Research Article

Coefficient Estimates for New Subclasses of Analytic and Bi-Univalent Functions Defined by Al-Oboudi Differential Operator

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We introduce and investigate two new subclasses $\mathcal{N}_{\Sigma}^{\delta, \mu}(n, \alpha, \lambda)$ and $\mathcal{N}_{\Sigma}^{\delta, \mu}(n, \beta, \lambda)$ of analytic and bi-univalent functions in the open unit disk $\mathbb{U}$. For functions belonging to these classes, we obtain estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$.

1. Introduction

Let $\mathbb{R} = (-\infty, \infty)$ be the set of real numbers, $\mathbb{C}$ the set of complex numbers, and

$$\mathbb{N} := \{1, 2, 3, \dots\} = \mathbb{N}_0 \setminus \{0\} \quad (1)$$

the set of positive integers.

Let $\mathcal{A}$ denote the class of all functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (2)$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C}, |z| < 1\}. \quad (3)$$

We also denote by $\mathcal{S}$ the class of all functions in the normalized analytic function class $\mathcal{A}$ which are univalent in $\mathbb{U}$.

For $f \in \mathcal{A}$, Al-Oboudi [1] introduced the following operator:

$$D_{\delta}^0 f(z) = f(z), \quad (4)$$

$$\begin{aligned} D_{\delta}^1 f(z) &= (1 - \delta) f(z) + \delta z f'(z) \\ &=: D_{\delta} f(z) \quad (\delta \geq 0) \end{aligned} \quad (5)$$

$$D_{\delta}^n f(z) = D_{\delta} (D_{\delta}^{n-1} f(z)) \quad (n \in \mathbb{N}). \quad (6)$$

If f is given by (2), then from (5) and (6) we see that

$$D_{\delta}^n f(z) = z + \sum_{k=2}^{\infty} [1 + (k-1)\delta]^n a_k z^k \quad (n \in \mathbb{N}_0), \quad (7)$$

with $D_{\delta}^n f(0) = 0$. When $\delta = 1$, we get Sălăgean's differential operator [2].

Since univalent functions are one-to-one, they are invertible and the inverse functions need not be defined on the entire unit disk $\mathbb{U}$. In fact, the Koebe one-quarter theorem [3] ensures that the image of $\mathbb{U}$ under every univalent function $f \in \mathcal{S}$ contains a disk of radius $1/4$. Thus, every function $f \in \mathcal{A}$ has an inverse f^{-1} , which is defined by

$$\begin{aligned} f^{-1}(f(z)) &= z \quad (z \in \mathbb{U}), \\ f(f^{-1}(w)) &= w \quad \left(|w| < r_0(f); r_0(f) \geq \frac{1}{4}\right). \end{aligned} \quad (8)$$

In fact, the inverse function f^{-1} is given by

$$\begin{aligned} f^{-1}(w) &= w - a_2 w^2 + (2a_2^2 - a_3) w^3 \\ &\quad - (5a_2^3 - 5a_2 a_3 + a_4) w^4 + \dots \end{aligned} \quad (9)$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. Let Σ denote the class of bi-univalent functions in $\mathbb{U}$ given by (2). For a brief history and

interesting examples of functions in the class Σ , see [4] (see also [5, 6]). In fact, the aforecited work of Srivastava et al. [4] essentially revived the investigation of various subclasses of the bi-univalent function class Σ in recent years; it was followed by such works as those by Frasin and Aouf [7], Çağlar et al. [8], Porwal and Darus [9], and others (see, for example, [10–13]).

The object of the present paper is to introduce two new subclasses of the function class Σ and find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in these new subclasses of the function class Σ .

Firstly, in order to derive our main results, we need the following lemma.

Lemma 1 (see [14]). *If $p \in \mathcal{P}$, then $|c_k| \leq 2$ for each k , where $\mathcal{P}$ is the family of all functions p analytic in $\mathbb{U}$ for which*

$$\Re(p(z)) > 0, \quad p(z) = 1 + c_1 z + c_2 z^2 + \dots \quad (10)$$

for $z \in \mathbb{U}$.

2. Coefficient Bounds for the Function Class

$$\mathcal{N}_{\Sigma}^{\delta, \mu}(n, \alpha, \lambda)$$

Definition 2. A function $f(z)$ given by (2) is said to be in the class $\mathcal{N}_{\Sigma}^{\delta, \mu}(n, \alpha, \lambda)$ if the following conditions are satisfied:

$$\begin{aligned} f &\in \Sigma, \\ \left| \arg \left((1 - \lambda) \left(\frac{D_{\delta}^n f(z)}{z} \right)^{\mu} \right. \right. \\ &\quad \left. \left. + \lambda (D_{\delta}^n f(z))' \left(\frac{D_{\delta}^n f(z)}{z} \right)^{\mu-1} \right) \right| < \frac{\alpha\pi}{2} \\ (0 < \alpha \leq 1, \lambda \geq 1, \mu \geq 0, \\ &\quad \delta \geq 0, n \in \mathbb{N}_0, z \in \mathbb{U}), \end{aligned} \quad (11)$$

$$\begin{aligned} \left| \arg \left((1 - \lambda) \left(\frac{D_{\delta}^n g(w)}{w} \right)^{\mu} \right. \right. \\ &\quad \left. \left. + \lambda (D_{\delta}^n g(w))' \left(\frac{D_{\delta}^n g(w)}{w} \right)^{\mu-1} \right) \right| < \frac{\alpha\pi}{2} \\ (0 < \alpha \leq 1, \lambda \geq 1, \mu \geq 0, \\ &\quad \delta \geq 0, n \in \mathbb{N}_0, w \in \mathbb{U}), \end{aligned}$$

where the function g is given by

$$\begin{aligned} g(w) &= w - a_2 w^2 + (2a_2^2 - a_3) w^3 \\ &\quad - (5a_2^3 - 5a_2 a_3 + a_4) w^4 + \dots \end{aligned} \quad (12)$$

and D_{δ}^n is the Al-Oboudi differential operator.

Remark 3. In Definition 2, if we choose

(i) $n = 0$, then we have the class

$$\mathcal{N}_{\Sigma}^{\delta, \mu}(0, \alpha, \lambda) = \mathcal{N}_{\Sigma}^{\mu}(\alpha, \lambda) \quad (13)$$

introduced by Çağlar et al. [8];

(ii) $n = 0$ and $\mu = 1$, then we have the class

$$\mathcal{N}_{\Sigma}^{\delta, 1}(0, \alpha, \lambda) = \mathcal{B}_{\Sigma}(\alpha, \lambda) \quad (14)$$

introduced by Frasin and Aouf [7];

(iii) $n = 0$, $\mu = 1$, and $\lambda = 1$, then we have the class

$$\mathcal{N}_{\Sigma}^{\delta, 1}(0, \alpha, 1) = \mathcal{H}_{\Sigma}^{\alpha} \quad (15)$$

introduced by Srivastava et al. [4];

(iv) $n = 0$, $\mu = 0$, and $\lambda = 1$, then we have the class

$$\mathcal{N}_{\Sigma}^{\delta, 0}(0, \alpha, 1) = \mathcal{S}_{\Sigma}^*[\alpha] \quad (16)$$

of strongly bi-starlike functions of order α , introduced by Brannan and Taha [5, 6];

(v) $\delta = 1$ and $\mu = 1$, then we have the class

$$\mathcal{N}_{\Sigma}^{1, 1}(n, \alpha, \lambda) = \mathcal{B}_{\Sigma}(n, \alpha, \lambda) \quad (17)$$

introduced by Porwal and Darus [9].

Theorem 4. *Let the function $f(z)$ given by the Taylor-Maclaurin series expansion (2) be in the function class*

$$\mathcal{N}_{\Sigma}^{\delta, \mu}(n, \alpha, \lambda) \quad (0 < \alpha \leq 1, \lambda \geq 1, \mu \geq 0, \delta \geq 0, n \in \mathbb{N}_0). \quad (18)$$

Then,

$$\begin{aligned} |a_2| &\leq (2\alpha) \\ &\quad \times \left(\left((1 + \delta)^{2n} (\lambda + \mu)^2 \right. \right. \\ &\quad \left. \left. + \alpha \left[2(1 + 2\delta)^n (2\lambda + \mu) \right. \right. \right. \\ &\quad \left. \left. \left. - (1 + \delta)^{2n} (\lambda^2 + 2\lambda + \mu) \right] \right)^{1/2} \right)^{-1} \end{aligned} \quad (19)$$

$$|a_3| \leq \frac{4\alpha^2}{(1 + \delta)^{2n} (\lambda + \mu)^2} + \frac{2\alpha}{(1 + 2\delta)^n (2\lambda + \mu)}. \quad (20)$$

Proof. First of all, it follows from conditions (11) that

$$(1 - \lambda) \left(\frac{D_{\delta}^n f(z)}{z} \right)^{\mu} + \lambda (D_{\delta}^n f(z))' \left(\frac{D_{\delta}^n f(z)}{z} \right)^{\mu-1}$$

$$\begin{aligned}
&= [p(z)]^\alpha \quad (z \in \mathbb{U}), \\
(1-\lambda) \left(\frac{D_\delta^n g(w)}{w} \right)^\mu + \lambda (D_\delta^n g(w))' \left(\frac{D_\delta^n g(w)}{w} \right)^{\mu-1} \\
&= [q(w)]^\alpha \quad (w \in \mathbb{U}),
\end{aligned} \tag{21}$$

respectively, where

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots, \tag{22}$$

$$q(w) = 1 + q_1 w + q_2 w^2 + \cdots$$

in $\mathcal{P}$. Now, upon equating the coefficients in (21), we get

$$(1+\delta)^n (\lambda + \mu) a_2 = \alpha p_1, \tag{23}$$

$$\begin{aligned}
(1+2\delta)^n (2\lambda + \mu) a_3 + (1+\delta)^{2n} (\mu-1) \left(\lambda + \frac{\mu}{2} \right) a_2^2 \\
= \alpha p_2 + \frac{\alpha(\alpha-1)}{2} p_1^2,
\end{aligned} \tag{24}$$

$$-(1+\delta)^n (\lambda + \mu) a_2 = \alpha q_1, \tag{25}$$

$$\begin{aligned}
-(1+2\delta)^n (2\lambda + \mu) a_3 + [4(1+2\delta)^n + (1+\delta)^{2n} (\mu-1)] \\
\times \left(\lambda + \frac{\mu}{2} \right) a_2^2 = \alpha q_2 + \frac{\alpha(\alpha-1)}{2} q_1^2.
\end{aligned} \tag{26}$$

From (23) and (25), we obtain

$$p_1 = -q_1, \tag{27}$$

$$2(1+\delta)^{2n} (\lambda + \mu)^2 a_2^2 = \alpha^2 (p_1^2 + q_1^2). \tag{28}$$

Also, from (24), (26), and (28), we find that

$$\begin{aligned}
[2(1+2\delta)^n + (1+\delta)^{2n} (\mu-1)] (2\lambda + \mu) a_2^2 \\
= \alpha (p_2 + q_2) + \frac{\alpha(\alpha-1)}{2} (p_1^2 + q_1^2) \\
= \alpha (p_2 + q_2) + \frac{\alpha-1}{\alpha} (1+\delta)^{2n} (\lambda + \mu)^2 a_2^2.
\end{aligned} \tag{29}$$

Therefore, we obtain

$$\begin{aligned}
a_2^2 &= (\alpha^2 (p_2 + q_2)) \\
&\times (2\alpha(1+2\delta)^n (2\lambda + \mu) \\
&+ (1+\delta)^{2n} [(\lambda + \mu)^2 - \alpha(\lambda^2 + 2\lambda + \mu)])^{-1}.
\end{aligned} \tag{30}$$

Applying Lemma 1 for the aforementioned equality, we get the desired estimate on the coefficient $|a_2|$ as asserted in (19).

Next, in order to find the bound on the coefficient $|a_3|$, we subtract (26) from (24). We thus get

$$\begin{aligned}
2(1+2\delta)^n (2\lambda + \mu) a_3 - 2(1+2\delta)^n (2\lambda + \mu) a_2^2 \\
= \alpha (p_2 - q_2) + \frac{\alpha(\alpha-1)}{2} (p_1^2 - q_1^2).
\end{aligned} \tag{31}$$

It follows from (27), (28), and (31) that

$$a_3 = \frac{\alpha^2 (p_1^2 + q_1^2)}{2(1+\delta)^{2n} (\lambda + \mu)^2} + \frac{\alpha (p_2 - q_2)}{2(1+2\delta)^n (2\lambda + \mu)}. \tag{32}$$

Applying Lemma 1 for the previous equality, we get the desired estimate on the coefficient $|a_3|$ as asserted in (20). $\square$

If we take $n = 0$ in Theorem 4, then we have the following corollary.

Corollary 5 (see [8]). *Let the function $f(z)$ given by the Taylor-Maclaurin series expansion (2) be in the class $\mathcal{N}_\Sigma^\mu(\alpha, \lambda)$ ($0 < \alpha \leq 1, \lambda \geq 1, \mu \geq 0$). Then,*

$$|a_2| \leq \frac{2\alpha}{\sqrt{(\lambda + \mu)^2 + \alpha(\mu + 2\lambda - \lambda^2)}}, \tag{33}$$

$$|a_3| \leq \frac{4\alpha^2}{(\lambda + \mu)^2} + \frac{2\alpha}{2\lambda + \mu}.$$

If we take $n = 0$ and $\mu = 1$ in Theorem 4, then we have the following corollary.

Corollary 6 (see [7]). *Let the function $f(z)$ given by the Taylor-Maclaurin series expansion (2) be in the class $\mathcal{B}_\Sigma(\alpha, \lambda)$ ($0 < \alpha \leq 1, \lambda \geq 1$). Then*

$$|a_2| \leq \frac{2\alpha}{\sqrt{(\lambda + 1)^2 + \alpha(1 + 2\lambda - \lambda^2)}}, \tag{34}$$

$$|a_3| \leq \frac{4\alpha^2}{(\lambda + 1)^2} + \frac{2\alpha}{2\lambda + 1}.$$

If we take $n = 0, \mu = 1$ and $\lambda = 1$ in Theorem 4, then we have the following corollary.

Corollary 7 (see [4]). *Let the function $f(z)$ given by the Taylor-Maclaurin series expansion (2) be in the class $\mathcal{H}_\Sigma^\alpha$ ($0 < \alpha \leq 1$). Then,*

$$\begin{aligned}
|a_2| &\leq \alpha \sqrt{\frac{2}{\alpha + 2}}, \\
|a_3| &\leq \frac{\alpha(3\alpha + 2)}{3}.
\end{aligned} \tag{35}$$

If we take $n = 0, \mu = 0$, and $\lambda = 1$ in Theorem 4, then we have the following corollary.

Corollary 8. *Let the function $f(z)$ given by the Taylor-Maclaurin series expansion (2) be in the class $\mathcal{S}_\Sigma^*[\alpha]$ ($0 < \alpha \leq 1$). Then,*

$$|a_2| \leq \frac{2\alpha}{\sqrt{1 + \alpha}}, \tag{36}$$

$$|a_3| \leq 4\alpha^2 + \alpha.$$

If we take $\delta = 1$ and $\mu = 1$ in Theorem 4, then we have the following corollary.

Corollary 9 (see [9]). Let the function $f(z)$ given by the Taylor-Maclaurin series expansion (2) be in the function class $\mathcal{B}_\Sigma(n, \alpha, \lambda)$ ($0 < \alpha \leq 1, \lambda \geq 1, n \in \mathbb{N}_0$). Then,

$$|a_2| \leq \frac{2\alpha}{\sqrt{4^n(\lambda+1)^2 + \alpha[2 \cdot 3^n(2\lambda+1) - 4^n(\lambda+1)^2]}}, \quad (37)$$

$$|a_3| \leq \frac{4\alpha^2}{4^n(\lambda+1)^2} + \frac{2\alpha}{3^n(2\lambda+1)}.$$

3. Coefficient Bounds for the Function Class

$$\mathcal{N}_\Sigma^{\delta, \mu}(n, \beta, \lambda)$$

Definition 10. A function $f(z)$ given by (2) is said to be in the class $\mathcal{N}_\Sigma^{\delta, \mu}(n, \beta, \lambda)$ if the following conditions are satisfied:

$$\begin{aligned} f &\in \Sigma, \\ \Re \left\{ (1-\lambda) \left(\frac{D_\delta^n f(z)}{z} \right)^\mu \right. \\ &\quad \left. + \lambda (D_\delta^n f(z))' \left(\frac{D_\delta^n f(z)}{z} \right)^{\mu-1} \right\} > \beta \\ (0 &\leq \beta < 1, \lambda \geq 1, \mu \geq 0, \\ \delta &\geq 0, n \in \mathbb{N}_0, z \in \mathbb{U}), \quad (38) \\ \Re \left\{ (1-\lambda) \left(\frac{D_\delta^n g(w)}{w} \right)^\mu \right. \\ &\quad \left. + \lambda (D_\delta^n g(w))' \left(\frac{D_\delta^n g(w)}{w} \right)^{\mu-1} \right\} > \beta \\ (0 &\leq \beta < 1, \lambda \geq 1, \mu \geq 0, \\ \delta &\geq 0, n \in \mathbb{N}_0, w \in \mathbb{U}), \end{aligned}$$

where the function g is defined by (12) and D_δ^n is the Al-Ouboudi differential operator.

Remark 11. In Definition 10, if we choose

(i) $n = 0$, then we have the class

$$\mathcal{N}_\Sigma^{\delta, \mu}(0, \beta, \lambda) = \mathcal{N}_\Sigma^\mu(\beta, \lambda) \quad (39)$$

introduced by Çağlar et al. [8];

(ii) $n = 0$ and $\mu = 1$, then we have the class

$$\mathcal{N}_\Sigma^{\delta, 1}(0, \beta, \lambda) = \mathcal{B}_\Sigma(\beta, \lambda) \quad (40)$$

introduced by Frasin and Aouf [7];

(iii) $n = 0, \mu = 1$, and $\lambda = 1$, then we have the class

$$\mathcal{N}_\Sigma^{\delta, 1}(0, \beta, 1) = \mathcal{H}_\Sigma(\beta) \quad (41)$$

introduced by Srivastava et al. [4];

(iv) $n = 0, \mu = 0$, and $\lambda = 1$, then we have the class

$$\mathcal{N}_\Sigma^{\delta, 0}(0, \beta, 1) = \mathcal{S}_\Sigma^*(\beta) \quad (42)$$

of bi-starlike functions of order β , introduced by Brannan and Taha [5, 6];

(v) $\delta = 1$ and $\mu = 1$, then we have the class

$$\mathcal{N}_\Sigma^{1, 1}(n, \beta, \lambda) = \mathcal{H}_\Sigma(n, \beta, \lambda) \quad (43)$$

introduced by Porwal and Darus [9].

Theorem 12. Let the function $f(z)$ given by the Taylor-Maclaurin series expansion (2) be in the function class

$$\mathcal{N}_\Sigma^{\delta, \mu}(n, \beta, \lambda) \quad (0 < \alpha \leq 1, \lambda \geq 1, \mu \geq 0, \delta \geq 0, n \in \mathbb{N}_0). \quad (44)$$

Then,

$$|a_2| \leq \min \left\{ \frac{2(1-\beta)}{(1+\delta)^n(\lambda+\mu)}, \right. \\ \left. \begin{aligned} &((4(1-\beta)) \\ &\times (|2(1+2\delta)^n + (1+\delta)^{2n}(\mu-1)| \\ &\times (2\lambda+\mu)^{-1})^{1/2} \end{aligned} \right\} \quad (45)$$

$$|a_3| \leq \min \left\{ \frac{4(1-\beta)^2}{(1+\delta)^{2n}(\lambda+\mu)^2} + \frac{2(1-\beta)}{(1+2\delta)^n(2\lambda+\mu)}, \right. \\ \left. \begin{aligned} &((1-\beta)[|4(1+2\delta)^n + (1+\delta)^{2n}(\mu-1)| \\ &+ (1+\delta)^{2n}|\mu-1|]) \\ &\times ((1+2\delta)^n|2(1+2\delta)^n + (1+\delta)^{2n}(\mu-1)| \\ &\times (2\lambda+\mu)^{-1}) \end{aligned} \right\}. \quad (46)$$

Proof. First of all, it follows from conditions (38) that

$$\begin{aligned} (1-\lambda) \left(\frac{D_\delta^n f(z)}{z} \right)^\mu + \lambda (D_\delta^n f(z))' \left(\frac{D_\delta^n f(z)}{z} \right)^{\mu-1} \\ = \beta + (1-\beta)p(z) \quad (z \in \mathbb{U}), \\ (1-\lambda) \left(\frac{D_\delta^n g(w)}{w} \right)^\mu + \lambda (D_\delta^n g(w))' \left(\frac{D_\delta^n g(w)}{w} \right)^{\mu-1} \\ = \beta + (1-\beta)q(w) \quad (w \in \mathbb{U}), \end{aligned} \quad (47)$$

respectively, where

$$\begin{aligned} p(z) &= 1 + p_1 z + p_2 z^2 + \dots, \\ q(w) &= 1 + q_1 w + q_2 w^2 + \dots \end{aligned} \quad (48)$$

in $\mathcal{P}$. Now, upon equating the coefficients in (47), we get

$$(1 + \delta)^n (\lambda + \mu) a_2 = (1 - \beta) p_1, \quad (49)$$

$$(1 + 2\delta)^n (2\lambda + \mu) a_3 + (1 + \delta)^{2n} (\mu - 1) \left(\lambda + \frac{\mu}{2} \right) a_2^2 = (1 - \beta) p_2, \quad (50)$$

$$-(1 + \delta)^n (\lambda + \mu) a_2 = (1 - \beta) q_1, \quad (51)$$

$$-(1 + 2\delta)^n (2\lambda + \mu) a_3 + [4(1 + 2\delta)^n + (1 + \delta)^{2n} (\mu - 1)] \times \left(\lambda + \frac{\mu}{2} \right) a_2^2 = (1 - \beta) q_2. \quad (52)$$

From (49) and (51), we obtain

$$p_1 = -q_1, \quad (53)$$

$$2(1 + \delta)^{2n} (\lambda + \mu)^2 a_2^2 = (1 - \beta)^2 (p_1^2 + q_1^2). \quad (54)$$

Also, from (50) and (52), we have

$$[2(1 + 2\delta)^n + (1 + \delta)^{2n} (\mu - 1)] \times (2\lambda + \mu) a_2^2 = (1 - \beta) (p_2 + q_2). \quad (55)$$

Therefore, from equalities (54) and (55) we find that

$$|a_2|^2 \leq \frac{(1 - \beta)^2 (|p_1|^2 + |q_1|^2)}{2(1 + \delta)^{2n} (\lambda + \mu)^2}, \quad (56)$$

$$\leq \frac{(1 - \beta) (|p_2| + |q_2|)}{[2(1 + 2\delta)^n + (1 + \delta)^{2n} (\mu - 1)] (2\lambda + \mu)},$$

respectively, and applying Lemma 1, we get the desired estimate on the coefficient $|a_2|$ as asserted in (45).

Next, in order to find the bound on the coefficient $|a_3|$, we subtract (52) from (50). We thus get

$$2(1 + 2\delta)^n (2\lambda + \mu) a_3 - 2(1 + 2\delta)^n (2\lambda + \mu) a_2^2 = (1 - \beta) (p_2 - q_2), \quad (57)$$

which, upon substitution of the value of a_2^2 from (54), yields

$$a_3 = \frac{(1 - \beta)^2 (p_1^2 + q_1^2)}{2(1 + \delta)^{2n} (\lambda + \mu)^2} + \frac{(1 - \beta) (p_2 - q_2)}{2(1 + 2\delta)^n (2\lambda + \mu)}. \quad (58)$$

On the other hand, by using the (55) into (57), it follows that

$$a_3 = (1 - \beta) \times (2(1 + 2\delta)^n [2(1 + 2\delta)^n + (1 + \delta)^{2n} (\mu - 1)] \times (2\lambda + \mu))^{-1} \times \{ [4(1 + 2\delta)^n + (1 + \delta)^{2n} (\mu - 1)] p_2 - (1 + \delta)^{2n} (\mu - 1) q_2 \}. \quad (59)$$

Applying Lemma 1 for (58) and (59), we get the desired estimate on the coefficient $|a_3|$ as asserted in (46). $\square$

If we take $n = 0$ in Theorem 12, then we have the following corollary.

Corollary 13 (see [8]). *Let the function $f(z)$ given by the Taylor-Maclaurin series expansion (2) be in the class $\mathcal{N}_\Sigma^\mu(\beta, \lambda)$ ($0 \leq \beta < 1, \lambda \geq 1, \mu \geq 0$). Then,*

$$|a_2| \leq \min \left\{ \sqrt{\frac{4(1 - \beta)}{(\mu + 1)(2\lambda + \mu)}}, \frac{2(1 - \beta)}{\lambda + \mu} \right\},$$

$$|a_3| \leq \begin{cases} \min \left\{ \frac{4(1 - \beta)}{(\mu + 1)(2\lambda + \mu)}, \frac{4(1 - \beta)^2}{(\lambda + \mu)^2} + \frac{2(1 - \beta)}{2\lambda + \mu} \right\}, & 0 \leq \mu < 1 \\ \frac{2(1 - \beta)}{2\lambda + \mu}, & \mu \geq 1. \end{cases} \quad (60)$$

If we take $n = 0$ and $\mu = 1$ in Theorem 12, then we have the following corollary.

Corollary 14. *Let the function $f(z)$ given by the Taylor-Maclaurin series expansion (2) be in the class $\mathcal{B}_\Sigma(\beta, \lambda)$ ($0 \leq \beta < 1, \lambda \geq 1$). Then,*

$$|a_2| \leq \min \left\{ \sqrt{\frac{2(1 - \beta)}{2\lambda + 1}}, \frac{2(1 - \beta)}{\lambda + 1} \right\}, \quad (61)$$

$$|a_3| \leq \frac{2(1 - \beta)}{2\lambda + 1}.$$

Remark 15. Corollary 14 provides an improvement of the following estimates obtained by Frasin and Aouf [7].

Corollary 16 (see [7]). *Let the function $f(z)$ given by the Taylor-Maclaurin series expansion (2) be in the class $\mathcal{B}_\Sigma(\beta, \lambda)$ ($0 \leq \beta < 1, \lambda \geq 1$). Then,*

$$|a_2| \leq \sqrt{\frac{2(1 - \beta)}{2\lambda + 1}}, \quad (62)$$

$$|a_3| \leq \frac{4(1 - \beta)^2}{(\lambda + 1)^2} + \frac{2(1 - \beta)}{2\lambda + 1}.$$

If we take $n = 0$, $\mu = 1$, and $\lambda = 1$ in Theorem 12, then we have the following corollary.

Corollary 17. Let the function $f(z)$ given by the Taylor-Maclaurin series expansion (2) be in the class $\mathcal{H}_\Sigma(\beta)$ ($0 \leq \beta < 1$). Then,

$$|a_2| \leq \begin{cases} \sqrt{\frac{2(1-\beta)}{3}}, & 0 \leq \beta \leq \frac{1}{3} \\ 1-\beta, & \frac{1}{3} \leq \beta < 1, \end{cases} \quad (63)$$

$$|a_3| \leq \frac{2(1-\beta)}{3}.$$

Remark 18. Corollary 17 provides an improvement of the following estimates obtained by Srivastava et al. [4].

Corollary 19 (see [4]). Let the function $f(z)$ given by the Taylor-Maclaurin series expansion (2) be in the class $\mathcal{H}_\Sigma(\beta)$ ($0 \leq \beta < 1$). Then,

$$|a_2| \leq \sqrt{\frac{2(1-\beta)}{3}}, \quad (64)$$

$$|a_3| \leq \frac{(1-\beta)(5-3\beta)}{3}. \quad (65)$$

If we take $n = 0$, $\mu = 0$, and $\lambda = 1$ in Theorem 12, then we have the following corollary.

Corollary 20. Let the function $f(z)$ given by the Taylor-Maclaurin series expansion (2) be in the class $\mathcal{S}_\Sigma^*(\beta)$ ($0 \leq \beta < 1$). Then,

$$|a_2| \leq \sqrt{2(1-\beta)},$$

$$|a_3| \leq \begin{cases} 2(1-\beta), & 0 \leq \beta \leq \frac{3}{4} \\ (1-\beta)(5-4\beta), & \frac{3}{4} \leq \beta < 1. \end{cases} \quad (66)$$

Remark 21. Corollary 20 provides an improvement of the following estimates obtained by Brannan and Taha [5, 6] (see also [10, Corollary 3.2]).

Corollary 22 (see [5, 6]). Let the function $f(z)$ given by the Taylor-Maclaurin series expansion (2) be in the class $\mathcal{S}_\Sigma^*(\beta)$ ($0 \leq \beta < 1$). Then,

$$|a_2| \leq \sqrt{2(1-\beta)}, \quad (67)$$

$$|a_3| \leq 2(1-\beta).$$

If we take $\delta = 1$ and $\mu = 1$ in Theorem 12, then we have the following corollary.

Corollary 23. Let the function $f(z)$ given by the Taylor-Maclaurin series expansion (2) be in the function class $\mathcal{H}_\Sigma(n, \beta, \lambda)$ ($0 < \alpha \leq 1, \lambda \geq 1, n \in \mathbb{N}_0$). Then,

$$|a_2| \leq \min \left\{ \frac{2(1-\beta)}{2^n(\lambda+1)}, \sqrt{\frac{2(1-\beta)}{3^n(2\lambda+1)}} \right\}, \quad (68)$$

$$|a_3| \leq \frac{2(1-\beta)}{3^n(2\lambda+1)}.$$

Remark 24. Corollary 23 provides an improvement of the following estimates obtained by Porwal and Darus [9].

Corollary 25 (see [9]). Let the function $f(z)$ given by the Taylor-Maclaurin series expansion (2) be in the function class $\mathcal{H}_\Sigma(n, \beta, \lambda)$ ($0 < \alpha \leq 1, \lambda \geq 1, n \in \mathbb{N}_0$). Then,

$$|a_2| \leq \sqrt{\frac{2(1-\beta)}{3^n(2\lambda+1)}}, \quad (69)$$

$$|a_3| \leq \frac{4(1-\beta)^2}{4^n(\lambda+1)^2} + \frac{2(1-\beta)}{3^n(2\lambda+1)}.$$

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