi + \varepsilon)$, and $\text{supp } \theta = [-\pi - \varepsilon, \pi + \varepsilon]$, where $0 < \varepsilon < 1/4$. Moreover, let

$$\widehat{\phi}_i(\xi) = \theta(\xi + k_i\pi), \quad k_i \in \mathbb{Z}, \quad i = 1, 2, \dots, r, \quad (10)$$

and $\Phi = (\phi_1, \phi_2, \dots, \phi_r)^T$.

- (1) If $|k_2 - k_1| = 2$ and $|k_i - k_j| \geq 2$ for different $i, j \leq r$, then

$$\text{rank} [\widehat{\Phi}(\xi + 2j\pi)]_{j \in \mathbb{Z}} = r, \quad \xi \in \mathbb{R}. \quad (11)$$

- (2) If $|k_2 - k_1| = 2$ and, at least for k_{i_1} and k_{i_2} , it holds that $|k_{i_1} - k_{i_2}| = 1$, where $1 \leq i_1, i_2 \leq r$, then $\text{rank}[\widehat{\Phi}(\xi + 2j\pi)]_{j \in \mathbb{Z}}$ is not a constant function on $\mathbb{R}$.

Proof. By the Paley-Wiener theorem, $\phi_i \in \mathcal{S}(\mathbb{R}) \subset W_\mu^1(\mathbb{R})$, $i = 1, \dots, r$. All possible cases are described in the following lemmas.

Lemma 6. Let $\Phi = (\phi_{k_1}, \phi_{k_2})^T$, $k_2 - k_1 = 2$, $k_1, k_2 \in \mathbb{Z}$. The rank of matrix $[\widehat{\Phi}(\xi + 2j\pi)]_{j \in \mathbb{Z}}$ is a constant function on $\mathbb{R}$ and equals 2.

Proof. We have the next two cases.

(1°) If $\xi \in (-\pi - \varepsilon - k_1\pi + 2\ell\pi, -\pi + \varepsilon - k_1\pi + 2\ell\pi)$, $\ell \in \mathbb{Z}$, for matrix $[\widehat{\Phi}(\xi + 2j\pi)]_{j \in \mathbb{Z}}$ we obtain $2 \times \infty$ matrix

$$\begin{bmatrix} \cdots & 0 & 0 & a^1 & b^1 & 0 & \cdots \\ \cdots & 0 & a^2 & b^2 & 0 & 0 & \cdots \end{bmatrix}, \quad (12)$$

for some $a^i, b^i > 0$, $i = 1, 2$. It is obvious that $\text{rank}[\widehat{\Phi}(\xi + 2j\pi)]_{j \in \mathbb{Z}} = 2$, $\xi \in (-\pi - \varepsilon - k_1\pi + 2\ell\pi, -\pi + \varepsilon - k_1\pi + 2\ell\pi)$, $\ell \in \mathbb{Z}$.

(2°) For $\xi \in [-\pi + \varepsilon - k_1\pi + 2\ell\pi, \pi - \varepsilon - k_1\pi + 2\ell\pi]$, $\ell \in \mathbb{Z}$, there are only two nonzero values a^1 and a^2 which are in different columns of matrix $[\widehat{\Phi}(\xi + 2j\pi)]_{j \in \mathbb{Z}}$. Since

$$[\widehat{\Phi}(\xi + 2j\pi)]_{j \in \mathbb{Z}} = \begin{bmatrix} \cdots & 0 & 0 & a^1 & 0 & \cdots \\ \cdots & 0 & a^2 & 0 & 0 & \cdots \end{bmatrix}_{2 \times \infty}, \quad (13)$$

it has the rank 2 for all $\xi \in [-\pi + \varepsilon - k_1\pi + 2\ell\pi, \pi - \varepsilon - k_1\pi + 2\ell\pi]$, $\ell \in \mathbb{Z}$.

We conclude that the rank of matrix $[\widehat{\Phi}(\xi + 2j\pi)]_{j \in \mathbb{Z}}$, $\Phi = (\phi_{k_1}, \phi_{k_2})^T$, $k_2 - k_1 = 2$, $k_1, k_2 \in \mathbb{Z}$, is a constant function on $\mathbb{R}$ and equals 2. $\square$

Lemma 7. The rank of matrix $[\widehat{\Phi}(\xi + 2j\pi)]_{j \in \mathbb{Z}}$ is not a constant function on $\mathbb{R}$ if $\Phi = (\phi_{k_1}, \phi_{k_2})^T$, $k_2 - k_1 = 1$, $k_1, k_2 \in \mathbb{Z}$.

Proof. We have four different cases for matrix $[\widehat{\Phi}(\xi + 2j\pi)]_{j \in \mathbb{Z}}$. Suppose, without losing generality, that $k_1 \in 2\mathbb{Z}$.

(1°) If $\xi \in (-\pi - \varepsilon + 2\ell\pi, -\pi + \varepsilon + 2\ell\pi)$, $\ell \in \mathbb{Z}$, then

$$[\widehat{\Phi}(\xi + 2j\pi)]_{j \in \mathbb{Z}} = \begin{bmatrix} \cdots & 0 & a^1 & b^1 & 0 & \cdots \\ \cdots & 0 & a^2 & 0 & 0 & \cdots \end{bmatrix}, \quad (14)$$

$$a^1, a^2, b^1 > 0,$$

and $\text{rank}[\widehat{\Phi}(\xi + 2j\pi)]_{j \in \mathbb{Z}} = 2$, for all $\xi \in (-\pi - \varepsilon + 2\ell\pi, -\pi + \varepsilon + 2\ell\pi)$, $\ell \in \mathbb{Z}$.

(2°) For $\xi \in [-\pi + \varepsilon + 2\ell\pi, -\varepsilon + 2\ell\pi]$, $\ell \in \mathbb{Z}$, nonzero values a^1 and a^2 are in the same column of matrix $[\widehat{\Phi}(\xi + 2j\pi)]_{j \in \mathbb{Z}}$. For any choice of a 2×2 matrix, we get that the determinant equals 0. So we obtain

$$\text{rank} \begin{bmatrix} \cdots & 0 & a^1 & 0 & \cdots \\ \cdots & 0 & a^2 & 0 & \cdots \end{bmatrix} = 1, \quad (15)$$

for all $\xi \in [-\pi + \varepsilon + 2\ell\pi, -\varepsilon + 2\ell\pi]$, $\ell \in \mathbb{Z}$.

(3°) If $\xi \in (-\varepsilon + 2\ell\pi, \varepsilon + 2\ell\pi)$, $\ell \in \mathbb{Z}$, then

$$[\widehat{\Phi}(\xi + 2j\pi)]_{j \in \mathbb{Z}} = \begin{bmatrix} \cdots & 0 & 0 & a^1 & 0 & \cdots \\ \cdots & 0 & b^2 & a^2 & 0 & \cdots \end{bmatrix}, \quad (16)$$

for some $a^1, a^2, b^2 > 0$, has the rank 2, for all $\xi \in (-\varepsilon + 2\ell\pi, \varepsilon + 2\ell\pi)$, $\ell \in \mathbb{Z}$.

(4°) For $\xi \in [\varepsilon + 2\ell\pi, \pi - \varepsilon + 2\ell\pi]$, $\ell \in \mathbb{Z}$, there are two non-zero values a^1 and b^2 in different columns of matrix $[\widehat{\Phi}(\xi + 2j\pi)]_{j \in \mathbb{Z}}$ and the block with these elements determines the rank 2 for all $\xi \in [-\varepsilon + 2\ell\pi, \pi - \varepsilon + 2\ell\pi]$, $\ell \in \mathbb{Z}$.

Considering possible cases, we conclude that $\text{rank}[\widehat{\Phi}(\xi + 2j\pi)]_{j \in \mathbb{Z}}$, $\Phi = (\phi_{k_1}, \phi_{k_2})^T$, $k_2 - k_1 = 1$, $k_1, k_2 \in \mathbb{Z}$, depends on $\xi \in \mathbb{R}$ and equals 1 or 2. This rank is a nonconstant function on $\mathbb{R}$. $\square$

Proof of Theorem 5. (1) Using Lemmas 6 and 7, it is obvious that if $|k_2 - k_1| = 2$ and $|k_i - k_j| \geq 2$ for different $i, j \leq r$, then the position of the first non-zero element in each row of matrix $[\widehat{\Phi}(\xi + 2j\pi)]_{j \in \mathbb{Z}}$ is unique for each row. So the rank of matrix $[\widehat{\Phi}(\xi + 2j\pi)]_{j \in \mathbb{Z}}$ is a constant function on $\mathbb{R}$ and equals r for all $\xi \in \mathbb{R}$.

(2) If $|k_2 - k_1| = 2$ and, at least for k_{i_1} and k_{i_2} , it holds that $|k_{i_1} - k_{i_2}| = 1$, $1 \leq i_1, i_2 \leq r$, then, in the row with the index i_2 (suppose, without losing generality, that $i_2 \in 2\mathbb{Z} + 1$), we will have a new column with a non-zero element for $\xi \in (-\pi - \varepsilon + 2\ell\pi, -\pi + \varepsilon + 2\ell\pi)$, $\ell \in \mathbb{Z}$, but for $\xi \in [\varepsilon + 2\ell\pi, \pi - \varepsilon + 2\ell\pi]$, $\ell \in \mathbb{Z}$, the positions of all non-zero elements in that row will appear in the previous columns. It is obvious that the rank of the matrix $[\widehat{\Phi}(\xi + 2j\pi)]_{j \in \mathbb{Z}}$ depends on $\xi \in \mathbb{R}$ and is not the same for all $\xi \in \mathbb{R}$. $\square$

As a consequence of Theorems 3 and 5(1), we have the following result.

Theorem 8. *Let the functions θ and Φ satisfy all the conditions of Theorem 5(1). Then space $V_\mu^p(\Phi)$ is closed in L_μ^p for any $p \in [1, \infty]$, and the family $\{\phi_i(\cdot - j) \mid j \in \mathbb{Z}, 1 \leq i \leq r\}$ is a p -Riesz basis for $V_\mu^p(\Phi)$ for any $p \in [1, \infty]$.*

The following theorem is a generalisation of Theorem 5 and can be proved in the same way, so we omit the proof.

Theorem 9. *Let $\theta \in C_0^\infty(\mathbb{R})$ be a positive function such that $\theta(x) > 0$, $x \in (a, b)$, $b > a$, and supported by $[a, b]$ where $b - a > 2\pi$. Moreover, let*

$$\widehat{\phi}_i(\xi) = \theta(\xi + k_i\pi), \quad k_i \in \mathbb{Z}, \quad i = 1, 2, \dots, r, \quad r \in \mathbb{N}, \quad (17)$$

and $\Phi = (\phi_1, \phi_2, \dots, \phi_r)^T$.

(1) If $|k_2 - k_1| = 2$ and $|k_i - k_j| \geq 2$ for different $i, j \leq r$, then

$$\text{rank} [\widehat{\Phi}(\xi + 2j\pi)]_{j \in \mathbb{Z}} = r, \quad \xi \in \mathbb{R}. \quad (18)$$

(2) If $|k_2 - k_1| = 2$ and, at least for k_{i_1} and k_{i_2} , it holds that $|k_{i_1} - k_{i_2}| = 1$, where $1 \leq i_1, i_2 \leq r$, then $\text{rank}[\widehat{\Phi}(\xi + 2j\pi)]_{j \in \mathbb{Z}}$ is not a constant function on $\mathbb{R}$.

4. Construction of Frames Using Several Band-Limited Functions

Firstly, we consider two smooth functions with proper compact supports.

Lemma 10. *Let $\theta \in C_0^\infty(\mathbb{R})$, $\psi \in C_0^\infty(\mathbb{R})$ be positive functions such that*

$$\begin{aligned} \theta(x) &> 0, & x &\in (-\varepsilon, 2\pi + \varepsilon), \\ \psi(x) &> 0, & x &\in (\varepsilon, 2\pi - \varepsilon), \\ \text{supp } \theta &= [-\varepsilon, 2\pi + \varepsilon], \\ \text{supp } \psi &= [\varepsilon, 2\pi - \varepsilon], & 0 < \varepsilon < \frac{1}{4}. \end{aligned} \quad (19)$$

Moreover, let $\widehat{\phi}_1(\xi) = \theta(\xi)$, $\widehat{\phi}_2(\xi) = \psi(\xi)$, $\xi \in \mathbb{R}$, and $\Phi = (\phi_1, \phi_2)^T$. Then $\text{rank}[\widehat{\Phi}(\xi + 2j\pi)]_{j \in \mathbb{Z}} = 1$, $\xi \in \mathbb{R}$.

Proof. Note that $\phi_i \in \mathcal{S}(\mathbb{R}) \subset W_\mu^1(\mathbb{R})$, $i = 1, 2$.

We have the following two cases.

(1°) If $\xi \in (-\varepsilon + 2\ell\pi, \varepsilon + 2\ell\pi)$, $\ell \in \mathbb{Z}$, then matrix

$$\begin{aligned} [\widehat{\Phi}(\xi + 2j\pi)]_{j \in \mathbb{Z}} &= \begin{bmatrix} \cdots & 0 & a & b & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & \cdots \end{bmatrix}, \\ a, b &> 0, \end{aligned} \quad (20)$$

has a constant rank equal to 1.

(2°) For $\xi \in (\varepsilon + 2\ell\pi, 2\pi - \varepsilon + 2\ell\pi)$, $\ell \in \mathbb{Z}$, the rank of matrix

$$[\widehat{\Phi}(\xi + 2j\pi)]_{j \in \mathbb{Z}} = \begin{bmatrix} \cdots & 0 & c & 0 & \cdots \\ \cdots & 0 & d & 0 & \cdots \end{bmatrix}, \quad (21)$$

where c, d are non-zero values, is equal to 1. An equivalent matrix is obtained for $\xi = \varepsilon + 2\ell\pi$ and $\xi = -\varepsilon + 2\ell\pi$, so we conclude that $\text{rank}[\widehat{\Phi}(\xi + 2j\pi)]_{j \in \mathbb{Z}} = 1$, for $\xi \in [\varepsilon + 2\ell\pi, 2\pi - \varepsilon + 2\ell\pi]$, $\ell \in \mathbb{Z}$.

Considering these two cases, the rank of matrix $[\widehat{\Phi}(\xi + 2j\pi)]_{j \in \mathbb{Z}}$ is a constant function on $\mathbb{R}$, $\text{rank}[\widehat{\Phi}(\xi + 2j\pi)]_{j \in \mathbb{Z}} = 1$, $\xi \in \mathbb{R}$. $\square$

Using functions θ and ψ from Lemma 10, in the next lemma we construct the p -frame with four functions.

Lemma 11. *Let the functions θ and ψ satisfy all the conditions of Lemma 10. Moreover, let*

$$\begin{aligned} \widehat{\phi}_k(\xi) &= \theta(\xi + 2k\pi), & \widehat{\phi}_{k+2}(\xi) &= \psi(\xi + 2k\pi), \\ k &= 0, 1, \end{aligned} \quad (22)$$

and $\Phi = (\phi_0, \phi_1, \phi_2, \phi_3)^T$. Then $\text{rank}[\widehat{\Phi}(\xi + 2j\pi)]_{j \in \mathbb{Z}} = 2$, $\xi \in \mathbb{R}$.

Proof. The proof is similar to the proof of Lemma 10.

(1°) If $\xi \in (-\varepsilon + 2\ell\pi, \varepsilon + 2\ell\pi)$, $\ell \in \mathbb{Z}$, then matrix

$$[\widehat{\Phi}(\xi + 2j\pi)]_{j \in \mathbb{Z}} = \begin{bmatrix} \cdots & 0 & 0 & a^1 & b^1 & 0 & \cdots \\ \cdots & 0 & a^2 & b^2 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & \cdots \end{bmatrix}, \quad (23)$$

where $a^i, b^i > 0$, $i = 1, 2$, has a constant rank equal to 2.

(2°) For $\xi \in [\varepsilon + 2\ell\pi, 2\pi - \varepsilon + 2\ell\pi]$, $\ell \in \mathbb{Z}$, we have

$$\text{rank} [\widehat{\Phi}(\xi + 2j\pi)]_{j \in \mathbb{Z}} = \begin{bmatrix} \cdots & 0 & 0 & c^1 & 0 & \cdots \\ \cdots & 0 & 0 & d^1 & 0 & \cdots \\ \cdots & 0 & c^2 & 0 & 0 & \cdots \\ \cdots & 0 & d^2 & 0 & 0 & \cdots \end{bmatrix} = 2, \quad (24)$$

where $c^i, d^i > 0$, $i = 1, 2$.

We conclude that the rank of matrix $[\widehat{\Phi}(\xi + 2j\pi)]_{j \in \mathbb{Z}}$ is a constant function on $\mathbb{R}$ equal to 2. $\square$

Lemma 10 can be easily generalised for an even number of functions ϕ_i , $i = 0, \dots, 2r - 1$, with compactly supported $\widehat{\phi}_i$, $i = 0, \dots, 2r - 1$. The proof of the next theorem is similar to the previous proofs.

Theorem 12. *Let the functions θ and ψ satisfy all the conditions of Lemma 10. Moreover, let*

$$\begin{aligned} \widehat{\phi}_k(\xi) &= \theta(\xi + 2k\pi), & \widehat{\phi}_{k+r}(\xi) &= \psi(\xi + 2k\pi), \\ k &= 0, \dots, r - 1, & r &\in \mathbb{N}, \end{aligned} \quad (25)$$

and $\Phi = (\phi_0, \phi_1, \dots, \phi_{2r-1})^T$.

The following statements hold.

- (1°) $\text{rank}[\widehat{\Phi}(\xi + 2j\pi)]_{j \in \mathbb{Z}} = r$ for all $\xi \in \mathbb{R}$.
- (2°) $V_\mu^p(\Phi)$ is closed in L_μ^p for any $p \in [1, \infty]$.
- (3°) $\{\phi_i(\cdot - j) \mid j \in \mathbb{Z}, 0 \leq i \leq 2r - 1\}$ is a p -frame for $V_\mu^p(\Phi)$ for any $p \in [1, \infty]$.

Now we consider three functions with compact supports.

Lemma 13. *Let the function θ satisfies all the conditions of Lemma 10, and let $\tau \in C_0^\infty(\mathbb{R})$ and $\omega \in C_0^\infty(\mathbb{R})$ be positive functions such that*

$$\begin{aligned} \tau(x) &> 0, & x &\in (\varepsilon, \pi - \varepsilon) \cup (\pi + \varepsilon, 2\pi - \varepsilon), \\ \omega(x) &> 0, & x &\in (-3\pi - \varepsilon, -\pi + \varepsilon), \\ \text{supp } \tau &= [\varepsilon, \pi - \varepsilon] \cup [\pi + \varepsilon, 2\pi - \varepsilon], \\ \text{supp } \omega &= [-3\pi - \varepsilon, -\pi + \varepsilon], & 0 < \varepsilon < \frac{1}{4}. \end{aligned} \quad (26)$$

Moreover, let $\widehat{\phi}_1(\xi) = \theta(\xi)$, $\widehat{\phi}_2(\xi) = \tau(\xi)$, $\widehat{\phi}_3(\xi) = \omega(\xi)$, $\xi \in \mathbb{R}$, and $\Phi = (\phi_1, \phi_2, \phi_3)^T$. Then $\text{rank}[\widehat{\Phi}(\xi + 2j\pi)]_{j \in \mathbb{Z}} = 2$, $\xi \in \mathbb{R}$.

Proof. We have four different forms for matrix $[\widehat{\Phi}(\xi + 2j\pi)]_{j \in \mathbb{Z}}$ and in all we show that the rank of matrix is equal to 2.

Now we will show all possible cases. Denote with a^i , $i = 1, 2, 3$, and b^i , $i = 1, 2$, some positive values.

(1°) Consider

$$\begin{aligned} [\widehat{\Phi}(\xi + 2j\pi)]_{j \in \mathbb{Z}} &= \begin{bmatrix} \cdots & 0 & a^1 & b^1 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & a^2 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & \cdots \end{bmatrix}, \\ \xi &\in (-\varepsilon + 2\ell\pi, \varepsilon + 2\ell\pi). \end{aligned} \quad (27)$$

(2°) Consider

$$\begin{aligned} [\widehat{\Phi}(\xi + 2j\pi)]_{j \in \mathbb{Z}} &= \begin{bmatrix} \cdots & 0 & b^1 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & a^2 & 0 & 0 & \cdots \\ \cdots & 0 & a^3 & 0 & 0 & 0 & \cdots \end{bmatrix}, \\ \xi &\in [\varepsilon + 2\ell\pi, \pi - \varepsilon + 2\ell\pi]. \end{aligned} \quad (28)$$

(3°) Consider

$$\begin{aligned} [\widehat{\Phi}(\xi + 2j\pi)]_{j \in \mathbb{Z}} &= \begin{bmatrix} \cdots & 0 & b^1 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & a^2 & b^2 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & \cdots \end{bmatrix}, \\ \xi &\in (\pi - \varepsilon + 2\ell\pi, \pi + \varepsilon + 2\ell\pi). \end{aligned} \quad (29)$$

(4°) Consider

$$\begin{aligned} [\widehat{\Phi}(\xi + 2j\pi)]_{j \in \mathbb{Z}} &= \begin{bmatrix} \cdots & 0 & b^1 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & b^2 & 0 & \cdots \\ \cdots & 0 & a^3 & 0 & 0 & 0 & \cdots \end{bmatrix}, \\ \xi &\in [\pi + \varepsilon + 2\ell\pi, 2\pi - \varepsilon + 2\ell\pi]. \end{aligned} \quad (30)$$

$\square$

Remark 14. In Lemma 13 the support of the function ω must have an empty intersection with the supports of θ and τ . In the opposite case, that is, $\text{supp } \theta \cap \text{supp } \tau \cap \text{supp } \omega \neq \emptyset$, the rank of the matrix $[\widehat{\Phi}(\xi + 2j\pi)]_{j \in \mathbb{Z}}$ is a non-constant function on $\mathbb{R}$.

Lemma 13 can be easily generalised for functions ϕ_i , $i = 0, \dots, 3r - 1$, with compactly supported $\widehat{\phi}_i$, $i = 0, \dots, 3r - 1$. The proof of the next theorem is similar to the previous proofs.

Theorem 15. *Let the functions θ , τ , and ω satisfy all the conditions of Lemma 13. Moreover, let*

$$\begin{aligned} \widehat{\phi}_k(\xi) &= \theta(\xi + 2k\pi), \\ \widehat{\phi}_{k+r}(\xi) &= \tau(\xi + 2k\pi), \\ \widehat{\phi}_{k+2r}(\xi) &= \omega(\xi + 2k\pi), \end{aligned} \quad (31)$$

$k = 0, \dots, r - 1$, $r \in \mathbb{N}$, and $\Phi = (\phi_0, \phi_1, \dots, \phi_{3r-1})^T$.

The following statements hold.

- (1°) $\text{rank}[\widehat{\Phi}(\xi + 2j\pi)]_{j \in \mathbb{Z}} = 2r$ for all $\xi \in \mathbb{R}$.
- (2°) $V_\mu^p(\Phi)$ is closed in L_μ^p for any $p \in [1, \infty]$.
- (3°) $\{\phi_i(\cdot - j) \mid j \in \mathbb{Z}, 0 \leq i \leq 3r - 1\}$ is a p -frame for $V_\mu^p(\Phi)$ for any $p \in [1, \infty]$.

5. Construction of Frames of Functions with Finite Regularities and Compact Supports

We will recall the well-known construction of the B -spline functions in order to justify the rank properties of the corresponding matrices.

Let $H(x)$, $x \in \mathbb{R}$, be the characteristic function of the semiaxis $x \geq 0$; that is, $H(x) = 0$ if $x < 0$ and $H(x) = 1$ if $x \geq 0$ (Heaviside's function). We construct a sequence $\{\phi_n\}_{n \in \mathbb{N}}$ in the following way. Let $\phi_1(x) := (H(x) - H(x-a))/a$, $a > 0$, $\phi_2 := \phi_1 * \phi_1$, $\phi_3 := \phi_1 * \phi_1 * \phi_1, \dots$; that is,

$$\phi_n := \underbrace{\phi_1 * \phi_1 * \dots * \phi_1}_{n-1 \text{ times}}, \quad n \in \mathbb{N}, \quad (32)$$

where $*$ denotes the convolution of the functions.

We obtain

$$\begin{aligned} \phi_2(x) &= \frac{1}{a^2} (xH(x) - 2(x-a)H(x-a) \\ &\quad + (x-2a)H(x-2a)), \\ \phi_3(x) &= \frac{1}{2!a^3} (x^2H(x) - 3(x-a)^2H(x-a) + 3(x-2a)^2 \\ &\quad \times H(x-2a) - (x-3a)^2H(x-3a)), \\ \phi_4(x) &= \frac{1}{3!a^4} (x^3H(x) - 4(x-a)^3H(x-a) \\ &\quad + 6(x-2a)^3H(x-2a) - 4(x-3a)^3 \\ &\quad \times H(x-3a) + (x-4a)^3H(x-4a)). \end{aligned} \quad (33)$$

Continuing in this manner, for all $n \in \mathbb{N}$, we have

$$\begin{aligned} \phi_n(x) &= \frac{1}{a^n(n-1)!} \\ &\times \left(\binom{n}{0} x^{n-1}H(x) - \binom{n}{1} (x-a)^{n-1} \right. \\ &\quad \times H(x-a) + \binom{n}{2} (x-2a)^{n-1} \\ &\quad \times H(x-2a) - \binom{n}{3} (x-3a)^{n-1}H(x-3a) \\ &\quad + \dots + (-1)^{n-1} \binom{n}{n-1} (x-(n-1)a)^{n-1} \\ &\quad \times H(x-(n-1)a) + (-1)^n \binom{n}{n} \\ &\quad \times (x-na)^{n-1}H(x-na) \Big). \end{aligned} \quad (34)$$

Calculating the Fourier transform of functions ϕ_n , $n \in \mathbb{N}$, we get

$$\begin{aligned} \widehat{\phi}_1(\xi) &= \frac{-i}{a} \nu \cdot p \cdot \left(\frac{1}{\xi} \right) (e^{ia\xi} - 1), \\ \widehat{\phi}_2(\xi) &= \frac{(-i)^2}{a^2} \nu \cdot p \cdot \left(\frac{1}{\xi^2} \right) (e^{ia\xi} - 1)^2, \\ \widehat{\phi}_3(\xi) &= \frac{(-i)^3}{a^3} \nu \cdot p \cdot \left(\frac{1}{\xi^3} \right) (e^{ia\xi} - 1)^3. \end{aligned} \quad (35)$$

Continuing in this manner, we obtain $\widehat{\phi}_n(\xi) = (-i)^n / a^n \nu \cdot p \cdot (1/\xi^n)(e^{ia\xi} - 1)^n$, $n \in \mathbb{N}$, where $\nu \cdot p \cdot$ denotes the principal value.

Let $\Phi = (\phi_1, \phi_2, \dots, \phi_r)^T$, $r \in \mathbb{N}$. matrix $[\widehat{\Phi}(\xi + 2j\pi)]_{j \in \mathbb{Z}}$ has for all $\xi \in \mathbb{R}$ the same rank as matrix

$$R(\xi) = \begin{bmatrix} \cdots & \alpha_{-4\pi} \beta_{-4\pi} & \alpha_{-2\pi} \beta_{-2\pi} & \alpha_0 \beta_0 & \alpha_{2\pi} \beta_{2\pi} & \alpha_{4\pi} \beta_{4\pi} & \cdots \\ \cdots & \alpha_{-4\pi}^2 \beta_{-4\pi}^2 & \alpha_{-2\pi}^2 \beta_{-2\pi}^2 & \alpha_0^2 \beta_0^2 & \alpha_{2\pi}^2 \beta_{2\pi}^2 & \alpha_{4\pi}^2 \beta_{4\pi}^2 & \cdots \\ \cdots & \alpha_{-4\pi}^3 \beta_{-4\pi}^3 & \alpha_{-2\pi}^3 \beta_{-2\pi}^3 & \alpha_0^3 \beta_0^3 & \alpha_{2\pi}^3 \beta_{2\pi}^3 & \alpha_{4\pi}^3 \beta_{4\pi}^3 & \cdots \\ \cdots & \alpha_{-4\pi}^4 \beta_{-4\pi}^4 & \alpha_{-2\pi}^4 \beta_{-2\pi}^4 & \alpha_0^4 \beta_0^4 & \alpha_{2\pi}^4 \beta_{2\pi}^4 & \alpha_{4\pi}^4 \beta_{4\pi}^4 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & \alpha_{-4\pi}^r \beta_{-4\pi}^r & \alpha_{-2\pi}^r \beta_{-2\pi}^r & \alpha_0^r \beta_0^r & \alpha_{2\pi}^r \beta_{2\pi}^r & \alpha_{4\pi}^r \beta_{4\pi}^r & \cdots \end{bmatrix}, \quad (36)$$

where $\alpha_k^m = \nu \cdot p \cdot (1/(\xi - k))^m$ and $\beta_k^m = (e^{ia(\xi - k)} - 1)^m$. Since $\text{rank } R(\xi) = r$, $\xi \in \mathbb{R}$, we have the next result.

Theorem 16. Let $\Phi = (\phi_k, \phi_{k+1}, \dots, \phi_{k+(r-1)})^T$, for $k \in \mathbb{Z}$, $r \in \mathbb{N}$. Then $V_\mu^p(\Phi)$ is closed in L_μ^p for any $p \in [1, \infty]$ and $\{\phi_{k+s}(\cdot - j) \mid j \in \mathbb{Z}, 0 \leq s \leq r-1\}$ is a p -Riesz basis for $V_\mu^p(\Phi)$ for any $p \in [1, \infty]$.

Remark 17. Let k be a positive integer. We refer to [23] for the L^p -approximation order k . Shift-invariant spaces generated by a finite number of compactly supported functions in $L^p(\mathbb{R})$, $1 \leq p \leq \infty$, were studied in [23] by Jia, who gave a characterization of the approximation order providing such shift-invariant spaces. Theorem 3 in [23] shows that the shift-invariant space generated with the family of splines, which we constructed in Section 5, provides L^p -approximation order k .

Remark 18. (1) We refer to [3, 20] for the γ -dense set $X = \{x_j \mid j \in J\}$. Let $\phi_k(x) = \mathcal{F}^{-1}(\theta(\cdot - k\pi))(x)$, $x \in \mathbb{R}$. Following the notation of [20], we put $\psi_{x_j} = \phi_{x_j}$, where $\{x_j \mid j \in J\}$ is γ -dense set determined by $f \in V^2(\phi) = V^2(\mathcal{F}^{-1}(\theta))$. Theorems 3.1, 3.2, and 4.1 in [20] give conditions and explicit form of $C_p > 0$ and $c_p > 0$ such that inequality $c_p \|f\|_{L_\mu^p} \leq (\sum_{j \in J} |\langle f, \psi_{x_j} \rangle \mu(x_j)|^p)^{1/p} \leq C_p \|f\|_{L_\mu^p}$ holds. This inequality guarantees the feasibility of a stable and continuous reconstruction algorithm in the signal spaces $V_\mu^p(\Phi)$ [20].

(2) Since the spectrum of the Gram matrix $[\widehat{\Phi}, \widehat{\Phi}](\xi)$, where Φ is defined in Theorem 16, is bounded and bounded away from zero (see [7]), it follows that the family $\{\phi(\cdot - j) \mid j \in \mathbb{Z}\}$ forms a p -Riesz basis for $V_\mu^p(\Phi)$.

(3) Frames of the above sections may be useful in applications since they satisfy assumptions of Theorems 3.1 and 3.2 in [4]. They show that error analysis for sampling and reconstruction can be tolerated or that the sampling and reconstruction problem in shift-invariant space is robust with respect to appropriate set of functions $\phi_{k_1}, \dots, \phi_{k_r}$.

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