Research Article

Frequency-Uniform Decomposition, Function Spaces $X_{p,q}^s$, and Applications to Nonlinear Evolution Equations

Shaolei Ru¹ and Jiecheng Chen²

¹ Department of Mathematics, Zhejiang University, Hangzhou 310027, China

² Department of Mathematics, Zhejiang Normal University, Jinhua 321004, China

Correspondence should be addressed to Jiecheng Chen; jcchen@zjnu.edu.cn

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By combining frequency-uniform decomposition with $L^p(\ell^q)$, we introduce a new class of function spaces (denoted by $X_{p,q}^s$). Moreover, we study the Cauchy problem for the generalized NLS equations and Ginzburg-Landau equations in $L^r(0, T; X_{p,1}^s)$.

1. Introduction and Notation

In this paper, we introduce a new space (denoted by $X_{p,q}^s$) by combining frequency-uniform decomposition with $L^p(\ell^q)$. In this new space, we discuss the semilinear estimates, Schwartz estimates, embedding properties between $X_{p,q}^s$ and Triebel-Lizorkin space $F_{p,q}^s$, and so forth. Moreover, we study the well-posedness of NLS equations and Ginzburg-Landau equations in $L^r(0, T; X_{p,1}^s)$.

In recent decades, the well-posedness of Schrödinger equations and Ginzburg-Landau equations is developed quickly. In this paper, we mainly study the local Cauchy problem of NLS equations and Ginzburg-Landau equations in this new space $X_{p,q}^s$. Many people have studied these problems in Bessel potential and Besov spaces, for example, see [1–9], and so forth. We know that Besov spaces and Triebel spaces are generated by combining Littlewood-Paley decomposition with $\ell^q(L^p)$ and $L^p(\ell^q)$ (one can refer to [10] for the properties of Besov and Triebel spaces.). Next, we give a brief introduction of these spaces (see [10, 11]).

To define the Littlewood-Paley decomposition, it is convenient to consider a radial function $\psi : \mathbb{R}^n \rightarrow [0, 1]$ such that

$$\psi(\xi) = \begin{cases} 1, & |\xi| \leq 1; \\ \text{smooth}, & 1 < |\xi| < 2; \\ 0, & |\xi| \geq 2, \end{cases} \quad (1)$$

and to define the sequence $\{\varphi_k\}_0^\infty$ by

$$\begin{aligned} \varphi_k(\xi) &= \varphi(2^{-k}\xi) - \varphi(2^{-k+1}\xi), \quad k \in \mathbb{N}; \\ \varphi_0(\xi) &= 1 - \sum_{k=1}^{\infty} \varphi_k(\xi) = \psi(\xi). \end{aligned} \quad (2)$$

Since $\text{supp } \varphi \subset \{\xi : 2^{-1} \leq |\xi| \leq 2\}$, we can easily see that $\text{supp } \varphi_k \subset \{\xi : 2^{k-1} \leq |\xi| \leq 2^{k+1}\}$, $k \in \mathbb{N}$, and $\text{supp } \varphi_0 \subset \{\xi : |\xi| \leq 2\}$. Define

$$\Delta_k = F^{-1} \varphi_k F, \quad k \in \mathbb{N} \cup \{0\}, \quad (3)$$

where F is the Fourier transform and F^{-1} is the inverse Fourier transform. $\{\Delta_k\}_0^\infty$ is said to be Littlewood-Paley decomposition operator. Formally, we see that

$$\sum_{k=0}^{\infty} \Delta_k = I, \quad \text{in } \mathcal{S}'. \quad (4)$$

We now define the Besov space $B_{p,q}^s(\mathbb{R}^n)$ for $-\infty < s < \infty$ and $0 < p, q \leq \infty$ by

$$\begin{aligned} B_{p,q}^s(\mathbb{R}^n) &= \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{B_{p,q}^s} < \infty \right\}, \\ \|f\|_{B_{p,q}^s} &:= \left(\sum_{k=0}^{\infty} 2^{ksq} \|\Delta_k f\|_p^q \right)^{1/q}, \end{aligned} \quad (5)$$

and Triebel space $F_{p,q}^s(\mathbb{R}^n)$ for $-\infty < s < \infty$ and $0 < p < \infty$, $0 < q \leq \infty$ by

$$F_{p,q}^s(\mathbb{R}^n) = \left\{ f \in S'(\mathbb{R}^n) : \|f\|_{F_{p,q}^s} < \infty \right\}, \quad (6)$$

$$\|f\|_{F_{p,q}^s} := \left\| \left(\sum_{k=0}^{\infty} 2^{ksq} |\Delta_k f|^q \right)^{1/q} \right\|_p.$$

Besov and Triebel spaces had been formulated during 1960s–1980s, which have been widely applied in recent years. Besov spaces $B_{p,q}^s$ and Triebel spaces $F_{p,q}^s$ are introduced by combining Littlewood-Paley decomposition operator with the function spaces $\ell^q(L^p)$ and $L^p(\ell^q)$. Similar to Besov spaces and Triebel spaces, one can use frequency-uniform decomposition and $\ell^q(L^p)$, $L^p(\ell^q)$ to generate a new class of function spaces. Actually, the spaces introduced by frequency-uniform decomposition and $\ell^q(L^p)$ are modulation spaces. Many people have studied the well-posedness of evolution equations in these spaces (e.g., see [12–16], etc.). In this paper, we will consider the spaces generated by the frequency-uniform decomposition and $L^p(\ell^q)$. Firstly, we will recall the definition of the frequency-uniform decomposition and modulation spaces. In the 1930s, Wiener [17] first introduced the frequency-uniform decomposition. So, sometimes we call it Wiener decomposition of $\mathbb{R}^n$ that is roughly denoted by

$$\square_k \sim F^{-1} \chi_{Q_k} F, \quad k \in \mathbb{Z}^n, \quad (7)$$

where χ_E denotes the characteristic function on the set E . Since Q_k is a translation of Q_0 , $\square_k$ have the same localized structures in frequency space, which are said to be the frequency-uniform decomposition operators. Since χ_{Q_k} can not make differential operations, one needs to redefine it by smooth truncation function (see [12, 14, 18, 19]).

Now we give an exact definition on modulation spaces. We first denote $|\xi|_{\infty} := \max_{i=1,\dots,n} |\xi_i|$. Let $\rho \in S(\mathbb{R}^n) : \mathbb{R}^n \rightarrow [0, 1]$ be a smooth radial bump function adapted to $\{\xi : |\xi_i|_{\infty} \leq 1\}$ with $\rho(\xi) = 1$ if $|\xi|_{\infty} \leq 1/2$ and $\rho(\xi) = 0$ if $|\xi|_{\infty} \geq 1$. Let ρ_k be a translation of ρ :

$$\rho_k(\xi) = \rho(\xi - k), \quad k \in \mathbb{Z}^n. \quad (8)$$

Since $\rho_k(\xi) = 1$ in the unit closed cube $Q_k : \{\xi \in \mathbb{R}^n : |\xi - k|_{\infty} \leq 1/2\}$ and $\{Q_k\}_{k \in \mathbb{Z}^n}$ is a covering of $\mathbb{R}^n$, one has that for all $\xi \in \mathbb{R}^n$, $\sum_{k \in \mathbb{Z}^n} \rho_k(\xi) \geq 1$. Then, we write

$$\sigma_k(\xi) = \rho_k(\xi) \left(\sum_{k \in \mathbb{Z}^n} \rho_k(\xi) \right)^{-1}, \quad k \in \mathbb{Z}^n. \quad (9)$$

It is easy to see that

$$\begin{aligned} |\sigma_k(\xi)| &\geq c, \quad \xi \in Q_k; \\ \text{supp } \sigma_k &\subset \{\xi : |\xi - k|_{\infty} \leq 1\}; \\ \sum_{k \in \mathbb{Z}^n} \sigma_k(\xi) &\equiv 1, \quad \forall \xi \in \mathbb{R}^n; \\ |D^{\alpha} \sigma_k(\xi)| &\leq C_{|\alpha|}, \quad \forall \xi \in \mathbb{R}^n, \alpha \in (N \cup \{0\})^n. \end{aligned} \quad (***)$$

Hence, the set

$$\Upsilon_n = \left\{ \{\sigma_k\}_{k \in \mathbb{Z}^n} : \{\sigma_k\}_{k \in \mathbb{Z}^n} \text{ satisfy } (***) \right\} \quad (10)$$

is nonvoid. Let $\{\sigma_k\}_{k \in \mathbb{Z}^n} \in \Upsilon_n$ be a function sequence. Denote

$$\square_k := F^{-1} \sigma_k F, \quad k \in \mathbb{Z}^n, \quad (11)$$

which is said to be the frequency-uniform decomposition operator. For any $k \in \mathbb{Z}^n$, we set $|k| = |k_1| + \dots + |k_n|$, $\langle k \rangle = 1 + |k|$. For any $s \in \mathbb{R}$, $0 < p, q \leq \infty$, we denote

$$M_{p,q}^s(\mathbb{R}^n) = \left\{ f \in S'(\mathbb{R}^n) : \|f\|_{M_{p,q}^s} = \left(\sum_{k \in \mathbb{Z}^n} \langle k \rangle^{sq} \|\square_k f\|_{L^p}^q \right)^{1/q} < \infty \right\}. \quad (12)$$

$M_{p,q}^s := M_{p,q}^s(\mathbb{R}^n)$ is said to be a modulation space, which was first introduced by Feichtinger [19] in the case $1 \leq p, q \leq \infty$ (Appendix Theorem E shows the properties on $M_{p,q}^s(\mathbb{R}^n)$). For any $0 \leq s < \infty$, $0 < p, q \leq \infty$, we denote

$$E_{p,q}^s(\mathbb{R}^n) = \left\{ f \in S'(\mathbb{R}^n) : \|f\|_{E_{p,q}^s} = \left(\sum_{k \in \mathbb{Z}^n} 2^{|k|sq} \|\square_k f\|_{L^p}^q \right)^{1/q} < \infty \right\}. \quad (13)$$

$E_{p,q}^s := E_{p,q}^s(\mathbb{R}^n)$ was first introduced by Baoxiang et al. [12].

We now introduce a new class of function spaces (denoted by $X_{p,q}^s$) by combining the frequency-uniform decomposition with $L^p(\ell^q)$. For any $0 < p < \infty$, $0 < q \leq \infty$, $0 \leq s < \infty$, we denote

$$X_{p,q}^s(\mathbb{R}^n) := \left\{ f \in S'(\mathbb{R}^n) : \|f\|_{X_{p,q}^s} = \left\| \left(\sum_{k \in \mathbb{Z}^n} 2^{|k|sq} |\square_k f|^q \right)^{1/q} \right\|_{L^p} < \infty \right\}. \quad (14)$$

Moreover, we denote

$$L^r(\mathbb{R}, X_{p,q}^s) := \left\{ f \in S' : \left(\int_{\mathbb{R}} \|f\|_{X_{p,q}^s}^r dt \right)^{1/r} < \infty \right\}. \quad (15)$$

Main Results

Theorem 1. Assume $f(u) = u|u|^k$, $k = 2m$, $m \in \mathbb{Z}^+$, and $r \geq k + 2$; then for any initial data $u_0 \in X_{1,1}^s$, $0 \leq s < \infty$, there

exists $T^* := T^*(\|u_0\|_{X_{1,1}^s}) > 0$ such that the initial value problem

$$u(t) = S(t)u_0 - i \int_0^t S(t-\tau) f(u(\tau)) d\tau, \quad (\text{NLS equation}) \quad (16)$$

has a unique solution

$$u \in L_{\text{loc}}^r(0, T^*; X_{1,1}^s). \quad (17)$$

Moreover, if $T^* < \infty$, then

$$\|u\|_{L^r(0, T^*; X_{1,1}^s)} = \infty, \quad (18)$$

where $S(t) = F^{-1}e^{-it|\xi|^2}F$.

Theorem 2. Assume $f(u) = (b + i\beta)u|u|^k + \mu u$, $k = 2m$, $m \in \mathbb{Z}^+$, and $r \geq k + 2$. Assume also $a, b > 0$, $\alpha, \beta, \mu \in \mathbb{R}$. Then, for any initial data $u_0 \in X_{p,1}^s$, $1 \leq p < \infty$, $0 \leq s < \infty$, there exists $T^* := T^*(\|u_0\|_{X_{p,1}^s}) > 0$ such that the initial value problem (complex Ginzburg-Landau equation)

$$u(t) = U(t)u_0 - i \int_0^t U(t-\tau) f(u(\tau)) d\tau \quad (19)$$

has a unique solution

$$u \in L_{\text{loc}}^r(0, T^*; X_{p,1}^s). \quad (20)$$

Moreover, if $T^* < \infty$, then

$$\|u\|_{L^r(0, T^*; X_{p,1}^s)} = \infty, \quad (21)$$

where $U(t) = F^{-1}e^{-tP(\xi)}F$, $P(\xi) = 1 + (a + i\alpha)|\xi|^2$.

This paper is organized as follows. In Section 2, we will state some properties on $X_{p,q}^s$, which is useful to establish the following inclusions. Section 3 is devoted to considering the multiplication algebra of $X_{p,q}^s$. Some dispersive smooth effects for the Schrödinger and Ginzburg-Landau semigroups will be given in Section 4. Theorems 1 and 2 will be proved in Section 5.

2. The Properties on $X_{p,q}^s$

In order to study the Cauchy problem in $X_{p,q}^s$, we first give some properties on $X_{p,q}^s$ as follows.

Proposition 3. Letting $0 \leq s < \infty$, $0 < p < \infty$, $0 < q \leq \infty$, one has

(1) if $q_1 \leq q_2$, then

$$X_{p,q_1}^s \subset X_{p,q_2}^s; \quad (22)$$

(2) if $\varepsilon > 0$, then

$$X_{p,q_1}^{s+\varepsilon} \subset X_{p,q_2}^s; \quad (23)$$

(3) if $p < \infty$, then

$$E_{p,p \wedge q}^s \subset X_{p,q}^s \subset E_{p,p \vee q}^s, \quad (p \wedge q = \min(p, q), \quad p \vee q = \max(p, q)). \quad (24)$$

Proof. Because $\ell^p \subset \ell^{p+a}$, $a \geq 0$, we can get the results of (1) directly. For (2), we know that

$$\begin{aligned} & \left(\sum_{k \in \mathbb{Z}^n} 2^{|k|sq_2} |a_k|^{q_2} \right)^{1/q_2} \\ &= \left(\sum_{k \in \mathbb{Z}^n} 2^{-|k|\varepsilon q_2} 2^{|k|(s+\varepsilon)q_2} |a_k|^{q_2} \right)^{1/q_2} \\ &\leq \sup_{k \in \mathbb{Z}^n} 2^{|k|(s+\varepsilon)} |a_k|, \quad (\varepsilon > 0). \end{aligned} \quad (25)$$

Then, letting $a_k = \square_k f$, we get (2).

Finally, we prove (3). Let $b_k = 2^{|k|s} \square_k f$. The proof can be classified into the following two kinds of cases.

Case 1 ($q \leq p$). In this case, $\ell^q \subset \ell^p$. We have

$$\|b_k\|_{\ell^p(L^p)} \leq \|b_k\|_{L^p(\ell^q)} \leq \|b_k\|_{\ell^q(L^p)}. \quad (26)$$

Actually, noticing that $\ell^q \subset \ell^p$, we have $\|b_k\|_{\ell^p(L^p)} = \|b_k\|_{L^p(\ell^p)} \leq \|b_k\|_{L^p(\ell^q)}$. So, the first part of the above inequality holds. On the other hand, by Minkowski's inequality, we have

$$\begin{aligned} \|b_k\|_{L^p(\ell^q)} &= \left\| \sum_{k=0}^{\infty} |b_k|^q \right\|_{L^{p/q}}^{1/q} \\ &\leq \left(\sum_{k=0}^{\infty} \| |b_k|^q \|_{L^{p/q}} \right)^{1/q} = \|b_k\|_{\ell^q(L^p)}. \end{aligned} \quad (27)$$

This proves the second part.

Case 2 ($p \leq q$). By Minkowski's inequality and $\ell^p \subset \ell^q$, we have

$$\begin{aligned} & \|b_k\|_{\ell^q(L^p)} \\ &= \left(\left\| \int_{\mathbb{R}^n} |b_k|^p dx \right\|_{\ell^{q/p}} \right)^{1/p} \leq \left(\int_{\mathbb{R}^n} \| |b_k|^p \|_{\ell^{q/p}} dx \right)^{1/p} \\ &= \|b_k\|_{L^p(\ell^q)} \leq \|b_k\|_{\ell^p(L^p)}. \end{aligned} \quad (28)$$

This finishes the proof of this proposition. $\square$

Proposition 4 (completeness). For any $0 \leq s < \infty$, $0 < p < \infty$, $0 < q \leq \infty$, one has the following:

- (1) $X_{p,q}^s$ is a quasi-Banach space. Moreover, if $1 \leq p < \infty$, $1 \leq q \leq \infty$, then $X_{p,q}^s$ is a Banach space;
- (2) $S(\mathbb{R}^n) \subset X_{p,q}^0 \subset S'(\mathbb{R}^n)$;
- (3) if $0 < p, q < \infty$, then $\mathcal{S}(\mathbb{R}^n)$ is dense in $X_{p,q}^0$.

Proof

Step 1. In this step, we prove that $S(\mathbb{R}^n) \subset X_{p,q}^0 \subset S'(\mathbb{R}^n)$. Actually, by $S(\mathbb{R}^n) \subset E_{p,q}^0 \subset S'(\mathbb{R}^n)$ [12] and $E_{p,p \wedge q}^s \subset X_{p,q}^s \subset E_{p,p \vee q}^s$ (Proposition 3), we immediately obtain the result.

Step 2. $X_{p,q}^s$ is a quasinormed space. We prove the completeness. Let $\{f_\ell\}_{\ell=1}^\infty$ be a Cauchy sequence in $X_{p,q}^s$ (with respect to a fixed quasinorm in $X_{p,q}^s$). Proposition 3 shows that $\{f_\ell\}_{\ell=1}^\infty$ is also a Cauchy sequence in $E_{p,p \vee q}^s$. Because $E_{p,p \vee q}^s$ is a complete topological linear space, we can find a limit element $f \in E_{p,p \vee q}^s$. Then, $F^{-1}\sigma_k F f_\ell$ converges to $F^{-1}\sigma_k F f$ in $E_{p,p \vee q}^s$ if $\ell \rightarrow \infty$. On the other hand, $\{F^{-1}\sigma_k F f_\ell\}_{\ell=1}^\infty$ is a Cauchy sequence in $L^p(\mathbb{R}^n)$ ($X_{p,q}^s \subset E_{p,p \vee q}^s$, $s \geq 0$). By Appendix Theorem B, it is also a Cauchy sequence in $L^\infty(\mathbb{R}^n)$. This shows that the limiting element of $\{F^{-1}\sigma_k F f_\ell\}_{\ell=1}^\infty$ in $L^p(\mathbb{R}^n)$ (which is the same as in $L^\infty(\mathbb{R}^n)$) coincides with $\{F^{-1}\sigma_k F f\}$. Now it follows by standard arguments that f belongs to $X_{p,q}^s$ and that f_ℓ converges in $X_{p,q}^s$ to f . Hence, $X_{p,q}^s$ is complete.

Step 3. We prove that $\mathcal{S}(\mathbb{R}^n)$ is dense in $X_{p,q}^0(\mathbb{R}^n)$ if $0 < p < \infty$ and $0 < q < \infty$. Let $f \in X_{p,q}^0$; then we put

$$f_N(x) = \sum_{k=0}^N F^{-1}\sigma_k F f. \quad (29)$$

Of course, $f_N \in X_{p,q}^0(\mathbb{R}^n)$. Consequently (by Appendix Theorem F),

$$\begin{aligned} & \|f - f_N\|_{X_{p,q}^0(\mathbb{R}^n)} \\ & \leq c \left\| \left(\sum_{k=N}^{\infty} \sum_{r=-1}^1 |F^{-1}\sigma_k \sigma_{k+r} F f|^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)} \\ & \leq c \left\| \left(\sum_{k=N}^{\infty} |F^{-1}\sigma_k F f|^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)}. \end{aligned} \quad (30)$$

Lebesgue's bounded convergence theorem proves that the right-hand side of the above inequality tends to zero if $N \rightarrow \infty$. Hence, f_N approximates f in $X_{p,q}^0(\mathbb{R}^n)$. Next, we let $\varphi \in \mathcal{S}$ with $\varphi(0) = 1$ and $\text{supp } \varphi \subset \{y : |y| \leq 1\}$. Let $f_\delta(x) = \varphi(\delta x)f(x)$ with $0 < \delta < 1$. Then, $(f_N)_\delta \in \mathcal{S}(\mathbb{R}^n)$ approximates f_N in $L_p^\Omega := \{f \in L^p : \text{supp } \tilde{f} \subset \Omega\}$ with $\Omega = \{y : |y| \leq 2^{N+2}\}$ if $\delta \rightarrow 0$. However this is also an approximation of f_N in $X_{p,q}^0(\mathbb{R}^n)$. This proves that $\mathcal{S}(\mathbb{R}^n)$ is dense in $X_{p,q}^0(\mathbb{R}^n)$. $\square$

Proposition 5 (equivalent norm). *Assume $\{\sigma_k\}_{k \in \mathbb{Z}^n}$, $\{\phi_k\}_{k \in \mathbb{Z}^n} \in Y_n$, $0 < p < \infty$, $0 < q \leq \infty$. Then, $\{\sigma_k\}_{k \in \mathbb{Z}^n}$ and $\{\phi_k\}_{k \in \mathbb{Z}^n}$ generate equivalent norms on $X_{p,q}^s$.*

Proof. Firstly, we have the following translation equality:

$$(F^{-1}mFf)(x) = e^{ixk} [F^{-1}m(\cdot + k)F(e^{-iky}f(y))](x). \quad (31)$$

For the sake of convenience, we denote

$$\square_k^\sigma = F^{-1}\sigma_k F, \quad \square_k^\phi = F^{-1}\phi_k F. \quad (32)$$

Then, we have

$$\begin{aligned} \square_k^\sigma &= \sum_{|\ell|_\infty \leq 1} \square_k^\sigma \square_{k+\ell}^\phi, \\ \|\square_k^\sigma f\|_{L^p(\ell^q)} &\leq \left\| \sum_{|\ell|_\infty \leq 1} \square_k^\sigma (\square_{k+\ell}^\phi f) \right\|_{L^p(\ell^q)}. \end{aligned} \quad (33)$$

By Appendix Theorem F, we have

$$\begin{aligned} \|\square_k^\sigma (\square_{k+\ell}^\phi f)\|_{L^p(\ell^q)} &\leq \sup_k \|\sigma_k\|_{H^s} \|\square_{k+\ell}^\phi f\|_{L^p(\ell^q)}, \\ s &> n \left(\frac{1}{1 \wedge p} - \frac{1}{2} \right). \end{aligned} \quad (34)$$

Then, by $\|\sigma_k\|_{H^s} < C$, we have

$$\|\square_k^\sigma f\|_{L^p(\ell^q)} \leq \sum_{|\ell|_\infty \leq 1} \|\square_{k+\ell}^\phi f\|_{L^p(\ell^q)}. \quad (35)$$

By the above discussion, we get $\|f\|_{X_{p,q}^0}^{\{\sigma_k\}} \lesssim \|f\|_{X_{p,q}^0}^{\{\phi_k\}}$. $\square$

The inverse inequality can be proved similarly. Based on the above observations, one can also obtain that $\|f\|_{X_{p,q}^0}^{\{\sigma_k\}} \sim \|f\|_{X_{p,q}^0}^{\{\phi_k\}}$.

Theorem 6. *Assume $1 \leq p_2 \leq p_1 < \infty$, $1 \leq q \leq \infty$, $0 \leq s < \infty$; then one has*

$$\|u\|_{X_{p_1,q}^s} \lesssim \|u\|_{X_{p_2,q}^s}. \quad (36)$$

Proof. By the definition of $X_{p,q}^s$, we have

$$\begin{aligned} \|u\|_{X_{p_1,q}^s} &= \left\| \left(\sum_{i \in \mathbb{Z}^n} 2^{|i|qs} |\square_i u|^q \right)^{1/q} \right\|_{L^{p_1}} \\ &= \left\| \left(\sum_{i \in \mathbb{Z}^n} 2^{|i|qs} \left| \square_i \sum_{|\ell|_\infty \leq 1} \square_{i+\ell} u \right|^q \right)^{1/q} \right\|_{L^{p_1}} \\ &= \left\| \left(\sum_{i \in \mathbb{Z}^n} 2^{|i|qs} \left| \sigma_i^\vee * \left(\sum_{|\ell|_\infty \leq 1} \square_{i+\ell} u \right) \right|^q \right)^{1/q} \right\|_{L^{p_1}} \\ &\leq \left\| \left(\sum_{i \in \mathbb{Z}^n} 2^{|i|qs} \left| \sigma_i^\vee \right| * \left(\sum_{|\ell|_\infty \leq 1} \square_{i+\ell} u \right) \right|^q \right)^{1/q} \right\|_{L^{p_1}} \end{aligned}$$

$$\begin{aligned}
&= \left\| \left(\sum_{i \in \mathbb{Z}^n} 2^{|i|qs} \left\| \sigma_0^\vee |*| \left(\sum_{|\ell|_\infty \leq 1} \square_{i+\ell} u \right) \right\|^q \right)^{1/q} \right\|_{L^{p_1}} \\
&\leq \left\| \left(\sum_{i \in \mathbb{Z}^n} 2^{|i|qs} \left| \left(\sum_{|\ell|_\infty \leq 1} \square_{i+\ell} u \right) \right|^q \right)^{1/q} \right\|_{L^{p_2}} \\
&\lesssim \left\| \left(\sum_{i \in \mathbb{Z}^n} 2^{|i|qs} |\square_i u|^q \right)^{1/q} \right\|_{L^{p_2}}.
\end{aligned} \tag{37}$$

In this proof, we used $\|\sigma_0^\vee |*| f\|_{L^{p_1}} \leq \|f\|_{L^{p_2}}$ ($p_1 \geq p_2$, Young's inequality) and Appendix Theorem G. $\square$

Theorem 7. Assume $f \in L^2$; then one has

$$\|f\|_{L^2} \sim \|f\|_{X_{2,2}^0}. \tag{38}$$

Proof. We first prove $\|f\|_{L^2} \leq \|f\|_{X_{2,2}^0}$. Consider the following:

$$\begin{aligned}
\|f\|_{L^2} &= \left(\int_{\mathbb{R}^n} \left| \sum_{i \in \mathbb{Z}^n} \square_i f \right|^2 dx \right)^{1/2} \\
&= \left(\int_{\mathbb{R}^n} \left| \sum_{i \in \mathbb{Z}^n} \sigma_i Ff \right|^2 dx \right)^{1/2} \\
&\leq \left(\int_{\mathbb{R}^n} \sum_{i \in \mathbb{Z}^n} |\sigma_i Ff|^2 dx \right)^{1/2} \\
&= \left(\int_{\mathbb{R}^n} \sum_{i \in \mathbb{Z}^n} |F^{-1} \sigma_i Ff|^2 dx \right)^{1/2} \\
&= \left(\int_{\mathbb{R}^n} \left(\sum_{i \in \mathbb{Z}^n} |F^{-1} \sigma_i Ff|^2 \right)^{1/2} dx \right)^{1/2} = \|f\|_{X_{2,2}^0}.
\end{aligned} \tag{39}$$

The inverse inequality can be proved similarly ($\sum_{i \in \mathbb{Z}^n} |\sigma_i Ff|^2 \leq |\sum_{i \in \mathbb{Z}^n} \sigma_i Ff|^2$). Moreover, by the similar discussion as mentioned previously, we can also obtain that $\|(\sum_{k \in \mathbb{Z}^n} |F^{-1}(1 + |\xi|^2)^{s/2} \sigma_k Ff|^2)^{1/2}\|_{L^2} \sim \|f\|_{H_2^s}$. $\square$

Theorem 8. Let $0 < s < \infty$, $0 < p < \infty$, $0 < q \leq \infty$. Then, one has

$$X_{p,q}^s(\mathbb{R}^n) \subset C^\infty(\mathbb{R}^n). \tag{40}$$

Proof. One can obtain the result directly by Appendix Theorem D and Proposition 3. $\square$

Theorem 9. Let $0 < p < \infty$, $0 < q \leq 1$. One has

$$X_{p,q}^0(\mathbb{R}^n) \subset F_{p,q}^0(\mathbb{R}^n). \tag{41}$$

Proof. Let $a_k = \max(0, 2^{k-1} - \sqrt{n})$, $b_k = 2^{k+1} + \sqrt{n}$. We can easily find that if $|i| \in [a_k, b_k]$, then $\Delta_k \square_i f = 0$. For $p > 0$, $q \leq 1$, by Appendix Theorem F, we have

$$\begin{aligned}
&\|\Delta_k f\|_{L^p(\ell^q)} \\
&= \left\| \left(\sum_k |\Delta_k f|^q \right)^{1/q} \right\|_{L^p} \\
&\leq \left\| \left(\sum_k \left| \sum_{i \in \mathbb{Z}^n, |i| \in [a_k, b_k]} \Delta_k \square_i f \right|^q \right)^{1/q} \right\|_{L^p} \\
&\leq \left\| \left(\sum_k \sum_{i \in \mathbb{Z}^n, |i| \in [a_k, b_k]} |F^{-1}(\varphi_k \sigma_i) F \square_i f|^q \right)^{1/q} \right\|_{L^p} \\
&\leq \left\| \left(\sum_{i \in \mathbb{Z}^n} \sum_{k, |i| \in [a_k, b_k]} |F^{-1}(\varphi_k \sigma_i) F \square_i f|^q \right)^{1/q} \right\|_{L^p} \\
&\leq \left\| \left(\sum_{i \in \mathbb{Z}^n} |\square_i f|^q \right)^{1/q} \right\|_{L^p}.
\end{aligned} \tag{42}$$

This proves the theorem. $\square$

Theorem 10. Let $F_{p,q}^s(\mathbb{R}^n)$ be Triebel space; then one has

$$\begin{aligned}
F_{2,q}^s(\mathbb{R}^n) &\subset X_{2,q}^0(\mathbb{R}^n), \quad s > n\left(\frac{1}{q} - \frac{1}{2}\right), \\
&\quad 0 < q < 2, \\
X_{2,q}^0(\mathbb{R}^n) &\subset F_{2,q}^s(\mathbb{R}^n), \quad s < n\left(\frac{1}{q} - \frac{1}{2}\right), \\
&\quad 2 < q \leq \infty.
\end{aligned} \tag{43}$$

Proof. By Appendix Theorem A, we know that

$$H^s \subset E_{2,q}^0; \quad s > n\left(\frac{1}{q} - \frac{1}{2}\right), \quad 0 < q < 2. \tag{44}$$

Then, by the embedding between $E_{p,q}^s$ and $X_{p,q}^s$ (Proposition 3), we have (for any given $\varepsilon > 0$)

$$\begin{aligned}
F_{2,q}^{n(1/q-1/2)+\varepsilon} &\subset F_{2,2}^{n(1/q-1/2)+\varepsilon} \\
&= H^{n(1/q-1/2)+\varepsilon} \subset E_{2,q}^0 \subset X_{2,q}^0.
\end{aligned} \tag{45}$$

This proves the first result ($F_{2,2}^s \sim H^s$, [11]).

On the other hand, by Appendix Theorem A, we have

$$E_{2,q}^0 \subset H^s; \quad s < n\left(\frac{1}{q} - \frac{1}{2}\right), \quad 2 < q \leq \infty. \tag{46}$$

Then, by the embedding between $E_{p,q}^s$ and $X_{p,q}^s$ (Proposition 3), we have

$$X_{2,q}^0 \subset E_{2,q}^0 \subset H^{n(1/q-1/2)-\varepsilon} \subset F_{2,q}^{n(1/q-1/2)-\varepsilon}. \tag{47}$$

This implies the theorem. $\square$

Theorem 11. Let $s > n(1/q - 1/2)$, $0 < q < 2$. Then, one has

$$\|f\|_{X_{2,q}^0} \leq C_s \left(1 + [\ln(1 + \|f\|_{H^s})]^{(1/q-1/2)} \|f\|_{H^{n(1/q-1/2)}}\right). \quad (48)$$

Proof. By the definition of $X_{p,q}^s$, we have

$$\begin{aligned} \|f\|_{X_{2,q}^0} &= \left\| \left(\sum_{k \in \mathbb{Z}^n} |\square_k f|^q \right)^{1/q} \right\|_{L^2} \\ &\leq \left\| \left(\sum_{|k| \leq T} |\square_k f|^q \right)^{1/q} \right\|_{L^2} \\ &\quad + \left\| \left(\sum_{|k| > T} |\square_k f|^q \right)^{1/q} \right\|_{L^2} = A + B. \end{aligned} \quad (49)$$

Then, by Appendix Theorem F, Theorem 7 and Hölder's inequality, the term A can be dominated by

$$\begin{aligned} A &= \left\| \left(\sum_{|k| \leq T} \left| F^{-1} \left(\frac{(1 + |k|)^2}{1 + |\xi|^2} \right)^{n(1/q-1/2)/2} \right. \right. \\ &\quad \times \left. \left. \left(\frac{1 + |\xi|^2}{(1 + |k|)^2} \right)^{n(1/q-1/2)/2} \sigma_k Ff \right|^q \right)^{1/q} \right\|_{L^2} \\ &\leq \left\| \left(\sum_{|k| \leq T} \left| F^{-1} \left(\frac{1 + |\xi|^2}{(1 + |k|)^2} \right)^{n(1/q-1/2)/2} \sigma_k Ff \right|^q \right)^{1/q} \right\|_{L^2} \\ &\leq \left(\sum_{|k| \leq T} (1 + |k|)^{-n} \right)^{(1/q-1/2)} \\ &\quad \times \left\| \left(\sum_{|k| \leq T} \left| F^{-1} (1 + |\xi|^2)^{n(1/q-1/2)/2} \sigma_k Ff \right|^2 \right)^{1/2} \right\|_{L^2} \\ &\leq \left(\sum_{0 \leq i \leq T} (1 + i)^{-1} \right)^{(1/q-1/2)} \|f\|_{H^{n(1/q-1/2)}} \\ &\lesssim [\ln(1 + T)]^{(1/q-1/2)} \|f\|_{H^{n(1/q-1/2)}}. \end{aligned} \quad (50)$$

On the other hand, by the similar discussion as the above, we have

$$\begin{aligned} B &\leq \left(\sum_{|k| > T} (1 + |k|)^{-2sq/(2-q)} \right)^{(1/q-1/2)} \\ &\quad \times \left\| \left(\sum_{|k| > T} \left| F^{-1} (1 + |\xi|^2)^{s/2} \sigma_k Ff \right|^2 \right)^{1/2} \right\|_{L^2} \end{aligned}$$

$$\begin{aligned} &\leq \left(\sum_{i > T} \sum_{|k|=i} (1 + i)^{-2sq/(2-q)} \right)^{(1/q-1/2)} \|f\|_{H^s} \\ &\leq \left(\sum_{i > T} (1 + i)^{-2sq/(2-q)+n-1} \right)^{(1/q-1/2)} \|f\|_{H^s} \\ &\leq C_s T^{-s+n(1/q-1/2)} \|f\|_{H^s}. \end{aligned} \quad (51)$$

Let

$$T = \max \left(1, \|f\|_{H^s}^{1/(s-n(1/q-1/2))} \right). \quad (52)$$

From the above discussion, we get the result. $\square$

Remark 12. By Theorem 9, we have

$$X_{p,1}^0(\mathbb{R}^n) \subset F_{p,1}^0(\mathbb{R}^n). \quad (53)$$

Then, by Theorem 11, we have

$$H^s(\mathbb{R}^n) \subset X_{2,1}^0(\mathbb{R}^n) \subset F_{2,1}^0(\mathbb{R}^n), \quad s > \frac{n}{2}. \quad (54)$$

So, $X_{2,1}^0$ is an intermediate space between H^s ($s > n/2$) and $F_{2,1}^0$.

3. Multilinear Estimate in $X_{p,q}^s$

It is well known that $B_{p,q}^s$ is a multiplication algebra if $s > n/p$, see [2]. The regularity indices, for which $X_{p,q}^s$ constitutes a multiplication algebra, are quite different from those of Besov space.

Theorem 13. Let $0 \leq s < \infty$, $0 < p \leq p_1$, $p_2 < \infty$, $0 < q \leq \infty$. If $1/p = 1/p_1 + 1/p_2$, then one has

$$\|fg\|_{X_{p,q}^s} \leq C 2^{Cs} \|f\|_{X_{p_1,q \wedge 1}^s} \|g\|_{X_{p_2,q \wedge 1}^s}, \quad (55)$$

where C is independent of s, q .

Proof. Recall that we denote $k \in \mathbb{Z}^n$ and $|k| = |k_1| + \dots + |k_n|$. We have

$$\|fg\|_{X_{p,q}^s} = \left\| \left(\sum_{k \in \mathbb{Z}^n} 2^{|k|sq} |\square_k(fg)|^q \right)^{1/q} \right\|_{L^p}. \quad (56)$$

By $\sum_{k \in \mathbb{Z}^n} \sigma_k = 1$, we can obtain the decomposition of fg as follows:

$$fg = \sum_{i,j \in \mathbb{Z}^n} (\square_i f) (\square_j g). \quad (57)$$

Then, we have

$$\square_k(fg) = \sum_{i,j \in \mathbb{Z}^n} \square_k(\square_i f) (\square_j g). \quad (58)$$

It is easy to see that $F(\square_i f \square_j g) = (\sigma_i \hat{f}) * (\sigma_j \hat{g})$ and

$$\text{supp}(\sigma_i \hat{f}) * (\sigma_j \hat{g}) \subset \{\xi : |\xi - i - j| \leq 2\sqrt{2n}\}. \quad (59)$$

Then, we have

$$\begin{aligned} \square_k(\square_i f \square_j g) &= \square_k \left(\sum_{i \in \mathbb{Z}^n} \square_i f \square_{j(i,k)} g \right), \\ |k - i - j(i,k)| &\leq C(n). \end{aligned} \quad (60)$$

Case 1 ($q \leq 1$). Based on the previous observations and Appendix Theorem F, we have

$$\begin{aligned} \|fg\|_{X_{p,q}^s} &= \left\| \left(\sum_{k \in \mathbb{Z}^n} 2^{|k|sq} |\square_k(fg)|^q \right)^{1/q} \right\|_{L^p} \\ &\leq \left\| \left(\sum_{k \in \mathbb{Z}^n} 2^{|k|sq} \left| \square_k \left(\sum_{i \in \mathbb{Z}^n} \square_i f \square_{j(i,k)} g \right) \right|^q \right)^{1/q} \right\|_{L^p} \\ &\leq \left\| \left(\sum_{k \in \mathbb{Z}^n} \sum_{i \in \mathbb{Z}^n} 2^{|k|sq} |\square_i f \square_{j(i,k)} g|^q \right)^{1/q} \right\|_{L^p} \\ &\leq \left\| \left(\sum_{k \in \mathbb{Z}^n} \sum_{i \in \mathbb{Z}^n} 2^{|k-i-j(i,k)|sq} \left| (2^{|i|s} \square_i f) (2^{|j(i,k)|s} \square_{j(i,k)} g) \right|^q \right)^{1/q} \right\|_{L^p} \\ &\leq \sum_{t=0}^{C(n)} \left\| \left(\sum_{k \in \mathbb{Z}^n} \sum_{i \in \mathbb{Z}^n, j=k-i \pm t} 2^{|k-i-j|sq} \right. \right. \\ &\quad \times \left. \left. \left| (2^{|i|s} \square_i f) (2^{|j|s} \square_j g) \right|^q \right)^{1/q} \right\|_{L^p} \\ &\leq C2^{Cs} \left\| \left(\sum_{i \in \mathbb{Z}^n} 2^{|i|sq} |\square_i f|^q \right)^{1/q} \left(\sum_{j \in \mathbb{Z}^n} 2^{|j|sq} |\square_j g|^q \right)^{1/q} \right\|_{L^p} \\ &\leq C2^{Cs} \|f\|_{X_{p_1,q}^s} \|g\|_{X_{p_2,q}^s}, \end{aligned} \quad (61)$$

where we used $|k-i-j(i,k)| < C(n)$ and the following Young's inequality:

$$\|a_i * b_j\|_{\ell^p} \leq \|a_i\|_{\ell^{p_1}} \|b_j\|_{\ell^{p_2}}, \quad 1 + \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}. \quad (62)$$

Case 2 ($q > 1$). By the definition of $X_{p,q}^s$ and Appendix Theorem F, we have

$$\begin{aligned} \|fg\|_{X_{p,q}^s} &= \left\| \left(\sum_{k \in \mathbb{Z}^n} 2^{|k|sq} |\square_k(fg)|^q \right)^{1/q} \right\|_{L^p} \\ &\leq \left\| \sum_{k \in \mathbb{Z}^n} 2^{|k|s} |\square_k(fg)| \right\|_{L^p}, \quad (\ell^1 \subset \ell^q, \text{ if } q \geq 1) \\ &\leq \left\| \sum_{k \in \mathbb{Z}^n} 2^{|k|s} \left| \square_k \left(\sum_{i \in \mathbb{Z}^n} \square_i f \square_{j(i,k)} g \right) \right| \right\|_{L^p} \\ &\leq \left\| \sum_{k \in \mathbb{Z}^n} 2^{|k|s} \left| \sum_{i \in \mathbb{Z}^n} \square_i f \square_{j(i,k)} g \right| \right\|_{L^p} \\ &\leq \sum_{t=0}^{C(n)} \left\| \sum_{k \in \mathbb{Z}^n} 2^{|k|s} \left| \sum_{i \in \mathbb{Z}^n, j=k-i \pm t} (\square_i f \square_j g) \right| \right\|_{L^p} \\ &\leq \sum_{t=0}^{C(n)} \left\| \sum_{k \in \mathbb{Z}^n} 2^{|k-i-j|s} \left| \sum_{i \in \mathbb{Z}^n, j=k-i \pm t} (2^{|i|s} \square_i f) (2^{|j|s} \square_j g) \right| \right\|_{L^p} \\ &\leq C2^{Cs} \left\| \sum_{i \in \mathbb{Z}^n} |(2^{|i|s} \square_i f)| \sum_{j \in \mathbb{Z}^n} |(2^{|j|s} \square_j g)| \right\|_{L^p} \\ &\leq C2^{Cs} \|f\|_{X_{p_1,1}^s} \|g\|_{X_{p_2,1}^s}. \end{aligned} \quad (63)$$

By the above discussion, we can obtain the claimed results. $\square$

Corollary 14. Let $0 \leq s < \infty$, $0 < p \leq p_i < \infty$, $i = 1, \dots, N$, $0 < q \leq \infty$. If $1/p = 1/p_1 + \dots + 1/p_N$, then one has

$$\left\| \prod_{i=1}^N u_i \right\|_{X_{p,q}^s} \leq C^N 2^{CNs} \prod_{i=1}^N \|u_i\|_{X_{p_i,q \wedge 1}^s}, \quad (64)$$

where C is independent of s, q .

Proof. Let $1/p = 1/p_1 + 1/p^*$, where $1/p^* = 1/p_2 + \dots + 1/p_N$. Then, we have

$$\left\| \prod_{i=1}^N u_i \right\|_{X_{p,q}^s} \leq C2^{Cs} \|u_1\|_{X_{p_1,q \wedge 1}^s} \left\| \prod_{i=2}^N u_i \right\|_{X_{p^*,q \wedge 1}^s}. \quad (65)$$

By induction, we can easily see that the claimed result holds. $\square$

4. Smooth Effects of the Schrödinger and Ginzburg-Landau Semigroups

In this section, we will discuss a kind of Strichartz's estimates. This kind of estimates is first introduced by Strichartz in [20], then developed by [18, 21, 22], and so forth.

Put

$$\begin{aligned} S(t) &= F^{-1} e^{-it|\xi|^2} F, \\ U(t) &= F^{-1} e^{-tP(\xi)} F, \quad P(\xi) = 1 + (a + i\alpha) |\xi|^2, \quad (66) \\ a &> 0, \quad \alpha \in \mathbb{R}. \end{aligned}$$

Our aim is to derive the estimates of $S(t)$, $U(t)$ in the spaces $X_{p,q}^s$.

Theorem 15. Assume $2 \leq p < \infty$, $p' \leq q \leq p$, $1/p + 1/p' = 1$, $0 \leq s < \infty$; then one has

$$\|S(t) f\|_{X_{p,q}^s} \leq (1 + |t|)^{-n(1/2-1/p)} \|f\|_{X_{p',q}^s}. \quad (67)$$

Proof

Step 1. By Appendix Theorem C, for $2 \leq p < \infty$, we have

$$\|\square_k S(t) f\|_{L^p} \leq (1 + |t|)^{-n(1/2-1/p)} \|\square_k f\|_{L^p}. \quad (68)$$

Thus, for $0 \leq s < \infty$, we have

$$\|S(t) f\|_{E_{p,q}^s} \leq (1 + |t|)^{-n(1/2-1/p)} \|f\|_{E_{p',q}^s}. \quad (69)$$

Step 2. By Proposition 3, for $p' \leq q \leq p$, we have

$$\begin{aligned} \|S(t) f\|_{X_{p,q}^s} &\leq \|S(t) f\|_{E_{p,q}^s} \\ &\leq (1 + t)^{-n(1/2-1/p)} \|f\|_{E_{p',q}^s} \\ &\leq (1 + t)^{-n(1/2-1/p)} \|f\|_{X_{p',q}^s}. \end{aligned} \quad (70)$$

This proves the theorem. $\square$

Theorem 16. Assume $r \geq 1$, $0 \leq s < \infty$, $p' \leq q \leq p$, $2 \leq p < \infty$, $Af = \int_0^t S(t - \tau) f(\tau) d\tau$; then one has

$$\|Af\|_{L'(-T,T;X_{p,q}^s)} \leq T^{2/r} \|f\|_{L'(-T,T;X_{p',q}^s)}. \quad (71)$$

Proof. By Theorem 15, we have

$$\begin{aligned} \|Af\|_{L'(-T,T;X_{p,q}^s)} &\leq \left\| \int_0^t \|S(t - \tau) f(\tau)\|_{X_{p,q}^s} d\tau \right\|_{L'(-T,T)} \\ &\leq \left\| \int_0^t (1 + |t - \tau|)^{-n(1/2-1/p)} \|f(\tau)\|_{X_{p',q}^s} d\tau \right\|_{L'(-T,T)} \\ &\leq \left\| \int_0^\infty \chi_{\tau \in [0,t]} \|f(\tau)\|_{X_{p',q}^s} d\tau \right\|_{L'(-T,T)} \\ &\leq T^{1/r} \left(\int_{-T}^T \|f(\tau)\|_{L'(-T,T;X_{p',q}^s)}^r dt \right)^{1/r} \\ &\leq T^{2/r} \|f\|_{L'(-T,T;X_{p',q}^s)}. \end{aligned} \quad (72)$$

This proves the theorem. $\square$

Theorem 17. Letting $0 \leq s < \infty$, one has

$$\|S(t) f\|_{X_{1,1}^s} \leq (1 + |t|)^{n/2} \|f\|_{X_{1,1}^s}, \quad (73)$$

which holds for all $f \in X_{1,1}^s$.

Proof. By [18] (Appendix Theorem H), we have

$$\|\square_k S(t) f\|_{L^p} \leq (1 + |t|)^{n(1/2-1/p)} \|\square_k f\|_{L^p}, \quad 1 \leq p \leq \infty. \quad (74)$$

Then, we have

$$\begin{aligned} \|S(t) f\|_{X_{1,1}^s} &= \left\| \sum_{k \in \mathbb{Z}^n} 2^{k|s|} \|\square_k S(t) f\|_{L^1} \right\|_{L^1} \\ &= \sum_{k \in \mathbb{Z}^n} 2^{k|s|} \|\square_k S(t) f\|_{L^1} \\ &\leq (1 + |t|)^{n/2} \sum_{k \in \mathbb{Z}^n} 2^{k|s|} \|\square_k f\|_{L^1} \leq (1 + |t|)^{n/2} \|f\|_{X_{1,1}^s}. \end{aligned} \quad (75)$$

This proves the theorem. $\square$

Theorem 18. Assume $r \geq 1$, $0 \leq s < \infty$, $Af = \int_0^t S(t - \tau) f(\tau) d\tau$; then one has

$$\|Af\|_{L'(-T,T;X_{1,1}^s)} \leq (1 + T)^{n/2} T^{2/r} \|f\|_{L'(-T,T;X_{1,1}^s)}. \quad (76)$$

Proof. By Theorem 17, we have

$$\begin{aligned} \|Af\|_{L'(-T,T;X_{1,1}^s)} &\leq \left\| \int_0^t \|S(t - \tau) f(\tau)\|_{X_{1,1}^s} d\tau \right\|_{L'(-T,T)} \\ &\leq \left\| \int_0^t (1 + |t - \tau|)^{n/2} \|f(\tau)\|_{X_{1,1}^s} d\tau \right\|_{L'(-T,T)} \\ &\leq (1 + T)^{n/2} \left\| \int_0^\infty \chi_{\tau \in [0,t]} \|f(\tau)\|_{X_{1,1}^s} d\tau \right\|_{L'(-T,T)} \\ &\leq (1 + T)^{n/2} T^{1/r} \left(\int_{-T}^T \|f(\tau)\|_{L'(-T,T;X_{1,1}^s)}^r dt \right)^{1/r} \\ &\leq (1 + T)^{n/2} T^{2/r} \|f\|_{L'(-T,T;X_{1,1}^s)}. \end{aligned} \quad (77)$$

This proves the theorem. $\square$

Theorem 19. Assume $r \geq k + 2$, $0 \leq s < \infty$. Assume also $f(u) = u|u|^k$, $k = 2m$, $m \in \mathbb{Z}^+$, $A(f) = \int_0^t S(t - \tau) f(u(\tau)) d\tau$. Then, one has

$$\|A(f)\|_{L'(-T,T;X_{1,1}^s)} \leq (1 + T)^{n/2} T^{1-k/r} \|u\|_{L'(-T,T;X_{1,1}^s)}^{k+1}. \quad (78)$$

Proof. By the definition, Theorems 18 and 13, we have

$$\begin{aligned}
 & \|A(f)\|_{L^r(-T, T; X_{1,1}^s)} \\
 &= \left\| \int_0^t S(t-\tau) f(u(\tau)) d\tau \right\|_{L^r(-T, T; X_{1,1}^s)} \\
 &\leq (1+T)^{n/2} T^{2/r} \|f\|_{L^{r'}(-T, T; X_{1,1}^s)} \\
 &\leq (1+T)^{n/2} T^{2/r} \left(\int_{-T}^T \|f(u(\tau))\|_{X_{1,1}^s}^{r'} d\tau \right)^{1/r'} \\
 &\leq (1+T)^{n/2} T^{2/r} \left(\int_{-T}^T \|u(\tau)\|_{X_{k+1,1}^s}^{(k+1)r'} d\tau \right)^{1/r'} \\
 &\leq (1+T)^{n/2} T^{2/r} \left(\int_{-T}^T \|u(\tau)\|_{X_{1,1}^s}^r d\tau \right)^{(k+1)/r} T^{(r-k-2)/r} \\
 &\leq (1+T)^{n/2} T^{1-k/r} \|u\|_{L^r(-T, T; X_{1,1}^s)}^{k+1}.
 \end{aligned} \quad (79)$$

This proves the theorem. $\square$

Theorem 20. Let $0 < p < \infty$, $0 < q \leq \infty$, $0 \leq s < \infty$; then

$$\|U(t)f\|_{X_{p,q}^s} \leq C \|f\|_{X_{p,q}^s}, \quad (80)$$

which holds for all $f \in X_{p,q}^s$.

Proof.

Step 1 (see [12]). Let $L > n(n/\min(1, p, q) - 1/2)$ and $L \in \mathbb{Z}^+$; then we have $\|\sigma_k e^{-tP(\xi)}\|_{H^L} \leq 1$. Actually, in view of the Leibnitz rule, we have

$$\begin{aligned}
 & |D^L(\sigma_k e^{-tP(\xi)})| \\
 &\leq C \sum_{L^1+L^2=L} |D^{L_1} \sigma_k D^{L_2} e^{-tP(\xi)}| \\
 &\leq C \sum_{0 \leq s \leq L} |D^s e^{-tP(\xi+k)}| \chi_{|\xi|_\infty \leq 1}.
 \end{aligned} \quad (81)$$

Let $\beta = (\beta_1, \dots, \beta_q)$, and let $\Lambda_q^s = \{\beta : \beta_1 + \dots + \beta_q = s, \beta_1, \dots, \beta_q \geq 1\}$. It is easy to see that

$$D^s e^{f(\eta)} = e^{f(\eta)} \left\{ \sum_{q=1}^s \sum_{\Lambda_q^s} C_\beta \prod_{i=1}^q D^{\beta_i} f(\xi) \right\}. \quad (82)$$

Let $f(\xi) = -tP(\xi)$; one easily sees that

$$|D^{\beta_i} f(\xi)| \leq Ct(1 + |\xi + k|^2), \quad k \in \mathbb{Z}. \quad (83)$$

Hence, we have

$$\begin{aligned}
 |D^s e^{f(\eta)}| &\leq e^{-ct(1+|\xi+k|^2)} \sum_{q=1}^s (t(1 + |\xi+k|^2))^q \\
 &\leq e^{-ct(1+|\xi+k|^2)}.
 \end{aligned} \quad (84)$$

Combining the previous discussion, we have

$$\|\sigma_k e^{-tP(\xi)}\|_{H^L} \leq C e^{-ct(1+|k|^2)}. \quad (85)$$

Step 2. By Step 1, we have $\sup_k \|\sigma_k e^{-tP(\xi)}\|_{H^L} \leq 1$. Then by Appendix Theorem F, we can obtain that

$$\|U(t)f\|_{X_{p,q}^s} \leq C \sup_k \|\sigma_k e^{-tP(\xi)}\|_{H^s} \|f\|_{X_{p,q}^s} \leq \|f\|_{X_{p,q}^s}. \quad (86)$$

This proves the theorem. $\square$

Theorem 21. Assume $1 \leq p < \infty$, $1 \leq q \leq \infty$, $r \geq 1$, $0 \leq s < \infty$, $Af = \int_0^t U(t-\tau)f(\tau)d\tau$; then one has

$$\|Af\|_{L^r(-T, T; X_{p,q}^s)} \leq T^{2/r} \|f\|_{L^{r'}(-T, T; X_{p,q}^s)}. \quad (87)$$

Proof. By Theorem 20, we have

$$\begin{aligned}
 & \|Af\|_{L^r(-T, T; X_{p,q}^s)} \\
 &\leq \left\| \int_0^t \|U(t-\tau)f(\tau)\|_{X_{p,q}^s} d\tau \right\|_{L^r(-T, T)} \\
 &\leq \left\| \int_0^t \|f(\tau)\|_{X_{p,q}^s} d\tau \right\|_{L^r(-T, T)} \\
 &\leq \left\| \int_0^\infty \chi_{\tau \in [0, t]} \|f(\tau)\|_{X_{p,q}^s} d\tau \right\|_{L^r(-T, T)} \\
 &\leq T^{2/r} \|f\|_{L^{r'}(-T, T; X_{p,q}^s)}.
 \end{aligned} \quad (88)$$

This proves the theorem. $\square$

Theorem 22. Assume $b > 0$, $\beta, \mu \in \mathbb{R}$, $f(u) = (b + i\beta)u|u|^k + \mu u$, $A(f) = \int_0^t U(t-\tau)f(u(\tau))d\tau$. Assume also $k = 2m$, $r \geq k+2$, $m \in \mathbb{Z}^+$, $1 \leq p < \infty$, $1 \leq q \leq \infty$. Then, one has

$$\|A(f)\|_{L^r(0, T; X_{p,q}^s)} \leq T^{1-k/r} \|u\|_{L^r(0, T; X_{p,1}^s)}^{k+1} + T \|u\|_{L^r(0, T; X_{p,q}^s)}. \quad (89)$$

Proof. By Theorems 20 and 21 and the multilinear estimate, we have

$$\begin{aligned}
 & \|A(f)\|_{L^r(-T, T; X_{p,q}^s)} \\
 &\leq T^{2/r} \|f\|_{L^{r'}(-T, T; X_{p,q}^s)} \\
 &= T^{2/r} \left(\int_{-T}^T \|f(u(\tau))\|_{X_{p,q}^s}^{r'} d\tau \right)^{1/r'}
 \end{aligned}$$

$$\begin{aligned}
&\leq T^{2/r} \left(\int_{-T}^T \|u|u|^k\|_{X_{p,q}^s}^{r'} d\tau \right)^{1/r'} \\
&\quad + T^{2/r} \left(\int_{-T}^T \|u(\tau)\|_{X_{p,q}^s}^{r'} d\tau \right)^{1/r'} \\
&\leq T^{2/r} \left(\int_{-T}^T \|u(\tau)\|_{X_{(k+1)p,q}^s}^{(k+1)r'} d\tau \right)^{1/r'} \\
&\quad + T^{2/r} \left(\int_{-T}^T \|u(\tau)\|_{X_{p,q}^s}^{r'} d\tau \right)^{1/r'} \\
&\leq T^{2/r} \left(\int_{-T}^T \|u(\tau)\|_{X_{p,1}^s}^r d\tau \right)^{(k+1)/r} T^{(r-k-2)/r} \\
&\quad + T \|u\|_{L^r(0,T;X_{p,q}^s)} \\
&\leq T^{1-k/r} \|u\|_{L^r(0,T;X_{p,1}^s)}^{k+1} + T \|u\|_{L^r(0,T;X_{p,q}^s)}. \tag{90}
\end{aligned}$$

This proves the theorem. $\square$

5. Well-Posedness of Nonlinear Schrödinger Equation and Ginzburg-Landau Equation

In this section, we first study the well-posedness of the following Schrödinger equation:

$$iu_t + \Delta u = f(u); \quad u(0, x) = u_0(x). \tag{91}$$

The solution, $u(x, t)$, of the above Cauchy problem is given by:

$$u(t) = S(t) u_0 - i \int_0^t S(t-\tau) f(u(\tau)) d\tau, \tag{92}$$

where $S(t) = F^{-1} e^{it|\xi|^2} F$. Then, we study the well-posedness of the following complex Ginzburg-Landau equation:

$$\begin{aligned}
&u_t - (a + i\alpha) \Delta u + (b + i\beta) |u|^k u \\
&\quad + (\mu + 1) u = 0, \quad u(0, x) = u_0(x). \tag{93}
\end{aligned}$$

The solution, $u(x, t)$, of the above Cauchy problem is given by

$$u(t) = U(t) u_0 - i \int_0^t U(t-\tau) f(u(\tau)) d\tau, \tag{94}$$

where $a, b > 0$, $\alpha, \beta, \mu \in \mathbb{R}$, $U(t) = F^{-1} e^{-itP(\xi)} F$, and $P(\xi) = 1 + (a + i\alpha)|\xi|^2$.

6. Proof of Theorem 1

Proof. Fix $T > 0$, $\delta > 0$ to be chosen later. Let

$$\mathcal{D} = \left\{ u \in L^r(0, T; X_{1,1}^s) : \|u\|_{L^r(0,T;X_{1,1}^s)} < \delta \right\} \tag{95}$$

be equipped with the distance

$$d(u, v) = \|u - v\|_{L^r(0,T;X_{1,1}^s)}. \tag{96}$$

It is easy to know that $(\mathcal{D}, d)$ is a complete metric space. Now we consider the following map:

$$\mathfrak{F} : u(t) \longrightarrow S(t) u_0 - i \int_0^t S(t-\tau) f(u(\tau)) d\tau. \tag{97}$$

We will prove that there exists $T, \delta > 0$, such that $\mathfrak{F} : (\mathcal{D}, d) \rightarrow (\mathcal{D}, d)$ is a strict contraction map.

By the nonlinear term estimate and the semigroup estimate,

$$\begin{aligned}
\| |u|^k u \|_{X_{1,1}^s} &\leq \|u\|_{X_{(k+1),1}^s}^{k+1}, \\
\|S(t) u_0\|_{X_{1,1}^s} &\leq (1 + |t|)^{n/2} \|u_0\|_{X_{1,1}^s}, \tag{98}
\end{aligned}$$

$$\|A(f)\|_{L^r(0,T;X_{1,1}^s)} \leq (1 + T)^{n/2} T^{1-k/r} \|u\|_{L^r(0,T;X_{1,1}^s)}^{k+1},$$

we have

$$\begin{aligned}
\|\mathfrak{F}u\|_{L^r(0,T;X_{1,1}^s)} &\leq (1 + T)^{n/2} T^{1/r} \|u_0\|_{X_{1,1}^s} \\
&\quad + (1 + T)^{n/2} T^{1-k/r} \|u\|_{L^r(0,T;X_{1,1}^s)}^{k+1}. \tag{99}
\end{aligned}$$

Then, we let

$$\delta = 2C \|u_0\|_{X_{1,1}^s}, \quad T < 1. \tag{100}$$

Then, we choose T such that

$$2CT^{(1-k/r)} \delta^k \leq \frac{1}{2}. \tag{101}$$

Then, we have

$$\|\mathfrak{F}u\|_{L^r(0,T;X_{1,1}^s)} < \delta. \tag{102}$$

Moreover, we have that

$$\|\mathfrak{F}u - \mathfrak{F}v\|_{L^r(0,T;X_{1,1}^s)} \leq \frac{1}{2\|u - v\|_{L^r(0,T;X_{1,1}^s)}}. \tag{103}$$

In this way, we prove that $\mathfrak{F} : (\mathcal{D}, d) \rightarrow (\mathcal{D}, d)$ is a strict contraction map. By Banach's fixed-point theorem, there exists a unique solution $u \in \mathcal{D}$ that satisfies the conditions in the theorem. Using standard argument, we can extend the solution, considering the map

$$\mathfrak{F} : u(t) \longrightarrow S(t - T) u_T - i \int_T^t S(t - \tau) f(u(\tau)) d\tau, \tag{104}$$

where $u(T)$ is the solution that we have chosen. We claim $u(T) \in X_{1,1}^s$; one can refer to [23] for the details. In this way, we can find a maximum $T^* > 0$ which satisfies the conditions in the theorem. $\square$

7. Proof of Theorem 2

Proof. Fix $T > 0$, $\delta > 0$ to be chosen later. Let

$$\mathcal{D} = \left\{ u \in L^r(0, T; X_{p,1}^s) : \|u\|_{L^r(0,T;X_{p,1}^s)} < \delta \right\} \quad (105)$$

be equipped with the distance

$$d(u, v) = \|u - v\|_{L^r(0,T;X_{p,1}^s)}. \quad (106)$$

It is easy to know that $(\mathcal{D}, d)$ is a complete metric space. Now, we consider the following map:

$$\mathfrak{F} : u(t) \longrightarrow U(t)u_0 - i \int_0^t U(t-\tau) f(u(\tau)) d\tau. \quad (107)$$

We shall prove that there exists $T, \delta > 0$, such that $\mathfrak{F} : (\mathcal{D}, d) \rightarrow (\mathcal{D}, d)$ is a strict contraction map.

By the nonlinear term estimate and the semigroup estimate,

$$\begin{aligned} \| |u|^k u \|_{X_{p,1}^s} &\leq \|u\|_{X_{p(k+1),1}^{s+1}}, \\ \|U(t)u_0\|_{X_{p,1}^s} &\leq \|u_0\|_{X_{p,1}^s}, \\ \|A(f)\|_{L^r(0,T;X_{p,1}^s)} &\leq T^{1-k/r} \|u\|_{L^r(0,T;X_{p,1}^s)}^{k+1} \\ &\quad + T \|u\|_{L^r(0,T;X_{p,1}^s)}, \end{aligned} \quad (108)$$

we have

$$\begin{aligned} \|\mathfrak{F}u\|_{L^r(0,T;X_{p,1}^s)} &\leq T^{1/r} \|u_0\|_{X_{p,1}^s} + T^{1-k/r} \|u\|_{L^r(0,T;X_{p,1}^s)}^{k+1} \\ &\quad + T \|u\|_{L^r(0,T;X_{p,1}^s)}. \end{aligned} \quad (109)$$

Then, we let

$$\delta = 3C \|u_0\|_{X_{p,1}^s}, \quad T < \frac{1}{3}. \quad (110)$$

Then, we choose T such that

$$CT^{(1-k/r)} \delta^k \leq \frac{1}{3}. \quad (111)$$

Then, we have

$$\|\mathfrak{F}u\|_{L^r(0,T;X_{p,1}^s)} < \delta. \quad (112)$$

Moreover, we have that

$$\|\mathfrak{F}u - \mathfrak{F}v\|_{L^r(0,T;X_{p,1}^s)} \leq \frac{2}{3\|u - v\|_{L^r(0,T;X_{p,1}^s)}}. \quad (113)$$

In this way, we prove that $\mathfrak{F} : (\mathcal{D}, d) \rightarrow (\mathcal{D}, d)$ is a strict contraction map. By Banach's fixed-point theorem, there exists a unique solution $u \in \mathcal{D}$ that satisfies the conditions

in the theorem. Using standard argument, we can extend the solution considering the map

$$\mathfrak{F} : u(t) \longrightarrow U(t-T)u_T - i \int_T^t U(t-\tau) f(u(\tau)) d\tau, \quad (114)$$

where $u(T)$ is the solution that we have chosen. We claim $u(T) \in X_{p,1}^s$; one can refer to [23] for the details. In this way, we can find a maximum $T^* > 0$ which satisfies the conditions in the theorem. $\square$

Appendix

Theorem A. We have

$$\begin{aligned} H^s(\mathbb{R}^n) &\subset E_{2,q}^0(\mathbb{R}^n), \quad s > n\left(\frac{1}{q} - \frac{1}{2}\right), \quad 0 < q < 2, \\ L^2(\mathbb{R}^n) &= E_{2,2}^0(\mathbb{R}^n), \end{aligned} \quad (A.1)$$

$$E_{2,q}^0(\mathbb{R}^n) \subset H^s(\mathbb{R}^n), \quad s < n\left(\frac{1}{q} - \frac{1}{2}\right), \quad 2 < q \leq \infty.$$

Proof. One can refer to [12]. $\square$

Theorem B. Assume $0 < p \leq q \leq \infty$. Let $\Omega \subset \mathbb{R}^n$ be compact set, $\text{diam } \Omega < 2R$. Then, there exists $C(p, q, R) > 0$, such that

$$\|f\|_{L^q} \leq C \|f\|_{L^p}, \quad \forall f \in L_{\Omega}^p, \quad (A.2)$$

where $L_{\Omega}^p = \{f \in L^p : \text{supp } \hat{f} \subset \Omega\}$.

Proof. One can refer to [15, 18]. $\square$

Theorem C. Assume $s \in \mathbb{R}$, $2 \leq p < \infty$, $1/p + 1/p' = 1$. Then we have

$$\begin{aligned} \|\square_k S(t) f\|_{L^p} &\leq (1 + |t|)^{-n(1/2-1/p)} \|\square_k f\|_{L^{p'}} \\ \|S(t) f\|_{M_{p,q}^s} &\leq (1 + |t|)^{-n(1/2-1/p)} \|f\|_{M_{p',q}^s}. \end{aligned} \quad (A.3)$$

Proof. One can refer to [18]. $\square$

Theorem D (infinite smoothness). Let $0 < \lambda < \infty$, $0 < p, q \leq \infty$. Then, one has

$$E_{p,q}^{\lambda}(\mathbb{R}^n) \subset B_{p,q}^{\{s_k\}}(\mathbb{R}^n) \subset C^{\infty}(\mathbb{R}^n), \quad (A.4)$$

if $s_k \leq \lambda 2^k / 3k$, $k \in \mathbb{N}$.

Proof. One can refer to [12]. $\square$

Theorem E (embedding). Assume $s_1, s_2 \in \mathbb{R}$, $0 < p_1, p_2, q_1, q_2 \leq \infty$.

- (a) If $s_2 \leq s_1$, $p_1 \leq p_2$, $q_1 \leq q_2$, then $M_{p_1,q_1}^{s_1} \subset M_{p_2,q_2}^{s_2}$.
- (b) If $q_2 < q_1$, $s_1 - s_2 > n/q_2 - n/q_1$, then $M_{p,q_1}^{s_1} \subset M_{p,q_2}^{s_2}$.
- (c) If $0 \leq s < \infty$, $0 < p_1 \leq p_2 \leq \infty$, $0 < q_1 \leq q_2 \leq \infty$, then $E_{p_1,q_1}^s \subset E_{p_2,q_2}^s$.

Proof. One can refer to [12, 18, 24]. $\square$

Theorem F. Assume $0 < p < \infty$, $0 < q \leq \infty$; $\Omega = \{\Omega_k\}_{k=0}^\infty$ is a sequence with compact support in $\mathbb{R}^n$; let $d_k > 0$ be the diameter of Ω_k . If $x > n(n/\min(1, p, q) - 1/2)$, then there exists a constant C such that

$$\|F^{-1}M_k F f_k\|_{L^p(\ell^q)} \leq C \sup_i \|M_i(d_i \cdot)\|_{H^x} \|f_k\|_{L^p(\ell^q)}, \quad (\text{A.5})$$

where $\{f_k\}_{k=0}^\infty \in L^p_\Omega(\ell^q)$, $\{M_k(x)\}_{k=0}^\infty \subset H^x$.

Proof. One can refer to [11]. Note that $L^p_\Omega(\ell^q) = \{f \mid f = \{f_k\}_{k=0}^\infty \subset S', \text{ supp } F f_k \subset \Omega_k \text{ if } k = 0, 1, 2, \dots \text{ and } \|f_k\|_{L^p(\ell^q)} < \infty\}$. $\square$

Theorem G. Let $0 < p, q \leq \infty$ and $(X, \mu), (Y, \nu)$ be two measure space. Let T be a positive linear operator mapping $L^p(X)$ into $L^q(Y)$ (resp., into $L^{q,\infty}(Y)$) with norm A . Let B be a Banach space. Then, T has a B -valued extension $\tilde{T}$ that maps $L^p(X, B)$ into $L^q(Y, B)$ (resp., into $L^{q,\infty}(Y, B)$) with the same norm.

Proof. One can refer to [25]. $\square$

Theorem H. Let $s \in \mathbb{R}$, $1 \leq p \leq \infty$, $0 < q < \infty$; then one has

$$\begin{aligned} \|\square_k S(t) f\|_{L^p} &\leq (1 + |t|)^{n(1/2 - 1/p)} \|\square_k f\|_{L^p} \\ \|S(t) f\|_{M_{p,q}^s} &\leq (1 + |t|)^{n(1/2 - 1/p)} \|f\|_{M_{p,q}^s}. \end{aligned} \quad (\text{A.6})$$

Proof. One can refer to [18]. $\square$

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