Research Article

On a Class of Anisotropic Nonlinear Elliptic Equations with Variable Exponent

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Based on truncation technique and priori estimates, we prove the existence and uniqueness of weak solution for a class of anisotropic nonlinear elliptic equations with variable exponent \( p(x) \) growth. Furthermore, we also obtain that the weak solution is locally bounded and regular; that is, the weak solution is \( L^\infty_{\text{loc}}(\Omega) \) and \( C^{1, \alpha}(\Omega) \).

1. Introduction

Since the variable exponent spaces have reflected into a various range of applications such as non-Newtonian fluids, plasticity, image processing, and nonlinear elasticity [1–4], some authors began to study the various properties of variable exponent space and some nonlinear problems with variable exponent growth. Edmunds et al. [5], Fan and Zhao [6] obtained that the variable exponent space \( L^{p(x)}(\Omega) \) and \( W^{m, p(x)}(\Omega) \) are reflexive Banach spaces under suitable conditions on \( p(x) \). Later, Edmunds and Rákosník [7], Fan et al. [8] proved some continuous and compact Sobolev embedding theorems for the variable exponent spaces \( W^{k, p(x)}(\Omega) \). For the anisotropic variable exponents spaces, in 2008, Mihăilescu et al. [9] studied the eigenvalue problems for a class of anisotropic quasilinear elliptic equations with variable exponents. In 2011, Boureanu et al. [10] proved the existence of multiple solutions for a class of anisotropic elliptic equations with variable exponents. Recently, Stancu-Dumitru [11, 12] has studied the existence of nontrivial solutions for a class of nonhomogeneous anisotropic problem. In particular, Fan [13] established some embedding theorems for anisotropic variable exponent Sobolev spaces.

In this paper, we investigate the following anisotropic nonlinear elliptic equation:

\[-\partial_x a_i(x, u, \nabla u) + \sum_{i=1}^N T_i(x, \nabla u) = f - \partial_x h_i, \quad x \in \Omega, \]

\[u = 0, \quad x \in \partial \Omega,\]

where \( \Omega \subset \mathbb{R}^N \) with Lipschitz conditions boundary, \( a_i \) and \( T_i \) \((i = 1, 2, \ldots, N)\) are Carathéodory functions, \( h_i \) and \( f \) satisfy suitable conditions, and \( \partial_x a_i(x, u, \nabla u) \) and \( \partial_x h_i(x) \) are Einstein Sum; that is, \( \partial_x a_i(x, u, \nabla u) = \sum_{i=1}^N (\partial a_i(x, u, \nabla u)/\partial x_i), \quad \partial_x h_i(x) = \sum_{i=1}^N (\partial h_i(x)/\partial x_i). \) We usually use critical theory to obtain the existence of weak solutions. However, since the problem (1) has no variational structure, we cannot define the energy functional for the problem (1). Therefore, based on truncation technique and priori estimates, we prove the existence and uniqueness of weak solutions for the problem (1) in \( W^{1, p(x)}_0(\Omega) \). Furthermore, we obtain that the weak solution for the problem (1) is locally bounded.
In particular, for the special case
\[ -\partial_{x_i} a_i (x, \nabla u) = f - \partial_{x_i} h_i, \quad x \in \Omega, \]
\[ u = 0, \quad x \in \partial \Omega, \]   \hfill (2)
we obtain that the weak solution is $C^{1,\alpha}(\Omega)$. To our knowledge, the above two problems have not been deeply studied in the anisotropic variable exponent Sobolev spaces.

The paper is organized as follows. In Section 2, we recall some results on anisotropic variable exponent Sobolev spaces and state our main results. In Section 3, we prove the existence, uniqueness and locally bounded of weak solution for the problem (1). In Section 4, the regularity of weak solutions for the problem (2) is proved.

2. Preliminary and Main Results

This section is dedicated to a general overview on the $W^{1,p(x)}(\Omega)$ and $L^{p(x)}(\Omega)$; for a deeper treatment on these spaces, see [5, 7, 8, 13, 14].

Let
\[ C_+ (\overline{\Omega}) := \left\{ h : h \in C(\overline{\Omega}), h(x) > 1 \ \forall x \in \overline{\Omega} \right\}. \]   \hfill (3)

Set
\[ h^+ = \sup_{x \in \overline{\Omega}} h(x) \quad \text{and} \quad h^- = \inf_{x \in \overline{\Omega}} h(x), \]   \hfill (4)
for any $p(x) \in C_+ (\overline{\Omega})$, and we define the variable exponent Lebesgue space
\[ L^{p(x)}(\Omega) := \left\{ u : u \text{ is a measurable real } \right\}. \]   \hfill (5)

We define the norm of $L^{p(x)}(\Omega)$:
\[ \| u \|_{L^{p(x)}(\Omega)} = \inf \left\{ \mu > 0 \ ; \ \int_{\Omega} \left| \frac{u(x)}{\mu} \right|^{p(x)} dx < \infty \right\}. \]   \hfill (6)

From [6], we have the following:

1. $(L^{p(x)}(\Omega), \| \cdot \|_{p(x)})$ is a Banach space;
2. $L^{p(x)}(\Omega)$ is reflexive, if and only if $1 < p^ - \leq p^ + < \infty$;
3. Hölder inequality.

For all $u, v \in L^{p(x)}(\Omega)$,
\[ \int_{\Omega} uv dx \leq \left( \frac{1}{p^ -} + \frac{1}{p^ +} \right) \| u \|_{L^{p(x)}(\Omega)} \| v \|_{L^{p(x)}(\Omega)}^\star, \]   \hfill (7)
where $1/p(x) + 1/p^\star(x) = 1$, $L^{p(x)}(\Omega)$ is the conjugate space of $L^{p(x)}(\Omega)$.

Now, we recall some results on anisotropic variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ [13]; set
\[ L^{\infty}_{p(x)}(\Omega) := \left\{ p \in L^{\infty}(\Omega) : p(x) \geq 1 \quad \text{for a.e. } x \in \Omega \right\}. \]   \hfill (8)
Denote $\overline{p(x)} = (p_1(x), p_2(x), \ldots, p_N(x)) \in (L^{\infty}(\Omega))^N$ and define
\[ \forall x \in \Omega, \quad p_-(x) = \min \left\{ p_1(x), p_2(x), \ldots, p_N(x) \right\}, \]   \hfill (9)
\[ p_+(x) = \max \left\{ p_1(x), p_2(x), \ldots, p_N(x) \right\}. \]
The anisotropic variable exponent Sobolev space
\[ W^{1,p(x)}(\Omega) := \left\{ u \in L^{1,p(x)}(\Omega) : D_j u \in L^{p(x)}(\Omega) \quad \text{for } i = 1, 2, \ldots, N \right\}. \]   \hfill (10)
is a Banach space with respect to the norm
\[ \| u \|_{W^{1,p(x)}(\Omega)} = \| u \|_{L^{1,p(x)}(\Omega)} + \sum_{i=1}^N \| D_j u \|_{L^{p(x)}(\Omega)}. \]   \hfill (11)
If $p_i(x) \in L^{\infty}_{p(x)}(\Omega)$, for $i = 1, 2, \ldots, N$, $p_i^ > > 1$, then $W^{1,p(x)}(\Omega)$ is reflexive. We define $W^{1,0,p(x)}(\Omega)$ as the closure of $C^{\infty}_0(\Omega)$ with respect to the norm
\[ \| u \|_{W^{1,0,p(x)}(\Omega)} = \sum_{i=1}^N \left\| \frac{\partial u}{\partial x_i} \right\|_{L^{p_i(x)}(\Omega)}, \]   \hfill (12)
and $W^{1,0,p(x)}(\Omega)$ is a reflexive Banach space (see [9]).

Let
\[ p^-(x) = \frac{N}{\sum_{i=1}^N (1/p_i(x))}, \]   \hfill (13)
\[ p^\star(x) = \begin{cases} \frac{N p^-(x)}{N - p^-(x)} & \text{if } p^-(x) < N, \\ +\infty & \text{if } p^-(x) \geq N, \end{cases} \]
\[ p_{\infty}(x) = \max \left\{ \overline{p}(x), p_+(x) \right\}. \]
Hence, we have the following embedding theorem for $W^{1,0,p(x)}(\Omega)$.

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and $\overline{p(x)} = (p_1(\cdot), p_2(\cdot), \ldots, p_N(\cdot)) \in (C_+ (\overline{\Omega}))^N$.

**Theorem 1** (see [13, Theorem 2.5]).
\[ (i) \text{ If } q \in C_+ (\overline{\Omega}) \text{ and } \] \[ q(x) < p_{\infty}(x) \quad \forall x \in \overline{\Omega}, \]   \hfill (14)
then $W^{1,p(x)}(\Omega) \hookrightarrow L^q(\Omega)$. The embedding is compact.

(ii) If $\mathbf{p}(x) > N$ for all $x \in \Omega$, then there exists $\beta \in (0,1)$ such that $W^{1,p(x)}(\Omega) \hookrightarrow C^{0,\beta}(\Omega)$.

The embedding is also compact.

**Theorem 2** (see [13, Theorem 2.6]). Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and $\mathbf{p}(\cdot) = (p_1(\cdot), p_2(\cdot), \ldots, p_N(\cdot)) \in (C^0(\Omega))^N$.

Suppose that

$$p_i(x) < \mathbf{p}^-(x) \quad \forall x \in \Omega.$$  \hspace{1cm} (15)

Then one has

$$\|u\|_{L^{p_i}(\Omega)} \leq C \|\nabla u\|_{L^{p_i}(\Omega)} \quad \forall u \in W^{1,\mathbf{p}^-}(\Omega),$$  \hspace{1cm} (16)

where $C$ is a positive constant independent of $u \in W^{1,\mathbf{p}^-}(\Omega)$.

**Remark 3.** From [13], we know that if $p_i(x) < \mathbf{p}^-(x)$, for all $u \in W^{1,\mathbf{p}^-}(\Omega)$, for every $i = 1, 2, \ldots, N$, then $\|u\|_{L^{p_i}(\Omega)} \leq C\|\nabla u\|_{L^{p_i}(\Omega)}$, where $C$ is a positive constant independent of $u \in W^{1,\mathbf{p}^-}(\Omega)$.

Assume that $\alpha_i : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}$ and $\alpha : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ are Carathéodory functions and satisfy the following:

- **(A1)** for a.e. $x \in \Omega$, for all $\xi \in \mathbb{R}^N$, $s \in \mathbb{R}$, we have
  $$\lambda \sum_{i=1}^N |\xi_i |^{p_i(x)} \leq \sum_{i=1}^N a_i(x, s \xi_i) \xi_i \leq \sum_{i=1}^N |\xi_i |^{p_i(x)} \leq \sum_{i=1}^N |\xi_i |^{p_i(x)} \leq \sum_{i=1}^N a_i(x, s \xi_i) \xi_i \leq \sum_{i=1}^N |\xi_i |^{p_i(x)} \leq \sum_{i=1}^N |\xi_i |^{p_i(x)} \leq \sum_{i=1}^N a_i(x, s \xi_i) \xi_i \leq \sum_{i=1}^N |\xi_i |^{p_i(x)}.$$  \hspace{1cm} (17)

and $|a_i(x, s \xi_i)| \leq \beta |s|^{p_i(x)/p_i(x)-1} |\xi_i |^{p_i(x)-1}$, where $\lambda, \beta > 0$;

- **(A2)** for a.e. $x \in \Omega$, for all $\xi_i \neq \xi_j$, $\gamma > 0$ and $0 < \epsilon < 0$, $a_i$ satisfies
  $$\left[ a_i(x, s, \xi_i) - a_i(x, s, \xi_j) \right] (\xi_i - \xi_j) \leq \gamma (\epsilon + |\xi_i |^{p_i(x)-\epsilon}) |\xi_i - \xi_j |^{p_i(x)-\epsilon};$$  \hspace{1cm} (18)

- **(A3)** for a.e. $x \in \Omega$, for all $\xi \in \mathbb{R}^N$, $\zeta \in \mathbb{R}^N$, $i, j \in \{1, 2, \ldots, N\}$
  $$c_i |\xi |^2 \leq \sum_{i=1}^N \sum_{j=1}^N a_{ij}^i(x, s \xi_i \xi_j \xi_i \xi_j) \leq c_2 \left( 1 + \sum_{i=1}^N |\xi_i |^{p_i(x)-2} \right) |\xi |^2;$$  \hspace{1cm} (19)

where $a_{ij}^i(x, s, \xi) = \partial a_i(x, s \xi) / \partial \xi_j$;

- **(A4)** for a.e. $x \in \Omega$, $i = 1, 2, \ldots, N$, for some $k \in \{1, 2, \ldots, N\}$, for all $\xi \in \mathbb{R}^N$, $s \in \mathbb{R}$
  $$c_i |\xi |^2 \leq \sum_{i=1}^N a_{ik}^i(x, s \xi_i \xi_i \xi_i \xi_i) \leq c_4 \left( |\xi |^2 + |s|^{p_i(x)} \right),$$  \hspace{1cm} (20)

where $a_{ik}^i(x, s \xi_i \xi_i \xi_i \xi_i) = \partial a_i(x, s \xi_i \xi_i \xi_i \xi_i) / \partial x_k$;

(T1) for a.e. $x \in \Omega$, for all $\xi \in \mathbb{R}^N$, $s \in \mathbb{R}$,

$$\sum_{i=1}^N |T_i(x, \xi) | \leq c_2 |s|^{q(x)/2} \leq \sum_{i=1}^N m_i |\xi_i |^{p_i(x)-1},$$  \hspace{1cm} (21)

where $m_i \in L^{q_i(x)}(\Omega)$ with $1/b_i(x) = 1/p_\infty(x) - 1/p_i(x)$, $c_1, \ldots, c_4$ are positive constants.

We enumerate the hypotheses concerning $f$ and $h_i$,

- **(F1)** $f \in L^{1/p(x)}(\Omega)$;

- **(F2)** for a.e. $x \in \Omega$, $s \in \mathbb{R}$, for some $k \in \{1, 2, \ldots, N\}$, $|f_{x_i} | \leq c_6 |s|^{q(x)/2}$, where $f_{x_i} = \partial f / \partial x_i$;

- **(H1)** for $i = 1, 2, \ldots, N$, $h_i \in L^{1/p_i(x)}(\Omega)$;

- **(H2)** for a.e. $x \in \Omega$, $i = 1, 2, \ldots, N$, for some $k \in \{1, 2, \ldots, N\}$, $\zeta \in \mathbb{R}^N$, $s \in \mathbb{R}$,

$$c_7 |\zeta |^2 \leq \sum_{i=1}^N h_{x_i}^i \zeta_i \leq c_8 \left( |\zeta |^2 + |s|^{q(x)} \right),$$  \hspace{1cm} (22)

where $h_{x_i}^i = \partial h_i / \partial x_i$, $c_7, c_8, c_9$ are positive constants.

**Remark 4.** Now, we give a simple example. Let $k = 1, x_0 = (3, 0), x = (x_1, x_2) \in \Omega = B(x_0, 1) \subset \mathbb{R}^2$, $u(x) = x_3^2 + x_2^2$, $h_1(x) = x_1^2 + x_2^2 + x_3^2$, $T_1(x, \zeta) = x_1 + x_2 + \sum_{i=1}^2 \zeta_i$, $T_2(x, \zeta) = x_1 x_2 + |\zeta |^2$, $f(x) = (1/2)x_1^2 + x_1 x_2$. By a simple calculation, we obtain that $T_1$ satisfies (T1), $f$ satisfies (F2), and $T_2$ satisfies (H2), where $i = 1, 2$.

Now, we define the weak solution of the problem (1). A function $u \in W^{1,\mathbf{p}^-}(\Omega)$ is a weak solution of the problem (1), if for all $\varphi \in W^{1,\mathbf{p}^-}(\Omega)$,

$$\int_\Omega \left[ a_i(x, u, \nabla u) \varphi_{x_i} + T_i(x, \nabla u) \varphi \right] = \int_\Omega \left[ f \varphi + \sum_{i=1}^N h_i \varphi_{x_i} \right],$$  \hspace{1cm} (23)

where $\varphi_{x_i} = \partial \varphi / \partial x_i$.

**Theorem 5.** Let $\Omega \subset \mathbb{R}^N (N > 2)$ be a bounded open subset, for every $i = 1, 2, \ldots, N, x \in \Omega$, assume $\mathbf{p}(x) < N$, $q \in C^0(\Omega)$ and $q(x) < p_\infty = \mathbf{p}^-(x)$, $p_i(x)$ satisfies

$$2 \leq p_i(x) \leq \frac{2N r_i(x)}{N r_i(x) - 2r_i(x) - 2N}$$  \hspace{1cm} (24)

and

$$\sum_{i=1}^N (1/p_i(x)) < 2,$$  \hspace{1cm} (25)

where $r_i(x) \geq \max[N, p_\infty(x) p_i(x)/(p_\infty(x) - p_i(x))]$, $a_i$ satisfies the hypotheses (A1), (A2), the hypotheses (F1) and (H1), and $T_i$ satisfies (T1) and the following Lipschitz condition:

$$|T_i(x, \xi) - T_i(x, \xi')| \leq k_i(x) \left( |\xi | + |\xi' | \right)^{\delta(x)} |\xi - \xi'|$$  \hspace{1cm} (25)

where $k_i(x) \in L^{1/\delta(x)}(\Omega), 0 \leq \delta(x) \leq p_i(x)/N - p_i(x)/r_i(x) + (p_i(x) - 2)/2$. Then there exists a unique weak solution $u$ for the problem (1). Furthermore, $u \in L^{\infty}_{\text{loc}}(\Omega)$. 

Theorem 6. Let \( \Omega \subset \mathbb{R}^N \) (\( N > 2 \)) be a bounded open subset. Suppose (A3), (A4) and (F2), (H2) hold, for \( i = 1, 2, \ldots, N, \) \( \leq p_i(x) \leq 2N/(N-2), \) \( q(x) \in C_{0}^{1}(\Omega) \) and \( q(x) < \bar{p}^-(x). \) If \( u \) is a weak solution of the problem (2), then \( u \in C^{1,\alpha}(\Omega), \) for all \( 0 < \alpha < 1. \)

3. The Proof of Theorem 5

We consider the following problem:

\[-\partial_x q_i (x, v_n, \nabla v_n) + \sum_{i=1}^{N} T^i_n (x, \nabla v_n) = f - \nabla x_i h_i, \quad x \in \Omega, \quad u = 0, \quad x \in \partial \Omega,\]

(26)

where \( T^i_n(x, \nabla v) \) is the truncation at levels \( \pm n \) of \( T_i. \) Due to [15], we obtain that there exists a weak solution \( v_n \in W_{0}^{1,p(x)}(\Omega) \) for the problem (26).

Lemma 7. If \( \bar{p}(x) < N, (AI), (A2), (TI), (FI), \) and (H1) hold.

Assume \( v_n \in W_{0}^{1,p(x)}(\Omega) \) is a weak solution of the problem (26); then one has

\[
\sum_{i=1}^{N} \int_{\Omega} |\partial_x v_n|^{p_i(x)} \leq C. \quad (27)
\]

Proof. Let \( K \in \mathbb{R}^{+}; \) \( K \) will be chosen later. There exist \( m \) measurable subsets \( \Omega_1, \ldots, \Omega_m \) of \( \Omega \) and \( m \) functions \( v_1, \ldots, v_m \) such that \( i \neq j, \Omega_i \cap \Omega_j = \emptyset. \) For \( t \in \{1, 2, \ldots, m-1\}, \) \( |\Omega_t| = K, \) if \( t = m, |\Omega_t| \leq K. \) Let \( x \in \Omega : |\nabla v_i| \neq 0 \subset \Omega, \) \( \nabla v = \nabla v_i \) a.e. in \( \Omega, \) \( \nabla v = \nabla v_i + \nabla v_2 + \cdots + \nabla v_t = \nabla \psi \) in \( \Omega. \) For \( t \in \{1, 2, \ldots, m\}, \) if \( v_i \neq 0, \) \( \text{sign}(v) = \text{sign}(v_i). \)

Choose \( v_i \) as test function of the problem (26) and fix \( t \in \{1, 2, \ldots, m\}. \) Using Young inequality, Hölder inequality, the embedding theorem of the \( W_{0}^{1,p(x)}(\Omega), \) and the hypotheses (AI), (FI), and (H1), we have

\[
\sum_{i=1}^{N} \int_{\Omega} |\partial_x v_i|^{p(x)} \leq c_1 \left( \|f\|_{L_{p(x)}(\Omega)} A_t^{1/N} + \sum_{i=1}^{N} \int_{\Omega} |T^i(x, \nabla v)| v_i \right) + \sum_{i=1}^{N} \int_{\Omega} |h_i|^{p(x)} , \quad (28)
\]

where \( c_1 > 0 \) is a constant, \( A_t = \prod_{i=1}^{N} |\partial_x v_i|^{p(x)} |\Omega_i| \). By (TI), Young inequality and Hölder inequality, we obtain, for \( c_2 > 0, \)

\[
|\nabla v_i|^{p_i(x)-1} v_i \leq c_2 \left( \|f\|_{L_{p(x)}(\Omega)} A_t^{1/N} + \sum_{i=1}^{N} \int_{\Omega} |h_i|^{p(x)} \right) + \sum_{i=1}^{N} \int_{\Omega} |\partial_x v_i|^{p_i(x)} v_i \quad (29)
\]

Let \( K \) satisfy \( c_3 \sum_{i=1}^{N} K^{1/p_i(x)-1/p_i(x)} < 1; \) then we have

\[
\sum_{i=1}^{N} \int_{\Omega} |\partial_x v_i|^{p(x)} \leq c_3 \left( \|f\|_{L_{p(x)}(\Omega)} A_t^{1/N} + \sum_{i=1}^{N} \int_{\Omega} |h_i|^{p(x)} \right) + \sum_{i=1}^{N} \int_{\Omega} |\partial_x v_i|^{p_i(x)} + \sum_{i=1}^{N} \int_{\Omega} |\partial_x v_i|^{p_i(x)} + A_t^{1/N} . \quad (30)
\]
where \( c_3 > 0 \) is a constant. Let \( t = 1 \); we obtain
\[
\int_\Omega \left| \partial_x v_1 \right|^{p(x)} \\
\leq \sum_{i=1}^{N} \int_\Omega \left| \partial_x v_i \right|^{p(x)} \\
\leq c_3 \left[ \left\| f \right\|_{L^\infty(\Omega)} A_1^{1/N} + \sum_{i=1}^{N} \int_\Omega |h_i|^{p(x)} \\
+ \sum_{i=1}^{N} K^{1/p_i - 1/p_0} p_i A_i^{1/N} \right].
\]
(32)

Now, we recall the following classical inequality, for \( \alpha, \beta \geq 0 \), and \( \alpha \neq \beta \), we have
\[
\int_\Omega |u|^{p(x)} \leq \left( \frac{1}{\max \{ \alpha, \beta \}} \right)^{1/(\alpha - \beta)} \int_\Omega |u|^{\alpha p(x)} + \int_\Omega |u|^{\beta p(x)}.
\]
From (33), we have
\[
A_1 = \int_\Omega \left( \frac{1}{\max \{ \alpha, \beta \}} \right)^{1/(\alpha - \beta)} \int_\Omega |\partial_x v_i|^{p(x)} \leq \left( \frac{1}{\max \{ \alpha, \beta \}} \right)^{1/(\alpha - \beta)} \int_\Omega |\partial_x v_i|^{\alpha p(x)} + \int_\Omega |\partial_x v_i|^{\beta p(x)},
\]
(34)

Combining (32) with (34), there exists a constant \( c_4 \) such that
\[
\sum_{i=1}^{N} \int_\Omega |\partial_x v_i|^{p(x)} \leq c_4.
\]
(35)

Furthermore, put (35) in (31) and iterate on \( t \); we have
\[
\sum_{i=1}^{N} \int_\Omega |\partial_x v_i|^{p(x)} \\
\leq c_4 \left( \left\| f \right\|_{L^\infty(\Omega)} A_1^{1/N} + \sum_{i=1}^{N} \int_\Omega |h_i|^{p(x)} \\
+ 1 + \sum_{i=1}^{N} K^{1/p_i - 1/p_0} p_i A_i^{1/N} \right].
\]
(36)

Therefore, we obtain
\[
\sum_{i=1}^{N} \int_\Omega |\partial_x v_i|^{p(x)} \leq c_6,
\]
(37)

for some constant \( c_6 > 0 \).

Lemma 8. If \( p(x) < N \), and \( a_i \) satisfies the hypotheses (A1), (A2), the hypotheses (T1), (F1), and (H1) hold. Then there exists at least a weak solution for the problem (1).

Proof. By (27), we obtain that there exists a sequence \( u_n \) which is bounded in \( L^{p(x)}(\Omega) \); we have for \( i = 1, 2, \ldots, N \),
\[
\partial_x u_n \rightarrow \partial_x u \quad \text{a.e. in } \Omega \quad \text{for } i = 1, 2, \ldots, N.
\]
(38)

From (38), we obtain
\[
a_i (x, u_n, \nabla u_n) \rightarrow a_i (x, u, \nabla u) \quad \text{a.e. in } \Omega,
\]
(39)

\[
T_n^i (x, u_n) \rightarrow T_n^i (x, u) \quad \text{a.e. in } \Omega.
\]
(40)

By the hypotheses (A1) and (T1), for \( \kappa_i (x) \in [1, p'_i (x)] \) and \( c > 0 \), we have
\[
\int_\Omega \left| a_i (x, u_n, \nabla u_n) \right|^{\kappa_i (x)} \\
\leq c \left( \int_\Omega \left| f \right|^{p(x)} \right)^{\kappa_i (x)/p(x)} \\
+ \left( \int_\Omega \left| \partial_x u \right|^{p(x)} \right)^{\kappa_i (x)/p(x)} |\Omega|^{(p_i - \kappa_i)/p_i},
\]
(41)

for any \( \kappa_i (x) \in [1, p'_i (x)] \). Hence, we complete the proof of Lemma 8.

Lemma 9. If \( p(x) < N \), \( p_{co}(x) = p^*(x) \), \( a_i \) satisfies the hypotheses (A1), (A2), the hypotheses (F1) and (H1) hold, for \( i = 1, 2, \ldots, N, x \in \Omega \), \( p_i (x) \) satisfies
\[
2 \leq p_i (x) \leq \frac{2 N r_{1}(x)}{N r_{1}(x) - 2 r_{1}(x) - 2 N},
\]
(42)

where \( r_{1}(x) \geq \max \{ N, p_{co}(x) p_{i}(x)/(p_{co}(x) - p_{i}(x)) \} \), \( T_i \) satisfies (T1) and (25). Then there exists a unique weak solution for the problem (1).

Proof. Suppose that there exist two distinct weak solutions \( u \) and \( v \) for the problem (1). Denote \( \varphi = \max \{ u - v, 0 \} \), and for \( t \in [1, \sup \varphi] \), let \( \Omega_t = \{ x \in \Omega : t < \varphi < \sup \varphi \} \). Otherwise, if \( t \in [0, 1] \), we choose \( m = 1/t \) and let \( \Omega_m = \{ x \in \Omega : m < \varphi < \sup \varphi \} \). We choose
\[
\varphi_t = \begin{cases} \varphi - t, & \varphi > t, \\ 0, & \text{otherwise}, \end{cases}
\]
(43)

as test function. From (A2) and (25), we have
\[
\sum_{i=1}^{N} \int_{\Omega_t} \left| \partial_x \varphi_t \right|^{2} \left( \varepsilon + |\partial_x u| + |\partial_x v| \right)^{p_i(x) - 2} \\
\leq \frac{1}{\varepsilon} \sum_{i=1}^{N} \int_{\Omega_t} \kappa_i (x) \left( |\partial_x u| + |\partial_x v| \right)^{\delta_i(x)} |\partial_x \varphi_t| \varphi_t,
\]
(44)
when $\delta_i(x) \geq (p_i(x) - 2)/2$. By Young inequality, for $c$ independent on $t$, we obtain
\[
\sum_{i=1}^{N} \int_{\Omega_i} |\partial_x \varphi_i|^p \left(\epsilon + |\partial_x u| + |\partial_x v|\right)^{p(x) - 2} \leq c \sum_{i=1}^{N} \int_{\Omega_i} k_i(x)^2 \left( |\partial_x u| + |\partial_x v| \right)^{2 \delta_i(x)p(x) + 2} \varphi_i^2.
\]  
(45)

Then we have
\[
\frac{1}{C} \left( \int_{\Omega} \varphi_i^2 \right)^{2/N} \leq \sum_{i=1}^{N} \left( \int_{\Omega_i} |\partial_x \varphi_i|^p \left(\epsilon + |\partial_x u| + |\partial_x v|\right)^{p(x) - 2} \right)^{1/N} \leq c \sum_{i=1}^{N} \int_{\Omega_i} k_i(x)^2 \left( |\partial_x u| + |\partial_x v| \right)^{2 \delta_i(x)p(x) + 2} \varphi_i^2 \leq c_1 \left( \int_{\Omega} \varphi_i^2 \right)^{2/N} \times \sum_{i=1}^{N} \left( \int_{\Omega_i} k_i(x)^N \left( |\partial_x u| + |\partial_x v| \right)^{2 \delta_i(x)p(x) + 2(N/2)} \right)^{2/N},
\]
where $c_1$ is a constant independent on $t$. Hence, we have
\[
\frac{1}{C} \leq \sum_{i=1}^{N} \left( \int_{\Omega_i} k_i(x)^N \left( |\partial_x u| + |\partial_x v| \right)^{2 \delta_i(x)p(x) + 2(N/2)} \right)^{2/N}.
\]  
(46)

Since $N/r_i(x) + ((2 \delta_i(x) - p_i(x) + 2)/p_i(x))(N/2) \leq 1$, we have
\[
\lim_{r \to \sup \varphi} \int_{\Omega_i} k_i(x)^N \left( |\partial_x u| + |\partial_x v| \right)^{2 \delta_i(x)p(x) + 2(N/2)} = 0.
\]  
(48)

Therefore, (48) leads to a contradiction.

On the other hand, if $\delta_i(x) < (p_i(x) - 2)/2$, from (44), and Young inequality, for some constant $c_2$ (independent on $t$), we have
\[
\int_{\Omega_i} |\partial_x \varphi_i|^p \leq c_2 \sum_{i=1}^{N} \int_{\Omega_i} k_i(x)^2 \left( |\partial_x u| + |\partial_x v| \right)^{2 \delta_i(x)p(x) + 2} \varphi_i^2. \]
\]  
(49)

Argue as in (46), for some constant $c_3$ (independent on $t$), we have
\[
\frac{1}{c} \leq c_3 \sum_{i=1}^{N} \left( \int_{\Omega_i} k_i(x)^N \left( |\partial_x u| + |\partial_x v| \right)^{2 \delta_i(x)p(x)} \right)^{2/N}.
\]
(50)

Since $N/r_i(x) + N \delta_i(x)/p_i(x) \leq 1$, we obtain a contradiction. \hfill \Box

**Lemma 10.** If (A1), (TI), (Fl), and (HI) hold, for $i = 1, 2, \ldots, N$, $2 \leq p_i(x)$ and $\sum_{i=1}^{N}(1/p_i(x)) < 2$, $q \in C^0_0(\Omega)$ and $q(x) < p_{\infty} = \overline{p}(x)$, and $u$ is a weak solution of the problem (1), then $u \in L_{\text{loc}}^{\infty}(\Omega)$.

**Proof.** We denote $E = W_{0,1}^{1,0}(\Omega)$. Let $t > 0, \varphi = \max\{u - t, 0\}$, and choose $\varphi$ as test function to the problem (1). We have
\[
\sum_{i=1}^{N} \int_{\Omega} \left[ a_i(x, u, \nabla u) \varphi_x_i + T_i(x, \nabla u) \varphi \right]
\]  
\[
= \sum_{i=1}^{N} \int_{\Omega} \left[ f_{\varphi} + \sum_{i=1}^{N} h_i \varphi_x_i \right].
\]
(51)

Let $x_0 \in \Omega$, fix $k > 1$, $B(x_0, R)$ be a ball, and let $\delta \in (0,1)$, $\delta R < s < t < R$, and $A_{k,s} = \{ x \in B(x_0, t), u(x) > k \}$. Using (51), (A1), (Fl), (HI), and Hölder inequality, we obtain
\[
\sum_{i=1}^{N} \int_{A_{k,s}} |\partial_x u|^p \leq \sum_{i=1}^{N} \int_{A_{k,s}} |\partial_x \varphi|^p \leq c \left[ \|f\|_{L^q(\Omega)} \|u - k\|_{L^p(\Omega)} \right] + \sum_{i=1}^{N} \int_{A_{k,s}} |T_i(x, \nabla u)| |u - k|
\]
\[
+ \sum_{i=1}^{N} \int_{A_{k,s}} h_i |\partial_x u|.
\]
(52)

For $2 \leq p_i(x)$ and (TI), by Young inequality, we have
\[
\sum_{i=1}^{N} \int_{A_{k,s}} |T_i(x, \nabla u)| |u - k|
\]
\[
\leq \sum_{i=1}^{N} \left( \frac{1}{2} \int_{A_{k,s}} |u - k|^2 + \frac{1}{2} \int_{A_{k,s}} |T_i(x, \nabla u)|^2 \right)
\]
\[
\leq \frac{1}{2} \sum_{i=1}^{N} \int_{A_{k,s}} |u - k|^p + \frac{1}{2} \sum_{i=1}^{N} \int_{A_{k,s}} |u|^{q(x)}
\]
\[
\leq c \left( \int_{A_{k,s}} |u - k|_{\overline{p}(x)}^p \right)^{1/2} + \frac{1}{2} \sum_{i=1}^{N} \int_{A_{k,s}} |\partial_x u|^2
\]
\[
\leq \frac{1}{2} \sum_{i=1}^{N} \int_{A_{k,s}} h_i^2 + \frac{1}{2} \sum_{i=1}^{N} \int_{A_{k,s}} |\partial_x \varphi|^p(x).
\]
(53)

As $2 \leq p_i(x)$ and $\sum_{i=1}^{N}(1/p_i(x)) < 2$, if $\overline{p}(x) < N$, there exists $\epsilon = 1/2 - \overline{p}(x)/\overline{p}(x) > 0$ such that
\[
c \left( \int_{A_{k,s}} h_i^2 \right)^{1/2} |A_{k,s}|^{1/2} \leq c |A_{k,s}|^{\overline{p}(x)/\overline{p}(x) + \epsilon}.
\]
(54)
If $p(x) \geq N$, a similar estimate is true; just only choose a suitable $p(x)$.

Due to $q \in C^0_+ (\Omega)$ and $q(x) < p_{co}(x) = \overline{p}(x)$, we have

$$\|f\|_{L^p(\Omega)} \leq c \int_{A_{kj}} |u|^q(x).$$

(55)

From

$$\int_{A_{kj}} |u|^q(x) = \int_{A_{kj}} |(u - k) + k|^q(x),$$

$$\leq c \int_{A_{kj}} |u - k|^q(x) + c k^q |A_{kj}|$$

(56)

$$\leq c \int_{A_{kj}} \frac{|u - k|^q(x)}{t - s} + c k^q |A_{k,l}|,$$

and (52)–(55), we obtain

$$\int_{A_{kj}} |\partial_x u|^q(x) \leq \frac{1}{2} \int_{A_{kj}} |\partial_x u|^p(x) + c \int_{A_{kj}} \frac{|u - k|^q(x)}{t - s} + c k^q |A_{k,l}| + c |A_{kj}| |k|^q(x)/p(x) + \epsilon.$$ (57)

Using Lemma 3.1 in [17], and (57), we have

$$\int_{A_{kj}} |\partial_x u|^q(x) \leq c \int_{A_{kj}} \frac{|u - k|^q(x)}{R - \delta R} + c k^q |A_{k,l}| + c |A_{kj}| |k|^q(x)/p(x) + \epsilon.$$ (58)

Then, combining (60) with (61)-(62), we obtain

$$\sum_{i=1}^{N} \int_{\Omega} \Delta^k_m a^i (x, Du) \phi_i = \int_{\Omega} \Delta^k_m f \phi - \sum_{i=1}^{N} \int_{\Omega} \Delta^k_m h_i \phi_i = 0.$$ (60)

For the definition of $\Delta^k_m$ and [19], we have

$$\Delta^k_m a^i (x, Du) = \frac{1}{m} \int_0^1 \frac{d}{dt} \left[ a^i \left( x + t me_k, Du + t m \Delta^k_m Du \right) \right] dt$$

(61)

$$= \int_0^1 \left[ a^i_{s_k} + \sum_{j=1}^{N} a^i_{t_j} D \left( \Delta^k_m u \right) \right] dt$$

(62)

where $e^i = \int_0^1 a^i_{s_k} dt$, $g_{ij} = \int_0^1 \sum_{j=1}^{N} a^i_{t_j} dt$, and

$$\Delta^k_m f = \frac{1}{m} \int_0^1 \frac{d}{dt} f \left( x + t me_k \right) dt$$

(63)

$$= \int_0^1 \frac{d}{dt} f \left( x + t me_k \right) dt = \int_0^1 h_i \phi dt = \Delta^k_m.$$

By Lemma 2.4 in [18], (58), we obtain that $u$ is bounded from above on $B(x_0, R/2)$. Note that $-u$ is also a weak solution of the problem (1), where $\tilde{a}_i (x, u, \nabla u) = a_i (x, -u, -\nabla u)$ and $\tilde{T}_i (x, \nabla u) = T_i (x, -\nabla u)$. Hence, $-u$ is also bounded from above on $B(x_0, R/2)$, and $u \in L^\infty (B(x_0, R/2))$. This implies that $u \in L^\infty (\Omega)$.

Proof of Theorem 5. From Lemmas 8, 9, and 10, we obtain that

$$\sum_{i=1}^{N} \int_{\Omega} \Delta^k_m a^i \phi_i = \int_{\Omega} \Delta^k_m f \phi - \sum_{i=1}^{N} \int_{\Omega} \Delta^k_m h_i \phi_i = 0.$$ (63)

Due to (A4), (F2), (H2), (63), Young inequality, we have

$$\sum_{i=1}^{N} \int_{\Omega} g_{ij} D \left( \Delta^k_m u \right) D \phi_i = \int_{\Omega} f \phi + \sum_{i=1}^{N} \int_{\Omega} H^i D \phi - \sum_{i=1}^{N} \int_{\Omega} e^i D \phi$$

(64)

$$\leq c \int_{\Omega} |D \phi|^2 + c \int_{\Omega} \phi^2 + c \int_{\Omega} |u|^{\phi(x)/2}$$

$$\leq c \int_{\Omega} |D \phi|^2 + c \int_{\Omega} \phi^2 + c \int_{\Omega} \phi^2.$$
that is,
\[
\sum_{i,j=1}^{N} \int_{\Omega} g_{ij} D_{j}(\Delta_{m}^{k} u) D_{i} \phi \leq c \int_{\Omega} |D\phi|^{2} + c \int_{\Omega} |u|^{q(x)} + c \int_{\Omega} \phi^{2}.
\]
(65)

Choose \( \phi = \psi^{2} \Delta_{m}^{k} u \), where \( \psi \in C_{0}^{1}(\Omega), |\psi| \leq 1 \), and (65), we obtain
\[
\sum_{i,j=1}^{N} \int_{\Omega} g_{ij} \psi^{2} D_{i}(\Delta_{m}^{k} u) D_{j}(\Delta_{m}^{k} u) \leq -2 \sum_{i,j=1}^{N} \int_{\Omega} \psi u g_{ij} D_{i} \psi D_{j}(\Delta_{m}^{k} u) + c \int_{\Omega} |D\psi|^{2} \Delta_{m}^{k} u
\]

\[+ c \int_{\Omega} |u|^{q(x)} + c \int_{\Omega} (\psi^{2} \Delta_{m}^{k} u)^{2}.
\]
(66)

From (66), (A3), Cauchy inequality and Young inequality, we have
\[
\sum_{i,j=1}^{N} \int_{\Omega} g_{ij} \psi^{2} D_{i}(\Delta_{m}^{k} u) D_{j}(\Delta_{m}^{k} u)
\]
\[\leq 2 \sum_{i,j=1}^{N} \int_{\Omega} |\psi u g_{ij} D_{i} \psi D_{j}(\Delta_{m}^{k} u)|
\]

\[+ c \int_{\Omega} |D\psi|^{2} \Delta_{m}^{k} u^{2} + c \int_{\Omega} |\psi|^{2} \Delta_{m}^{k} u^{2}
\]
\[\leq 2 \sum_{i,j=1}^{N} \int_{\Omega} g_{ij} \psi^{2} D_{i}(\Delta_{m}^{k} u) D_{j}(\Delta_{m}^{k} u) \leq \frac{1}{2} \left[ \int_{\Omega} \psi^{2} |D(\Delta_{m}^{k} u)|^{2} \right]^{1/2} + c \int_{\Omega} |D\psi|^{2} \Delta_{m}^{k} u^{2}
\]

\[+ c \int_{\Omega} |\psi|^{2} \Delta_{m}^{k} u^{2} \leq 2 \sum_{i,j=1}^{N} \left[ \frac{1}{2} \left( \int_{\Omega} g_{ij} \psi^{2} D_{i}(\Delta_{m}^{k} u) D_{j}(\Delta_{m}^{k} u) \right) + c \int_{\Omega} |D\psi|^{2} \Delta_{m}^{k} u^{2}
\]

\[+ c \int_{\Omega} |\psi|^{2} |D(\Delta_{m}^{k} u)|^{2} \leq 2 \sum_{i,j=1}^{N} \left[ \frac{1}{2} \int_{\Omega} g_{ij} \psi^{2} D_{i}(\Delta_{m}^{k} u) D_{j}(\Delta_{m}^{k} u) \right]^{1/2} \leq c \int_{\Omega} |\psi|^{2} |D(\Delta_{m}^{k} u)|^{2}
\]
(67)

By (A3), we obtain
\[
\int_{\Omega} \psi^{2} |D(\Delta_{m}^{k} u)|^{2}
\]
\[\leq c \int_{\Omega} \left[ 1 + \sum_{i=1}^{N} \int_{\Omega} |D_{i} u(x)|^{p_{i}-2} + c \int_{\Omega} |D_{i} u(x + me_{i})|^{p_{i}-2} \right]
\]

\[\times |D(\Delta_{m}^{k} u)|^{2} |D\psi|^{2} + c \int_{\Omega} |u|^{q(x)}.
\]
(68)

Assume \( p_{N} = p_{N}^{*} = \sup_{x \in \Omega} p_{i}(x) \); we have for \( i = 1, 2, \ldots, N \), \( p_{i}(x) \leq p_{N} \). For \( q(x) < p_{i}^{*}(x) \), let \( k = N \) and take \( m \to 0 \); then we have
\[
\int_{\Omega} \psi^{2} |D(\Delta_{N} u)|^{2} \leq c \int_{\Omega} \left( 1 + \sum_{i=1}^{N} |D_{i} u|^{p_{i}-2} \right) |D\psi|^{2} |D_{N} u|^{2}
\]
(69)

Hence, \( D_{N} u \in W^{1,2}_{loc}(\Omega) \). If we take \( \phi = \psi^{2}(\Delta_{N}^{k} u - r)^{+} \), where \( r > 0, |\psi| < 1, (\Delta_{N}^{k} u - r)^{+} = \max(\Delta_{N}^{k} u - r, 0) \). Since \( D_{N} u \in W^{1,2}_{loc}(\Omega), D_{N} u \in L^{1/(N-2)}_{loc}(\Omega) \), we have
\[
\int_{\Omega} \psi^{2} |D(\Delta_{N}^{k} u - r)|^{2}
\]
\[\leq c \int_{\Omega} \left( 1 + \sum_{i=1}^{N} |D_{i} u|^{p_{i}-2} \right) |D\psi|^{2} |(D_{N} u - r)^{+}|^{2}
\]
\[\leq c \int_{\Omega} \left( 1 + \sum_{i=1}^{N} |D_{i} u|^{p_{i}-2} \right) |D\psi|^{2} |(D_{N} u - r)^{+}|^{2}
\]
\[+ c \int_{\Omega} |D_{N} u|^{p_{N}-2} |D\psi|^{2} |(D_{N} u - r)^{+}|^{2}
\]
\[+ c \sum_{i=1}^{N} \int_{\Omega} \left( |D_{N} u - r|^{+} |D\psi|^{2} + c r^{p_{N}-2} \int_{\Omega} |D_{N} u - r|^{2} \right)^{2/(N-2)}
\]
\[\times \left( \int_{\Omega} |D_{N} u - r|^{+} \right)^{2/(N-2)}.
\]
(70)

For \( 2 \leq p_{i}(x) \leq 2N/(N - 2) \), we have
\[
\left( \int_{\Omega} |D_{N} u|^{p_{i}(x)-2} \right)^{2/(N-2)} \leq c \left( \int_{\Omega} |D_{N} u - k| \cap \sup \psi \right)^{2/(N-2)/(p_{i}-1)}
\]
(71)
If we fix $B(x_0, R)$, supp $\psi \subset B(x_0, R)$, where $R < 1$; let $r \geq 1$, $0 < \delta < 1$, and take $\psi \in C_0^\infty(B(x_0, R))$, $\psi \equiv 1$ on $B(x_0, \delta R)$; $0 \leq \psi \leq 1$, $|D\psi| \leq c/(R - \delta R)$, then we have
\[
\left\{ A_{r,R} \right\} \left| D(Nu - r) \right|^2 \leq c \int_{A_{r,R}} \left| D(Nu - r) \right|^2 (2N/(N-2)) \\
+ c\left( \int_{A_{r,R}} \left| D(Nu - r) \right| (2N/(N-2))(N-2)/N \right)^{N-2} \left| A_{r,R} \right|.
\]
Hence, (72) implies that $D_N u$ is in $L^2_{loc}(\Omega)$. So we obtain $u \in W^{1,\infty}(\Omega) \cap W^{2,2}_{0,loc}(\Omega)$. By De Giorgi-Moser regularity theorem, for any $\omega \subset \subset \Omega$ and $0 < \alpha < 1$, we have $D_N u \in C^{\alpha,\gamma}(\omega)$; then $u \in C^{\alpha,\gamma}(\Omega)$, for $0 < \alpha < 1$.

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References

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