Research Article

Linear Transformations between Multipartite Quantum Systems That Map the Set of Tensor Product of Idempotent Matrices into Idempotent Matrix Set

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Let H_n be the set of $n \times n$ complex Hermitian matrices and $\mathcal{P}_n$ (resp., $\mathcal{T}_n$) be the set of all idempotent (resp., tripotent) matrices in H_n . In l -partite quantum system $H_{m_1 \cdots m_l} = \otimes_1^l H_{m_i}$, $\otimes_1^l \mathcal{P}_{m_i}$ (resp., $\otimes_1^l \mathcal{T}_{m_i}$) denotes the set of all decomposable elements $\otimes_1^l A_i$ such that $A_i \in \mathcal{P}_{m_i}$ (resp., $A_i \in \mathcal{T}_{m_i}$). In this paper, linear maps ϕ from $H_{m_1 \cdots m_l}$ to H_n with $n \leq m_1 \cdots m_l$ such that $\phi(\otimes_1^l \mathcal{P}_{m_i}) \in \mathcal{P}_n$ are characterized. As its application, the structure of linear maps ϕ from $H_{m_1 \cdots m_l}$ to H_n with $n \leq m_1 \cdots m_l$ such that $\phi(\otimes_1^l \mathcal{T}_{m_i}) \in \mathcal{T}_n$ is also obtained.

1. Introduction

Let M_n be the vector space of $n \times n$ complex matrices and $H_n \subseteq M_n$ be the vector space of $n \times n$ complex Hermitian matrices. In quantum information theory, a l -partite system can be represented as the tensor product space $H_{m_1 \cdots m_l} \equiv \otimes_1^l H_{m_i}$, where $\otimes$ is the usual Kronecker product of matrices; a quantum state is represented as a positive semidefinite with trace one in H_n , see [1]. In quantum information science and quantum computing, it is important to understand, characterize, and construct different classes of maps on quantum states [2]. For example, entanglement is one of the main concepts in quantum information theory, and entangled state involves at least bipartite system or multipartite system [3]; to study entangled states, one should construct entanglement witnesses, which are special types of positive maps, see [4]. On this background, the research on the characterizations of maps leaving invariant, some important subsets or quantum properties attracted more and more researchers' attention, see ([1, 3, 5]). Especially, in [5], Lim characterized the linear and additive maps on tensor products of spaces of Hermitian matrices that carry the set of tensor product of rank one matrices into itself.

Preserver problem is a hot area in Banach algebra; there are many results about this area, see ([6–11]). Specially, the idempotent preservers and the rank one preservers play an important role; therefore, it is meaningful to study the two preservers. Chan et al. [12] first characterized linear transformations on M_n preserving idempotent matrices. In [13], the authors obtained the following result: linear map $T : H_n \rightarrow H_m$ with $n \leq m$ satisfies $T(\mathcal{P}_n) \subset \mathcal{P}_m$ if and only if $T = 0$ or

$$T(X) = QXQ^{-1} \quad \text{or} \quad T(X) = QX^T Q^{-1}, \quad \forall X \in H_n, \quad (1)$$

where Q is unitary matrix.

Since the study of linear transformations, preserving idempotents is important in many aspects of mathematics and physics, see [14–17], and inspired by the above, the purpose of this paper is to study linear maps from l -partite system to the space of Hermitian matrices that carry the set of tensor product of idempotent matrices into the set of idempotent matrices in the space of Hermitian matrices, that is, $\phi : H_{m_1 \cdots m_l} \rightarrow H_n$ with $n \leq m_1 \cdots m_l$ satisfying

$$\phi(\otimes_1^l \mathcal{P}_{m_i}) \subset \mathcal{P}_n, \quad (2)$$

where $\mathcal{P}_n$ is the set of all idempotent matrices in H_n , that is, $\mathcal{P}_n = \{A \in H_n | A^2 = A\}$ and $\otimes_1^l \mathcal{P}_{m_i}$ denotes the set of all decomposable elements $\otimes_1^l A_i$ such that $A_i \in \mathcal{P}_{m_i}$. As application, the forms of linear maps from l -partite system to the space of Hermitian matrices that carry the set of tensor product of tripotent matrices into the set of tripotent matrices in the space of Hermitian matrices are obtained, that is, $\phi : H_{m_1 \dots m_l} \rightarrow H_n$ with $n \leq m_1 \dots m_l$ satisfying

$$\phi(\otimes_1^l \mathcal{T}_{m_i}) \subset \mathcal{T}_n, \quad (3)$$

where $\mathcal{T}_n$ is the set of all tripotent matrices in H_n , that is, $\mathcal{T}_n = \{A \in H_n | A^3 = A\}$ and $\otimes_1^l \mathcal{T}_{m_i}$ denotes the set of all decomposable elements $\otimes_1^l A_i$ such that $A_i \in \mathcal{T}_{m_i}$.

Throughout this paper, we always assume integers $l \geq 1$ and $n, m_1, \dots, m_l \geq 2$. Let I_k be the $k \times k$ identity matrix, 0 be the zero matrix which order is omitted in different matrices just for simplicity, and X^T (resp., X^* , rank X) be the transpose (resp., conjugate transpose, rank) of X . $E_{ij}^{(n)}$ ($1 \leq i, j \leq n$) stands for the $n \times n$ matrix with 1 at the (i, j) th entry and 0 otherwise. $D_{ij}^{(n)} = E_{ij}^{(n)} + E_{ji}^{(n)}$ and $K_{ij}^{(n)} = \sqrt{-1}E_{ij}^{(n)} - \sqrt{-1}E_{ji}^{(n)}$, where $1 \leq i \neq j \leq n$. For real numbers a and b with $a \leq b$, let $[a, b]$ be the set of all integers between a and b . $\oplus$ is the usual direct sum of matrices. A linear map $\pi = \otimes_1^l \pi_i : H_{m_1 \dots m_l} \rightarrow H_{m_1 \dots m_l}$ is canonical if π satisfies $\pi(\otimes_1^l A_i) = \otimes_1^l \pi_i(A_i)$, where $\pi_i : H_{m_i} \rightarrow H_{m_i}$ satisfies $\pi_i(A_i) = A_i$ or $\pi_i(A_i) = A_i^T$ for all $A_i \in H_{m_i}$.

2. Main Results

Lemma 1 (see [18]). Suppose $P_1, \dots, P_k \in \mathcal{P}_n$ such that $P_i + P_j \in \mathcal{P}_n, \forall 1 \leq i < j \leq k$. Let $r_i = \text{rank } P_i$. Then, there exists a unitary matrix $U \in M_n$ such that

$$P_i = U \text{diag}(0, \dots, 0, 1, \dots, 1, 0, \dots, 0) U^*, \quad (4)$$

where $\text{diag}(0, \dots, 0, 1, \dots, 1, 0, \dots, 0)$ is the diagonal matrix in which all diagonal entries are zero except those in the $(r_1 + \dots + r_{i-1} + 1)$ st to the $(r_1 + \dots + r_i)$ th rows.

Lemma 2. Let $P \in \mathcal{P}_r$. Suppose $X, 0_s \oplus P \oplus 0_{n-r-s} - X \in \mathcal{P}_n$. Then, there exists $X_r \in \mathcal{P}_r$ such that $X = 0_s \oplus X_r \oplus 0_{n-r-s}$, where $0 \leq r, s \leq n$.

Proof. Let $T \in M_r$ be a unitary matrix satisfying $P = T(I_t \oplus 0)T^*$. Set

$$\begin{bmatrix} I_r & 0 & 0 \\ 0 & T^* & 0 \\ 0 & 0 & I_{n-r-s} \end{bmatrix} X \begin{bmatrix} I_r & 0 & 0 \\ 0 & T & 0 \\ 0 & 0 & I_{n-r-s} \end{bmatrix} = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{bmatrix}, \quad (5)$$

where $X_{11} \in H_s, X_{22} \in H_t$. Then,

$$\begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{bmatrix} \in \mathcal{P}_n, \quad (6)$$

$$\begin{bmatrix} 0_r & 0 & 0 \\ 0 & I_t & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{bmatrix} \in \mathcal{P}_n.$$

A straightforward computation shows that $X = 0_s \oplus [T(X_{22} \oplus 0_{r-t})T^*] \oplus 0_{n-r-s}$; therefore, the result holds. $\square$

Lemma 3. Let $A = P_r \oplus 0_s \oplus 0_{n-r-s}$ and $B = 0_r \oplus P_s \oplus 0_{n-r-s}$ with $P_r \in \mathcal{P}_r, P_s \in \mathcal{P}_s$. If

$$\alpha A + \beta B + \gamma X \in \mathcal{P}_n \quad \text{for any real } \gamma \text{ with } 0 \neq |\gamma| \leq \frac{1}{2}, \quad (7)$$

where $X \in M_n, \alpha$ and β are all roots of $x^2 - x + \gamma^2 = 0$, then, $X^2 = AB$, and X is of the form

$$\begin{bmatrix} 0_r & X_{12} \\ X_{21} & 0_s \end{bmatrix} \oplus 0_{n-r-s}, \quad (8)$$

where $X_{12} \in M_{r \times s}, X_{21} \in M_{s \times r}$.

Proof. It is clear that $A, B \in \mathcal{P}_n$ and $AB = BA = 0$. By a direct computation, one can obtain that

$$X = \alpha (AX + XA) + \beta (BX + XB) + \gamma (X^2 - A - B). \quad (9)$$

Replacing γ by $-\gamma$, we have $X^2 = A + B$ and

$$X = \alpha (AX + XA) + \beta (BX + XB). \quad (10)$$

Choosing $\gamma = 2/5$, then $\alpha = 1/5, \beta = 4/5$ or $\alpha = 4/5, \beta = 1/5$, this yields that

$$X = \frac{1}{5} (AX + XA) + \frac{4}{5} (BX + XB), \quad (11)$$

$$X = \frac{4}{5} (AX + XA) + \frac{1}{5} (BX + XB).$$

Thus, $X = AX + XA$ and $X = BX + XB$. This, together with the form of A and B , implies that X has the form $\begin{bmatrix} 0_r & X_{12} \\ X_{21} & 0_s \end{bmatrix} \oplus 0_{n-r-s}$, where $X_{12} \in M_{r \times s}, X_{21} \in M_{s \times r}$. $\square$

Lemma 4. Let $J = P_p \oplus -P_q \oplus 0_{n-p-q}$ with $P_p \in \mathcal{P}_p$ and $P_q \in \mathcal{P}_q$. Suppose $X \in \mathcal{T}_n$ satisfying $J - X \in \mathcal{T}_n$ and $J - 2X \in \mathcal{T}_n$. Then, $X = X_p \oplus -X_q \oplus 0_{n-p-q}$ with $X_p \in \mathcal{P}_p$ and $X_q \in \mathcal{P}_q$.

Proof. By a direct computation, we have

$$3X = J^2 X + JXJ + XJ^2, \quad (12)$$

$$3X = JX^2 + XJX + X^2 J. \quad (13)$$

Let $U \in M_p$ and $V \in M_q$ be unitary matrices satisfying $P_p = U(I_r \oplus 0_{p-r})U^*$ and $P_q = V(I_s \oplus 0_{q-s})V^*$, respectively. Then,

$$\begin{aligned} & \begin{bmatrix} U^* & 0 & 0 \\ 0 & V^* & 0 \\ 0 & 0 & I_{n-p-q} \end{bmatrix} J \begin{bmatrix} U & 0 & 0 \\ 0 & V & 0 \\ 0 & 0 & I_{n-p-q} \end{bmatrix} \\ &= \begin{bmatrix} I_r \oplus 0_{p-r} & 0 & 0 \\ 0 & -I_s \oplus 0_{q-s} & 0 \\ 0 & 0 & 0_{n-p-q} \end{bmatrix}. \end{aligned} \quad (14)$$

By (12), one can assume that

$$\begin{aligned} & \begin{bmatrix} U^* & 0 & 0 \\ 0 & V^* & 0 \\ 0 & 0 & I_{n-p-q} \end{bmatrix} X \begin{bmatrix} U & 0 & 0 \\ 0 & V & 0 \\ 0 & 0 & I_{n-p-q} \end{bmatrix} \\ &= \begin{bmatrix} X_r \oplus 0_{p-r} & 0 & 0 \\ 0 & -X_s \oplus 0_{q-s} & 0 \\ 0 & 0 & 0_{n-p-q} \end{bmatrix}, \end{aligned} \quad (15)$$

where $X_r \in H_r$ and $X_s \in H_s$. This, together with (13), implies that $X_r \in \mathcal{P}_r$ and $X_s \in \mathcal{P}_s$. Thus,

$$X = [U(X_r \oplus 0_{p-r})U^*] \oplus -[V(X_s \oplus 0_{q-s})V^*] \oplus 0_{n-p-q}, \quad (16)$$

as desired. $\square$

Lemma 5. Let ψ_1, ψ_2 be canonical maps on $H_{m_1 \cdots m_l}$, and U, V are invertible matrices of order $m_1 \cdots m_l$. If $\psi_1(X)U = V\psi_2(X)$ for all $X \in H_{m_1 \cdots m_l}$, then $\psi_1 = \psi_2$ and $U = V = \lambda I_{m_1 \cdots m_l}$ for some $\lambda \neq 0$.

Proof. Let us first point out a simple observation which will be used in our proof. If a matrix $U \in M_{mm}$ commutes with $I_m \otimes S$ for all real symmetric $S \in M_n$, then U has the form $W \otimes I_n$ with $W \in M_m$.

Since ψ_1, ψ_2 are canonical maps on $H_{m_1 \cdots m_l}$, we assume that

$$\psi_1 = \tau_1 \otimes \cdots \otimes \tau_l, \quad \psi_2 = \eta_1 \otimes \cdots \otimes \eta_l, \quad (17)$$

where τ_i and η_i are the identity maps $X \mapsto X$ or the transposition maps $X \mapsto X^T$ for $i \in [1, l]$.

Clearly, $\psi_i(I_{m_1 \cdots m_l}) = I_{m_1 \cdots m_l}$, $i = 1, 2$, hence $U = V$.

We prove $U = \lambda I_{m_1 \cdots m_l}$ by induction on l . The case of $l = 1$ is the well-known fact. We assume that the statement holds true for $l - 1$. We will prove it for l . For any real symmetric $S \in M_{m_l}$, since $\psi_i(I_{m_1 \cdots m_{l-1}} \otimes S) = I_{m_1 \cdots m_{l-1}} \otimes S$, $i = 1, 2$, it follows that $U = W \otimes I_{m_l}$ for some matrix $W \in M_{m_1 \cdots m_{l-1}}$. We define linear maps

$$\varphi_1 = \tau_1 \otimes \cdots \otimes \tau_{l-1}, \quad \varphi_2 = \eta_1 \otimes \cdots \otimes \eta_{l-1}. \quad (18)$$

It is easy to see that φ_1 and φ_2 are canonical maps on $H_{m_1 \cdots m_{l-1}}$. For any $Y \in H_{m_1 \cdots m_{l-1}}$, since

$$\begin{aligned} & (\varphi_1(Y)W) \otimes I_{m_l} = \psi_1(Y \otimes I_{m_l})U \\ &= U\psi_2(Y \otimes I_{m_l}) \\ &= (W\varphi_2(Y)) \otimes I_{m_l}, \end{aligned} \quad (19)$$

we have $\varphi_1(Y)W = W\varphi_2(Y)$, thus, by induction hypothesis, $W = \lambda I_{m_1 \cdots m_{l-1}}$. Since U is invertible, we have $\lambda \neq 0$. Thus, $U = \lambda I_{m_1 \cdots m_l}$ and $\psi_1(X) = (1/\lambda)\psi_2(X)U = (1/\lambda)U\psi_2(X) = \psi_2(X)$ for all $X \in H_{m_1 \cdots m_l}$. $\square$

Theorem 6. Suppose ϕ is a linear map from $H_{m_1 \cdots m_l}$ to H_n with $n \leq m_1 \cdots m_l$. Then, $\phi(\otimes_1^l \mathcal{P}_{m_i}) \subset \mathcal{P}_n$ if and only if either $\phi = 0$ or $n = m_1 \cdots m_l$, there exists a unitary matrix $U \in M_n$ and a canonical map π on $H_{m_1 \cdots m_l}$ such that

$$\phi(X) = U\pi(X)U^*, \quad \forall X \in H_{m_1 \cdots m_l}. \quad (20)$$

Proof. The sufficiency part is obvious. We prove the necessity part by induction on l . When $l = 1$, it is the result in [13], which has been given in the above introduction. We assume now that the result holds true for $l - 1$ and give the proof of the case l by the following five steps.

Step 1. Suppose $P \in \mathcal{P}_{m_l}$ and $\phi(P \otimes I_{m_2 \cdots m_l}) = 0_s \oplus I_r \oplus 0_{n-r-s}$. Then,

$$\phi(P \otimes X) = 0_s \oplus \psi(X) \oplus 0_{n-r-s}, \quad \forall X \in H_{m_2 \cdots m_l}, \quad (21)$$

where ψ is a linear map from $H_{m_2 \cdots m_l}$ to H_r satisfying $\psi(\otimes_2^l \mathcal{P}_{m_i}) \subset \mathcal{P}_r$.

Proof of Step 1. Set

$$\begin{aligned} \Gamma_0 &= \{I_{m_2} \otimes \cdots \otimes I_{m_l}\}, \\ \Gamma_1 &= \{P_2 \otimes I_{m_3 \cdots m_l} : P_2 \in \mathcal{P}_{m_2}\}, \\ \Gamma_2 &= \{P_2 \otimes P_3 \otimes I_{m_4 \cdots m_l} : P_i \in \mathcal{P}_{m_i}, i \in [2, 3]\}, \\ &\dots \\ \Gamma_{l-1} &= \{P_2 \otimes \cdots \otimes P_{m_l} : P_i \in \mathcal{P}_{m_i}, i \in [2, l]\}. \end{aligned} \quad (22)$$

We prove by induction on k that $\phi(P \otimes X) = 0_s \oplus \psi(X) \oplus 0_{n-r-s}$, $\forall X \in \Gamma_k, k \in [0, l - 1]$. Assume for a moment that we have already proved this. Then, when ϕ is linear, we can complete the proof of Step 1.

Now, we prove the assertion. The case of $k = 0$ is just the assumption. Then, we assume that our statement holds true for $k - 1$, and we consider the image of $P \otimes P_2 \otimes \cdots \otimes P_{k+1} \otimes I_{m_{k+2} \cdots m_l}$ for $P_i \in \mathcal{P}_{m_i}, i \in [2, k + 1]$. Because

$$\begin{aligned} & P \otimes P_2 \otimes \cdots \otimes P_k \otimes I_{m_{k+1}} \otimes I_{m_{k+2} \cdots m_l} \in \otimes_1^l \mathcal{P}_{m_i}, \\ & P \otimes P_2 \otimes \cdots \otimes P_k \otimes P_{k+1} \otimes I_{m_{k+2} \cdots m_l} \in \otimes_1^l \mathcal{P}_{m_i}, \end{aligned} \quad (23)$$

$$P \otimes P_2 \otimes \cdots \otimes P_k \otimes (I_{m_{k+1}} - P_{k+1}) \otimes I_{m_{k+2} \cdots m_l} \in \otimes_1^l \mathcal{P}_{m_i},$$

we obtain using the property of ϕ that

$$\begin{aligned} & \phi(P \otimes P_2 \otimes \cdots \otimes P_k \otimes I_{m_{k+1}} \otimes I_{m_{k+2} \cdots m_l}) \in \mathcal{P}_n, \\ & \phi(P \otimes P_2 \otimes \cdots \otimes P_k \otimes P_{k+1} \otimes I_{m_{k+2} \cdots m_l}) \in \mathcal{P}_n, \\ & \phi(P \otimes P_2 \otimes \cdots \otimes P_k \otimes I_{m_{k+1}} \otimes I_{m_{k+2} \cdots m_l}) \in \mathcal{P}_n, \\ & -\phi(P \otimes P_2 \otimes \cdots \otimes P_{k+1} \otimes I_{m_{k+2} \cdots m_l}) \in \mathcal{P}_n. \end{aligned} \quad (24)$$

We obtain by induction hypothesis and Lemma 2 that $\phi(P \otimes P_2 \otimes \cdots \otimes P_{k+1} \otimes I_{m_{k+2} \cdots m_l}) = 0_s \oplus X_r \oplus 0_{n-r-s}$, where $X_r \in \mathcal{P}_r$, as desired.

Step 2. If $m_1 \cdots m_l > n$. Then, $\phi = 0$.

Proof of Step 2. Since $E_{ii}^{(m_1)} \otimes I_{m_2 \cdots m_l} \in \otimes_1^l \mathcal{P}_{m_i}, \forall i \in [1, m_1]$ and $(E_{ii}^{(m_1)} + E_{jj}^{(m_1)}) \otimes I_{m_2 \cdots m_l} \in \otimes_1^l \mathcal{P}_{m_i}, \forall i \neq j \in [1, m_1]$, we obtain using the property of ϕ that

$$\begin{aligned} \phi(E_{ii}^{(m_1)} \otimes I_{m_2 \cdots m_l}) &\in \mathcal{P}_n, \quad \forall i \in [1, m_1], \\ \phi(E_{ii}^{(m_1)} \otimes I_{m_2 \cdots m_l}) + \phi(E_{jj}^{(m_1)} \otimes I_{m_2 \cdots m_l}) &\in \mathcal{P}_n, \quad (25) \\ \forall i \neq j &\in [1, m_1]. \end{aligned}$$

Denote by $r_i = \text{rank } \phi(E_{ii}^{(m_1)} \otimes I_{m_2 \cdots m_l}), \forall i \in [1, m_1]$. Using Lemma 1 and composing ϕ by a similarity transformation, we may obtain that

$$\phi(E_{ii}^{(m_1)} \otimes I_{m_2 \cdots m_l}) = \text{diag}(0, \dots, 0, 1, \dots, 1, 0, \dots, 0), \quad (26)$$

where $\text{diag}(0, \dots, 0, 1, \dots, 1, 0, \dots, 0)$ is the diagonal matrix in which all diagonal entries are zero except those in the $(r_1 + \cdots + r_{i-1} + 1)$ st to the $(r_1 + \cdots + r_i)$ th rows.

Since $n < m_1 \cdots m_l$, there exists some $r_i < m_2 \cdots m_l$. Without loss of generality, we may assume $r_1 < m_2 \cdots m_l$. Thus, $\phi(E_{11}^{(m_1)} \otimes I_{m_2 \cdots m_l}) = I_{r_1} \oplus 0_{n-r_1}$. By Step 1, we see that there exists a linear map ψ from $H_{m_2 \cdots m_l}$ to H_{r_1} satisfying $\psi(\otimes_2^l \mathcal{P}_{m_i}) \subset \mathcal{P}_{r_1}$ such that $\phi(E_{11}^{(m_1)} \otimes X) = \psi(X) \oplus 0_{n-r_1}$ for all $X \in H_{m_2 \cdots m_l}$. We obtain by induction hypothesis $\psi = 0$. Thus,

$$\phi(E_{11}^{(m_1)} \otimes X) = 0, \quad \forall X \in H_{m_2 \cdots m_l}. \quad (27)$$

For any $j \in [2, m_1]$, $P \in \otimes_2^l \mathcal{P}_{m_i}$ and real γ with $0 \neq |\gamma| \leq 1/2$, we have $(\alpha E_{11}^{(m_1)} + \beta E_{jj}^{(m_1)} + \gamma D_{1j}^{(m_1)}) \otimes P \in \otimes_1^l \mathcal{P}_{m_i}$, where α and β are all roots of $x^2 - x + \gamma^2 = 0$. We obtain using the property of ϕ , (27), and Lemma 3 that $\phi(E_{jj}^{(m_1)} \otimes P) = 0$ and $\phi(D_{1j}^{(m_1)} \otimes P) = 0$. By using the similar argument, we see that $\phi(D_{ij}^{(m_1)} \otimes P) = 0, \phi(K_{ij}^{(m_1)} \otimes P) = 0, \forall i \neq j \in [1, m_1]$. Thus,

$$\phi(A \otimes P) = 0, \quad \forall A \in H_{m_1}. \quad (28)$$

By the arbitrariness of $P \in \otimes_2^l \mathcal{P}_{m_i}$, we have

$$\phi(A \otimes X) = 0, \quad \forall A \in H_{m_1}, X \in H_{m_2 \cdots m_l}. \quad (29)$$

Therefore, $\phi = 0$.

We next always assume $m_1 \cdots m_l = n$ and $\phi \neq 0$.

Step 3. Suppose $F_1, \dots, F_{m_1} \in \mathcal{P}_{m_1}$ are of rank one with $F_i + F_j \in \mathcal{P}_{m_1}, \forall i \neq j \in [1, m_1]$. Then, there exists a unitary matrix U of order $m_1 \cdots m_l$ such that

$$\begin{aligned} \phi(F_k \otimes X) &= U [E_{kk}^{(m_1)} \otimes \pi_k(X)] U^*, \\ \forall X &\in H_{m_2 \cdots m_l}, \quad k \in [1, m_1], \end{aligned} \quad (30)$$

where π_k is a canonical map on $H_{m_2 \cdots m_l}, \forall k \in [1, m_1]$.

Proof of Step 3. Using the similar approach as in the proof of Step 2, we may show that $\text{rank } \phi(F_k \otimes I_{m_2 \cdots m_l}) = m_2 \cdots m_l, \forall k \in [1, m_1]$. In fact, if $\text{rank } \phi(F_k \otimes I_{m_2 \cdots m_l}) > m_2 \cdots m_l$ for some k , then, by Lemma 1, there exists $\text{rank } \phi(F_i \otimes I_{m_2 \cdots m_l}) < m_2 \cdots m_l$ for some i . Then, using the similar approach as in the proof of Step 2, we have $\phi = 0$, which is a contradiction, and there exist a unitary matrix Q such that $\phi(F_k \otimes I_{m_2 \cdots m_l}) = Q[E_{kk}^{(m_1)} \otimes I_{m_2 \cdots m_l}]Q^*$. Thus, applying Step 1, we obtain that

$$\phi(F_k \otimes X) = Q [E_{kk}^{(m_1)} \otimes \psi_k(X)] Q^*, \quad \forall X \in H_{m_2 \cdots m_l}, \quad (31)$$

where linear map $\psi_k : H_{m_2 \cdots m_l} \rightarrow H_{m_2 \cdots m_l}$ satisfying $\psi_k(\otimes_2^l \mathcal{P}_{m_i}) \subset \mathcal{P}_{m_2 \cdots m_l}$, for any $k \in [1, m_1]$. We obtain by induction hypothesis that there exists unitary matrices $U_k \in \mathcal{M}_{m_2 \cdots m_l}$ and canonical maps π_k on $H_{m_2 \cdots m_l}$ such that

$$\psi_k(X) = U_k \pi_k(X) U_k^*, \quad \forall X \in H_{m_2 \cdots m_l}, \quad k \in [1, m_1]. \quad (32)$$

Set $U = Q \text{diag}(U_1, \dots, U_{m_1})$, we complete the proof of Step 3.

By Step 3, we may assume that

$$\begin{aligned} \phi(E_{kk}^{(m_1)} \otimes X) &= E_{kk}^{(m_1)} \otimes \pi_k(X), \\ \forall X &\in H_{m_2 \cdots m_l}, \quad k \in [1, m_1], \end{aligned} \quad (33)$$

where π_k is canonical map on $H_{m_2 \cdots m_l}, \forall k \in [1, m_1]$.

Step 4. For $\forall i \neq j \in [1, m_1], \pi_i = \pi_j$ and there exists λ_{ij}, μ_{ij} with $|\lambda_{ij}| = |\mu_{ij}| = 1$ such that

$$\begin{aligned} \phi(D_{ij}^{(m_1)} \otimes X) &= (\lambda_{ij} E_{ij}^{(m_1)} + \bar{\lambda}_{ij} E_{ji}^{(m_1)}) \otimes \pi_1(X), \\ \forall X &\in H_{m_2 \cdots m_l}, \\ \phi(K_{ij}^{(m_1)} \otimes X) &= (\mu_{ij} E_{ij}^{(m_1)} + \bar{\mu}_{ij} E_{ji}^{(m_1)}) \otimes \pi_1(X), \\ \forall X &\in H_{m_2 \cdots m_l}. \end{aligned} \quad (34)$$

Proof of Step 4. Without loss of generality, we only prove the case of $i = 1, j = 2$.

Set $F_1 = (1/2)E_{11}^{(m_1)} + (1/2)E_{22}^{(m_1)} + (1/2)D_{12}^{(m_1)}, F_2 = (1/2)E_{11}^{(m_1)} + (1/2)E_{22}^{(m_1)} - (1/2)D_{12}^{(m_1)}, F_k = E_{kk}^{(m_1)}$ for $k \in [3, m_1]$ (if $m_1 \geq 3$). By Step 3, there exists a unitary matrix V and canonical maps η_k on $H_{m_2 \cdots m_l}$ such that

$$\begin{aligned} \phi(F_k \otimes X) &= V [E_{kk}^{(m_1)} \otimes \eta_k(X)] V^*, \\ \forall X &\in H_{m_2 \cdots m_l}, \quad k \in [1, m_1]. \end{aligned} \quad (35)$$

This, together with (33), implies that

$$E_{kk}^{(m_1)} \otimes I_{m_2 \cdots m_l} = V [E_{kk}^{(m_1)} \otimes I_{m_2 \cdots m_l}] V^*, \quad \forall k \in [3, m_1]. \quad (36)$$

Thus, we may set

$$V = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} \oplus \text{diag}(V_{33}, \dots, V_{m_1 m_1}), \quad (37)$$

where $V_{kk} \in \mathcal{M}_{m_2 \cdots m_l}, k \in [1, m_1]$.

On one hand, noting that

$$\begin{aligned} D_{12}^{(m_1)} &= 2F_1 - E_{11}^{(m_1)} - E_{22}^{(m_1)}, \\ D_{12}^{(m_1)} &= E_{11}^{(m_1)} + E_{22}^{(m_1)} - 2F_2, \end{aligned} \quad (38)$$

we obtain using (33) and (35) that for all $X \in H_{m_2 \dots m_l}$,

$$\begin{aligned} \phi(D_{12}^{(m_1)} \otimes X) &= \begin{bmatrix} 2V_{11}\eta_1(X) V_{11}^* - \pi_1(X) & 2V_{11}\eta_1(X) V_{21}^* \\ 2V_{21}\eta_1(X) V_{11}^* & 2V_{21}\eta_1(X) V_{21}^* - \pi_2(X) \end{bmatrix} \otimes 0, \end{aligned} \quad (39)$$

$$\begin{aligned} \phi(D_{12}^{(m_1)} \otimes X) &= \begin{bmatrix} \pi_1(X) - 2V_{12}\eta_2(X) V_{12}^* & -2V_{12}\eta_2(X) V_{22}^* \\ -2V_{22}\eta_2(X) V_{12}^* & \pi_2(X) - 2V_{22}\eta_2(X) V_{22}^* \end{bmatrix} \otimes 0. \end{aligned} \quad (40)$$

On the other hand, for any $P \in \otimes_1^l \mathcal{P}_{m_i}$ and real γ with $0 \neq |\gamma| \leq 1/2$, we have $(\alpha E_{11}^{(m_1)} + \beta E_{22}^{(m_1)} + \gamma D_{12}^{(m_1)}) \otimes P \in \otimes_1^l \mathcal{P}_{m_i}$, where α and β are all roots of $x^2 - x + \gamma^2 = 0$. We obtain using the property of ϕ , (33), and Lemma 3 that

$$[\phi(D_{12}^{(m_1)} \otimes I_{m_2 \dots m_l})]^2 \quad (41)$$

$$= \phi(E_{11}^{(m_1)} \otimes I_{m_2 \dots m_l}) + \phi(E_{22}^{(m_1)} \otimes I_{m_2 \dots m_l}),$$

$$\phi(D_{12}^{(m_1)} \otimes X) = \begin{bmatrix} 0_{m_1} & Y_{12} \\ Y_{12}^* & 0_{m_1} \end{bmatrix} \otimes 0, \quad \forall X \in H_{m_2 \dots m_l}, \quad (42)$$

where $Y_{12} \in M_{m_1}$. Choosing $X = I_{m_2 \dots m_l}$, we get using (39), (40), and (42) that $2V_{11}V_{11}^* = 2V_{22}V_{22}^* = 2V_{21}V_{21}^* = 2V_{12}V_{12}^* = I_{m_2 \dots m_l}$. Hence, by (39), (40), and (42) again, one can obtain that

$$\begin{aligned} \pi_1(X) (V_{11}^*)^{-1} V_{21}^* &= 2V_{11}\eta_1(X) V_{21}^* = -2V_{12}\eta_2(X) V_{22}^* = -V_{12}V_{22}^{-1}\pi_2(X). \end{aligned} \quad (43)$$

By Lemma 5, we have $\pi_1 = \pi_2$ and $(V_{11}^*)^{-1} V_{21}^* = \lambda_{12} I_{m_2 \dots m_l}$ with $\lambda_{12} \neq 0$. Hence,

$$\begin{aligned} \phi(D_{12}^{(m_1)} \otimes X) &= (\lambda_{12} E_{12}^{(m_1)} + \bar{\lambda}_{12} E_{21}^{(m_1)}) \otimes \pi_1(X), \\ \forall X \in H_{m_2 \dots m_l}, \quad \lambda_{12} &\neq 0. \end{aligned} \quad (44)$$

This, together with (41), implies that $|\lambda_{12}| = 1$. Similarly,

$$\begin{aligned} \phi(K_{12}^{(m_1)} \otimes X) &= (\mu_{12} E_{12}^{(m_1)} + \bar{\mu}_{12} E_{21}^{(m_1)}) \otimes \pi_1(X), \\ \forall X \in H_{m_2 \dots m_l}, \quad |\mu_{12}| &= 1. \end{aligned} \quad (45)$$

This completes the proof of Step 4.

Step 5. There exists $\varepsilon \in \{-1, 1\}$ such that $\mu_{ij} = \varepsilon \sqrt{-1} \lambda_{ij}$, $\forall i \neq j \in [1, m_1]$. If $m_1 \geq 3$, then $\lambda_{ij} \lambda_{jk} = \lambda_{ik}$, for distinct $i, j, k \in [1, m_1]$.

Proof of Step 5. Since

$$\begin{aligned} \left(\frac{1}{2} E_{ii}^{(m_1)} + \frac{1}{2} E_{jj}^{(m_1)} + \frac{\sqrt{2}}{4} D_{ij}^{(m_1)} + \frac{\sqrt{2}}{4} K_{ij}^{(m_1)} \right) \\ \otimes I_{m_2 \dots m_l} \in \otimes_1^l \mathcal{P}_{m_i}, \end{aligned} \quad (46)$$

we obtain using the property of ϕ , (33), and Step 4 that

$$\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{2}}{4} (\lambda_{ij} + \mu_{ij}) \\ \frac{\sqrt{2}}{4} (\bar{\lambda}_{ij} + \bar{\mu}_{ij}) & \frac{1}{2} \end{bmatrix} \in \mathcal{P}_2. \quad (47)$$

Hence, $(\lambda_{ij} + \mu_{ij})(\bar{\lambda}_{ij} + \bar{\mu}_{ij}) = 2$. It follows from $|\lambda_{ij}| = |\mu_{ij}| = 1$ that $\lambda_{ij}^2 = -\mu_{ij}^2$. Thus, there exists $\varepsilon_{ij} \in \{-1, 1\}$ such that $\mu_{ij} = \varepsilon_{ij} \sqrt{-1} \lambda_{ij}$.

If $m_1 \geq 3$, since

$$\begin{aligned} \frac{1}{3} (E_{ii}^{(m_1)} + E_{jj}^{(m_1)} + E_{kk}^{(m_1)} + D_{ij}^{(m_1)} + D_{ik}^{(m_1)} + D_{jk}^{(m_1)}) \\ \otimes I_{m_2 \dots m_l} \in \otimes_1^l \mathcal{P}_{m_i}, \end{aligned} \quad (48)$$

$$\begin{aligned} \frac{1}{3} (E_{ii}^{(m_1)} + E_{jj}^{(m_1)} + E_{kk}^{(m_1)} + D_{ij}^{(m_1)} + K_{ik}^{(m_1)} + K_{jk}^{(m_1)}) \\ \otimes I_{m_2 \dots m_l} \in \otimes_1^l \mathcal{P}_{m_i}, \end{aligned}$$

we obtain using the property of ϕ , (33), and Step 4 that

$$\frac{1}{3} \begin{bmatrix} 1 & \lambda_{ij} & \lambda_{ik} \\ \bar{\lambda}_{ij} & 1 & \lambda_{jk} \\ \bar{\lambda}_{ik} & \bar{\lambda}_{jk} & 1 \end{bmatrix} \in \mathcal{P}_3, \quad (49)$$

$$\frac{1}{3} \begin{bmatrix} 1 & \varepsilon_{ij} \lambda_{ij} \sqrt{-1} & \varepsilon_{ik} \lambda_{ik} \sqrt{-1} \\ -\varepsilon_{ij} \bar{\lambda}_{ij} \sqrt{-1} & 1 & \lambda_{jk} \\ -\varepsilon_{ik} \bar{\lambda}_{ik} \sqrt{-1} & \bar{\lambda}_{jk} & 1 \end{bmatrix} \in \mathcal{P}_3, \quad (50)$$

$$\frac{1}{3} \begin{bmatrix} 1 & \lambda_{ij} & \varepsilon_{ik} \lambda_{ik} \sqrt{-1} \\ \bar{\lambda}_{ij} & 1 & \varepsilon_{jk} \lambda_{jk} \sqrt{-1} \\ -\varepsilon_{ik} \bar{\lambda}_{ik} \sqrt{-1} & -\varepsilon_{jk} \bar{\lambda}_{jk} \sqrt{-1} & 1 \end{bmatrix} \in \mathcal{P}_3. \quad (51)$$

It follows from (49) that $\lambda_{ij} \lambda_{jk} = \lambda_{ik}$. This, together with (50), implies that $\varepsilon_{ij} = \varepsilon_{ik}$. Similarly, (51) implies that $\varepsilon_{ik} = \varepsilon_{jk}$. Thus, all numbers ε_{ij} have to be the same for any $i \neq j \in [1, m_1]$. This completes the proof of this step.

By Steps 3–5, we set $Q = \text{diag}(1, \lambda_{12}, \dots, \lambda_{1m_1})$, then Q is unitary such that

$$\begin{aligned}\phi(E_{kk}^{(m_1)} \otimes X) &= [QE_{kk}^{(m_1)} Q^*] \otimes \pi_1(X), \\ \forall X \in H_{m_2 \dots m_l}, \quad k \in [1, m_1], \\ \phi(D_{ij}^{(m_1)} \otimes X) &= [QD_{ij}^{(m_1)} Q^*] \otimes \pi_1(X), \\ \forall X \in H_{m_2 \dots m_l}, \quad i \neq j \in [1, m_1], \\ \phi(K_{ij}^{(m_1)} \otimes X) &= \varepsilon [QK_{ij}^{(m_1)} Q^*] \otimes \pi_1(X), \\ \forall X \in H_{m_2 \dots m_l}, \quad i \neq j \in [1, m_1].\end{aligned}\quad (52)$$

Thus, set $U = Q \otimes I_{m_2 \dots m_l}$, we have

$$\begin{aligned}\phi(A \otimes X) &= U [A \otimes \pi_1(X)] U^*, \\ \forall A \in H_{m_1}, \quad X \in H_{m_2 \dots m_l}\end{aligned}\quad (53)$$

or

$$\begin{aligned}\phi(A \otimes X) &= U [A^T \otimes \pi_1(X)] U^*, \\ \forall A \in H_{m_1}, \quad X \in H_{m_2 \dots m_l}.\end{aligned}\quad (54)$$

This completes the proof of Theorem 6. $\square$

Remark 7. When $l \geq 2$, the linear transformation that maps the set of tensor product of idempotent matrices into idempotent matrix set does not necessarily preserve idempotent matrices. For example, let

$$\begin{aligned}\phi(A_1 \otimes A_2 \otimes \dots \otimes A_l) &= A_1^T \otimes A_2 \otimes \dots \otimes A_l, \\ \forall A_i \in \mathcal{H}_{m_i}, \quad i \in [1, l], \\ C &= \frac{1}{2} (E_{11}^{(m_1)} \otimes E_{11}^{(m_2)} + E_{22}^{(m_1)} \otimes E_{22}^{(m_2)} + E_{12}^{(m_1)} \\ &\quad \otimes E_{12}^{(m_2)} + E_{21}^{(m_1)} \otimes E_{21}^{(m_2)}) \otimes I_{m_3 \dots m_l}.\end{aligned}\quad (55)$$

Then $\phi(\otimes_1^l \mathcal{P}_{m_i}) \subset \mathcal{P}_{m_1 \dots m_l}$ and $C \in \mathcal{P}_{m_1 \dots m_l}$, but

$$\begin{aligned}\phi(C) &= \frac{1}{2} (E_{11}^{(m_1)} \otimes E_{11}^{(m_2)} + E_{22}^{(m_1)} \otimes E_{22}^{(m_2)} + E_{21}^{(m_1)} \\ &\quad \otimes E_{12}^{(m_2)} + E_{12}^{(m_1)} \otimes E_{21}^{(m_2)}) \otimes I_{m_3 \dots m_l} \notin \mathcal{P}_{m_1 \dots m_l}.\end{aligned}\quad (56)$$

As the application of Theorem 6, we give the following theorem.

Theorem 8. Suppose ϕ is a linear map from $H_{m_1 \dots m_l}$ to H_n with $n \leq m_1 \dots m_l$. Then, $\phi(\otimes_1^l \mathcal{T}_{m_i}) \subset \mathcal{T}_n$ if and only if either $\phi = 0$ or $n = m_1 \dots m_l$, there exist a unitary matrix $U \in M_n$, $\varepsilon \in \{-1, 1\}$ and a canonical map π on $H_{m_1 \dots m_l}$ such that

$$\phi(X) = \varepsilon U \pi(X) U^*, \quad \forall X \in H_{m_1 \dots m_l}. \quad (57)$$

Proof. The sufficiency part is clear. We give the proof of the necessity part.

Set

$$\begin{aligned}\Gamma_0 &= \{I_{m_1} \otimes \dots \otimes I_{m_l}\}, \\ \Gamma_1 &= \{P_1 \otimes I_{m_2 \dots m_l} : P_1 \in \mathcal{P}_{m_1}\}, \\ \Gamma_2 &= \{P_1 \otimes P_2 \otimes I_{m_3 \dots m_l} : P_i \in \mathcal{P}_{m_i}, \quad i \in [1, 2]\}, \\ &\dots \\ \Gamma_l &= \{P_1 \otimes \dots \otimes P_{m_l} : P_i \in \mathcal{P}_{m_i}, \quad i \in [1, l]\}.\end{aligned}\quad (58)$$

It is obvious that $\Gamma_l = \otimes_1^l \mathcal{P}_{m_i}$ and $\Gamma_k \subset \otimes_1^l \mathcal{T}_{m_i}$, $\forall k \in [0, l]$.

Since $I_{m_1 \dots m_l} \in \otimes_1^l \mathcal{T}_{m_i}$, we have $\phi(I_{m_1 \dots m_l}) \in \mathcal{T}_n$. Without loss of generality, we may assume that $\phi(I_{m_1 \dots m_l}) = I_p \oplus -I_q \oplus 0_{n-p-q}$. We next prove by induction on k that

$$\phi(X) = \psi_p(X) \oplus -\psi_q(X) \oplus 0_{n-p-q}, \quad \forall X \in \Gamma_k, \quad (59)$$

where $\psi_p(X) \in \mathcal{P}_p$ and $\psi_q(X) \in \mathcal{P}_q$.

We assume that our statement holds true for $k-1$ and prove it for k . For any $X = P_1 \otimes \dots \otimes P_k \otimes I_{m_{k+1} \dots m_l} \in \Gamma_k$, choosing $Y = P_1 \otimes \dots \otimes P_{k-1} \otimes I_{m_k} \otimes I_{m_{k+1} \dots m_l} \in \Gamma_{k-1}$, we have by induction hypothesis that

$$\phi(Y) = \psi_p(Y) \oplus -\psi_q(Y) \oplus 0_{n-p-q}, \quad (60)$$

with $\psi_p(Y) \in \mathcal{P}_p$ and $\psi_q(Y) \in \mathcal{P}_q$. Since $X \in \otimes_1^l \mathcal{T}_{m_i}$, $Y - X \in \otimes_1^l \mathcal{T}_{m_i}$ and $Y - 2X \in \otimes_1^l \mathcal{T}_{m_i}$. Using the property of ϕ , we have $\phi(X) \in \mathcal{T}_n$, $\phi(Y) - \phi(X) \in \mathcal{T}_n$ and $\phi(Y) - 2\phi(X) \in \mathcal{T}_n$. By Lemma 4, we have

$$\phi(X) = \psi_p(X) \oplus -\psi_q(X) \oplus 0_{n-p-q}, \quad (61)$$

with $\psi_p(X) \in \mathcal{P}_p$ and $\psi_q(X) \in \mathcal{P}_q$. This implies that (59) holds.

As ϕ is linear, we can extend ψ_p to be a linear map from $H_{m_1 \dots m_l}$ to H_p and ψ_q to be a linear map from $H_{m_1 \dots m_l}$ to H_q , then

$$\phi(X) = \psi_p(X) \oplus -\psi_q(X) \oplus 0_{n-p-q}, \quad \forall X \in H_{m_1 \dots m_l}, \quad (62)$$

with $\psi_p(\otimes_1^l \mathcal{P}_{m_i}) \subset \mathcal{P}_p$ and $\psi_q(\otimes_1^l \mathcal{P}_{m_i}) \subset \mathcal{P}_q$, we obtain using Theorem 6 that

- (a) if $n < m_1 \dots m_l$, then $p, q < m_1 \dots m_l$, $\psi_p = 0$, $\psi_q = 0$, thus, $\phi = 0$, and
- (b) if $\phi \neq 0$, then $n = m_1 \dots m_l$ and $p = n$ or $q = n$, thus $\phi = \psi_p$ or $\phi = -\psi_q$.

This completes the proof of Theorem 8. $\square$

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