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Research Article

Wijsman Orlicz Asymptotically Ideal ϕ-Statistical Equivalent Sequences

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An ideal I is a family of subsets of positive integers $\mathbb{N}$ which is closed under taking finite unions and subsets of its elements. In this paper, we introduce a new definition of asymptotically ideal ϕ-statistical equivalent sequence in Wijsman sense and present some definitions which are the natural combination of the definition of asymptotic equivalence, statistical equivalent, ϕ-statistical equivalent sequences in Wijsman sense. Finally, we introduce the notion of Cesaro Orlicz asymptotically ϕ-equivalent sequences in Wijsman sense and establish their relationship with other classes.

1. Preliminaries and Notations

Let us start with a basic definition from the literature. Let $K \subseteq \mathbb{N}$, the set of all natural numbers, and $K_n = \{k \leq n : k \in K\}$. Then, the *natural density* of K is defined by $\delta(K) = \lim_n n^{-1}|K_n|$ if the limit exists, where the vertical bars indicate the number of elements in the enclosed set.

Fast [1] presented the following definition of statistical convergence for the sequences of real numbers. The sequence $x = (x_k)$ is said to be *statistically convergent* to L if for every $\varepsilon > 0$, the set $K_\varepsilon := \{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}$ has natural density zero, that is, for each $\varepsilon > 0$,

$$\lim_n \frac{1}{n} \left|\{k \leq n : |x_k - L| \geq \varepsilon\}\right| = 0. \tag{1}$$

In this case, we write $\mathcal{S}\text{-}\lim x = L$ or $x_k \to L(\mathcal{S})$ and $\mathcal{S}$ denotes the set of all statistically convergent sequences. Note that every convergent sequence is statistically convergent but not conversely. In 1985, Fridy [2] presented the notion of statistically Cauchy sequence and determined that it is equivalent to statistical convergence. Some basic properties related to the concept of statistical convergence were studied in [3–5] where many important references can be found.

Kostyrko et al. [6] introduced the notion of I-convergence with the help of an admissible ideal Iwhich denotes the ideal of subsets of $\mathbb{N}$, which is a generalization of statistical convergence. Quite recently, Das et al. [7] unified these two approaches to introduce new concepts such as I-statistical convergence and I-lacunary statistical convergence and investigated some of their consequences. For more applications of ideals we refer to [8–21] where many important references can be found.

A family of sets $I \subset P(\mathbb{N})$ (power sets of $\mathbb{N}$) is said to be an *ideal* if and only if

(a) $\phi \in I$,

(b) for each $A, B \in I$, we have $A \cup B \in I$,

(c) for each $A \in I$ and each $B \subset A$, we have $B \in I$.

A nonempty family of sets $\mathcal{F} \subset P(\mathbb{N})$ is said to be a *filter* on $\mathbb{N}$ if and only if

(a) $\phi \notin \mathcal{F}$,

(b) for each $A, B \in \mathcal{F}$, we have $A \cap B \in \mathcal{F}$,

(c) for each $A \in \mathcal{F}$ and each $B \supset A$, we have $B \in \mathcal{F}$.

An ideal I is called a nontrivial ideal if $I \neq \phi$ and $\mathbb{N} \notin I$. Clearly $I \subset P(\mathbb{N})$ is a non-trivial ideal if and only if $\mathcal{F} = \mathcal{F}(I) = \{\mathbb{N} - A : A \in I\}$ is a filter on $\mathbb{N}$.

A non-trivial ideal $I \subset P(\mathbb{N})$ is called

(a) *admissible* if and only if $\{\{x\} : x \in \mathbb{N}\} \subset I$,

(b) *maximal* if there cannot exist any non-trivial ideal $J \neq I$ containing I as a subset.

Recall that a sequence $x = (x_k)$ of points in $\mathbb{R}$ is said to be I-convergent to a real number ℓ if for every $\varepsilon >$, $\{k \in \mathbb{N} : |x_k - \ell| \geq \varepsilon\} \in I$ (see [6]). In this case, we write $I\text{-}\lim x_k = \ell$.

Throughout the paper, we denote that I is an admissible ideal of subsets of $\mathbb{N}$, unless otherwise stated.

Pobyvanets [22] introduced the concept of asymptotically regular matrices, which preserve the asymptotic equivalence of two nonnegative numbers sequences. Marouf [23] presented definitions for asymptotically equivalent and asymptotic regular matrices. Li [24] introduced the concept asymptotic equivalent sequences using summability. Patterson [25] extend these concepts by presenting an asymptotically statistical equivalent analog of these definitions and natural regularity conditions for nonnegative summability matrices. Patterson and Savaş [26] introduced the concept of an asymptotically lacunary statistical equivalent sequences of real numbers. Braha [27] presented the notion of Δ^m-lacunary statistical equivalent real sequences. Savaş [28] introduced the concept of asymptotically lacunary statistical equivalent sequences via ideals. Kumar and Sharma [29] introduced the generalized equivalent sequences of real numbers using ideals and studied some basic properties of this notion.

The concept of the convergence of sequences of points has been extended by several authors to the convergence of sequences of sets. One of these extensions considered in this paper is the concept of Wijsman convergence. The concept of Wijsman statistical convergence which is the implementation of the concept of statistical convergence to sequences of sets is presented by Nuray and Rhoades [30]. Hazarika and Esi [31], introduced the concept of statistical almost λ-convergence of sequences of sets. For more works on convergence of sequences of sets, we refer to [32–38].

Now we recall the definitions that are being used throughout the paper.

Definition 1 (see [23]). Two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically equivalent if

$$\lim_k \frac{x_k}{y_k} = 1, \tag{2}$$

denoted by $x \sim y$.

Definition 2 (see [25]). Two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically statistical equivalent of multiple L provided that, for every $\varepsilon > 0$,

$$\lim_n \frac{1}{n} \left| \left\{ k \leq n : \left| \frac{x_k}{y_k} - L \right| \geq \varepsilon \right\} \right| = 0, \tag{3}$$

denoted by $x \overset{\mathcal{S}^L}{\sim} y$ and simply asymptotically statistical equivalent if $L = 1$.

Definition 3 (see [29]). Two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be strongly asymptotically I-equivalent of multiple L provided that, for each $\varepsilon > 0$,

$$\left\{ n \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^{n} \left| \frac{x_k}{y_k} - L \right| \geq \varepsilon \right\} \in I, \tag{4}$$

denoted by $x \overset{I-[C_1]^L}{\sim} y$ and simply strongly asymptotically I-equivalent if $L = 1$.

Definition 4 (see [29]). Two non-negative sequences $x = (x_k)$ and $y = (y_k)$ are said to be I-asymptotically statistical equivalent of multiple L provided that, for every $\varepsilon > 0$ and for every $\delta > 0$,

$$\left\{ n \in \mathbb{N} : \frac{1}{n} \left| \left\{ k \leq n : \left| \frac{x_k}{y_k} - L \right| \geq \varepsilon \right\} \right| \geq \delta \right\} \in I, \tag{5}$$

denoted by $(x_k) \overset{I-\mathcal{S}^L}{\sim} (y_k)$ and simply I-asymptotically statistical equivalent if $L = 1$.

Definition 5 (see [7]). A sequence (x_k) of real numbers is said to be I-statistically convergent to a real number x_0 for each $\varepsilon > 0$ and $\delta > 0$,

$$\left\{ n \in \mathbb{N} : \frac{1}{n} \left| \{ k \leq n : |x_k - x_0| \geq \varepsilon \} \right| \geq \delta \right\} \in I. \tag{6}$$

In this case, we write $I - \mathcal{S} - \lim x_k = x_0$.

Let (X, ρ) be a metric space. For any point $x \in X$ and any non-empty subset $A \subset X$, the distance from x to A is defined by

$$d(x, A) = \inf_{y \in A} \rho(x, y). \tag{7}$$

Definition 6 (see [33]). Let (X, ρ) be a metric space. For any non-empty closed subsets $A, A_k \subset X$ $(k \in \mathbb{N})$, we say that the sequence (A_k) is Wijsman convergent to A if $\lim_k d(x, A_k) = d(x, A)$ for each $x \in X$. In this case, we write $W\text{-}\lim A_k = A$.

The concepts of Wijsman statistical convergence and boundedness for the sequence (A_k) were given by Nuray and Rhoades [30] as follows.

Definition 7 (see [30]). Let (X, ρ) be a metric space. For any non-empty closed subsets $A, A_k \subset X$ $(k \in \mathbb{N})$, we say that the sequence (A_k) is Wijsman statistical convergent to A if the sequence $(d(x, A_k))$ is statistically convergent to $d(x, A)$, that is, for $\varepsilon > 0$ and for each $x \in X$

$$\lim_n \frac{1}{n} \left| \{ k \leq n : |d(x, A_k) - d(x, A)| \geq \varepsilon \} \right| = 0. \tag{8}$$

In this case, we write $W\mathcal{S}\text{-}\lim_k A_k = A$ or $A_k \to A(W\mathcal{S})$.

The sequence (A_k) is bounded if $\sup_k d(x, A_k) < \infty$ for each $x \in X$. The set of all bounded sequences of sets is denoted by L_∞.

Ulusu and Nuray in [39] defined asymptotically equivalent and asymptotically statistical equivalent sequences of sets as follows.

Definition 8. Let (X, ρ) be a metric space. For any non-empty closed subsets $A_k, B_k \subseteq X$ such that $d(x, A_k) > 0$ and $d(x, B_k) > 0$ for each $x \in X$. We say that the sequences (A_k)

and (B_k) are asymptotically equivalent (Wijsman sense) if, for each $x \in X$,

$$\lim_k \frac{d(x, A_k)}{d(x, B_k)} = 1, \tag{9}$$

denoted by $(A_k) \sim (B_k)$.

Definition 9. Let (X, ρ) be a metric space. For any non-empty closed subsets $A_k, B_k \subseteq X$ such that $d(x, A_k) > 0$ and $d(x, B_k) > 0$ for each $x \in X$. We say that the sequences (A_k) and (B_k) are asymptotically statistical equivalent (Wijsman sense) of multiple L if, for every $\varepsilon > 0$ and for each $x \in X$,

$$\lim_n \frac{1}{n} \left| \left\{ k \le n : \left| \frac{d(x, A_k)}{d(x, B_k)} - L \right| \ge \varepsilon \right\} \right| = 0, \tag{10}$$

denoted by $(A_k) \overset{W\mathcal{S}^L}{\sim} (B_k)$, and simply asymptotically statistical equivalent (Wijsman sense) if $L = 1$.

2. Wijsman Orlicz Asymptotically Ideal ϕ-Statistical Equivalent Sequences

In this section, we define the notion of Cesaro Orlicz asymptotically ϕ-statistical equivalent and Orlicz asymptotically ideal ϕ-statistical equivalent sequences in Wijsman sense and establish some interesting relationships between these notions.

Let P denote the space whose elements are finite sets of distinct positive integers. Given any element σ of P, we denote by $p(\sigma)$ the sequence $\{p_n(\sigma)\}$ such that $p_n(\sigma) = 1$ for $n \in \sigma$ and $p_n(\sigma) = 0$ otherwise. Further,

$$P_s = \left\{ \sigma \in P : \sum_{n=1}^{\infty} p_n(\sigma) \le s \right\}; \tag{11}$$

that is, P_s is the set of those σ whose support has cardinality at most s, and we get

$$\begin{aligned} \Phi = \{ & \phi = (\phi_n) : 0 < \phi_1 \le \phi_n \le \phi_{n+1}, n\phi_{n+1} \\ & \le (n+1)\phi_n \}. \end{aligned} \tag{12}$$

Let (X, ρ) be a metric space. For any non-empty closed subsets $A_k \subset X$ $(k \in \mathbb{N})$. We define

$$\tau_s = \frac{1}{\phi_s} \sum_{k \in \sigma, \sigma \in P_s} d(x, A_k). \tag{13}$$

Now, we give the following definitions.

Definition 10. Let (X, ρ) be a metric space. For any non-empty closed subsets $A, A_k \subset X$ $(k \in \mathbb{N})$, we say that the sequence (A_k) is ϕ-summable to A (Wijsman sense) if $\lim_s (1/\phi_s) \sum_{k \in \sigma, \sigma \in P_s} d(x, A_k) = d(x, A)$. In this case, we write $W\phi - \lim_k A_k = A$.

Definition 11. Let (X, ρ) be a metric space. For any non-empty closed subsets $A, A_k \subset X$ $(k \in \mathbb{N})$, we say that the sequence (A_k) is strongly ϕ-summable to A (Wijsman sense) if

$$\lim_{s \to \infty} \frac{1}{\phi_s} \sum_{k \in \sigma, \sigma \in P_s} \left| d(x, A_k) - d(x, A) \right| = 0. \tag{14}$$

In this case, we write $W[\phi] - \lim_k A_k = A$ or $A_k \xrightarrow{W[\phi]} A$ and $W[\phi]$ denote the set of all Wijsman strongly ϕ-summable sequences.

Definition 12. Let $E \subseteq \mathbb{N}$. The number

$$\delta_\phi(E) = \lim_{s \to \infty} \frac{1}{\phi_s} \left| \{ k \in \sigma, \sigma \in P_s : k \in E \} \right| \tag{15}$$

is said to be the ϕ-density of E. It is clear that $\delta_\phi(E) \le \delta(E)$.

Definition 13. Let (X, ρ) be a metric space. For any non-empty closed subsets $A, A_k \subset X$ $(k \in \mathbb{N})$, we say that the sequence (A_k) is ϕ-statistical convergent to A (Wijsman sense) if, for each $\varepsilon > 0$,

$$\lim_{s \to \infty} \frac{1}{\phi_s} \left| \{ k \in \sigma, \sigma \in P_s : \left| d(x, A_k) - d(x, A) \right| \ge \varepsilon \} \right| = 0. \tag{16}$$

In this case we write $W\mathcal{S}_\phi - \lim_k A_k = A$ or $A_k \xrightarrow{W\mathcal{S}_\phi} A$ and $W\mathcal{S}_\phi$ denote the set of all Wijsman ϕ-statistically convergent sequences.

Definition 14. Let (X, ρ) be a metric space. For any non-empty closed subsets $A_k, B_k \subseteq X$ such that $d(x, A_k) > 0$ and $d(x, B_k) > 0$ for each $x \in X$. We say that the sequences (A_k) and (B_k) are

(i) strongly asymptotically ϕ-equivalent (Wijsman sense) of multiple L provided that

$$\lim_s \frac{1}{\phi_s} \sum_{k \in \sigma, \sigma \in P_s} \left| \frac{d(x, A_k)}{d(x, B_k)} - L \right| = 0, \tag{17}$$

denoted by $(A_k) \overset{W[\phi]^L}{\sim} (B_k)$, and simply Wijsman strongly asymptotically ϕ-equivalent if $L = 1$;

(ii) asymptotically ϕ-statistical equivalent (Wijsman sense) of multiple L provided that, for every $\varepsilon > 0$,

$$\lim_s \frac{1}{\phi_s} \left| \left\{ k \in \sigma, \sigma \in P_s : \left| \frac{d(x, A_k)}{d(x, B_k)} - L \right| \ge \varepsilon \right\} \right| = 0, \tag{18}$$

denoted by $(A_k) \overset{W\mathcal{S}_\phi^L}{\sim} (B_k)$, and simply Wijsman asymptotically ϕ-statistical equivalent if $L = 1$;

(iii) I-asymptotically ϕ-statistical equivalent (Wijsman sense) of multiple L provided that, for every $\varepsilon > 0$ and for every $\delta > 0$,

$$\left\{s \in \mathbb{N} : \frac{1}{\phi_s}\left|\left\{k \in \sigma, \sigma \in P_s : \left|\frac{d(x, A_k)}{d(x, B_k)} - L\right| \geq \varepsilon\right\}\right| \geq \delta\right\} \in I \tag{19}$$

denoted by $(A_k) \overset{I-W\mathcal{S}_\phi^L}{\sim} (B_k)$, and simply Wijsman I-asymptotically ϕ-statistical equivalent if $L = 1$.

Recall that an Orlicz function is a function $M : [0, \infty) \to [0, \infty)$ which is continuous, nondecreasing, and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. An Orlicz function M is said to satisfy the Δ_2-*condition* for all values of u, if there exists a constant $K > 0$ such that $M(2u) \leq KM(u)$, $u \geq 0$. Note that if $0 < \lambda < 1$, then $M(\lambda x) \leq \lambda M(x)$ for all $x \geq 0$ (see [40]).

Now, we give the following definitions.

Definition 15. Let (X, ρ) be a metric space. For any non-empty closed subsets $A_k, B_k \subseteq X$ such that $d(x, A_k) > 0$ and $d(x, B_k) > 0$ for each $x \in X$. We say that the sequences (A_k) and (B_k) are as follows.

(i) Cesaro Orlicz asymptotically equivalent (Wijsman sense) of multiple L provided that

$$\lim_n \frac{1}{n}\sum_{k=1}^{n} M\left(\left|\frac{d(x, A_k)}{d(x, B_k)} - L\right|\right) = 0, \tag{20}$$

denoted by $(A_k) \overset{W[C_1]^L(M)}{\sim} (B_k)$, and simply Wijsman Cesaro Orlicz asymptotically equivalent if $L = 1$.

(ii) Cesaro Orlicz I-asymptotically equivalent of multiple L provided that, for every $\delta > 0$

$$\left\{n \in \mathbb{N} : \frac{1}{n}\sum_{k=1}^{n} M\left(\left|\frac{d(x, A_k)}{d(x, B_k)} - L\right|\right) \geq \delta\right\} \in I, \tag{21}$$

denoted by $(A_k) \overset{I-W[C_1]^L(M)}{\sim} (B_k)$ and simply Wijsman Cesaro Orlicz I-asymptotically equivalent if $L = 1$.

(iii) Orlicz asymptotically ϕ-equivalent (Wijsman sense) of multiple L provided that

$$\lim_s \frac{1}{\phi_s}\sum_{k\in\sigma, \sigma\in P_s} M\left(\left|\frac{d(x, A_k)}{d(x, B_k)} - L\right|\right) = 0, \tag{22}$$

denoted by $(A_k) \overset{W[\phi]^L(M)}{\sim} (B_k)$, and simply Wijsman Orlicz asymptotically ϕ-equivalent if $L = 1$.

(iv) Orlicz I-asymptotically ϕ-equivalent (or $I - [\phi]$-equivalent) (Wijsman sense) of multiple L provided that, for every $\delta > 0$,

$$\left\{s \in \mathbb{N} : \frac{1}{\phi_s}\sum_{k\in\sigma, \sigma\in P_s} M\left(\left|\frac{d(x, A_k)}{d(x, B_k)} - L\right|\right) \geq \delta\right\} \in I, \tag{23}$$

denoted by $(A_k) \overset{I-W[\phi]^L(M)}{\sim} (B_k)$, and simply Wijsman Orlicz I-asymptotically ϕ-equivalent if $L = 1$.

(v) Orlicz asymptotically ϕ-statistical equivalent (Wijsman sense) of multiple L provided that, for every $\varepsilon > 0$,

$$\lim_s \frac{1}{\phi_s}\left|\left\{k \in \sigma, \sigma \in P_s : M \times \left(\left|\frac{d(x, A_k)}{d(x, B_k)} - L\right|\right) \geq \varepsilon\right\}\right| = 0, \tag{24}$$

denoted by $(A_k) \overset{W\mathcal{S}_\phi^L(M)}{\sim} (B_k)$, and simply Wijsman Orlicz asymptotically ϕ-statistical equivalent if $L = 1$.

(vi) Orlicz I-asymptotically ϕ-statistical equivalent (Wijsman sense) of multiple L provided that, for every $\varepsilon > 0$ and for every $\delta > 0$

$$\left\{s \in \mathbb{N} : \frac{1}{\phi_s}\left|\left\{k \in \sigma, \sigma \in P_s : M \times \left(\left|\frac{d(x, A_k)}{d(x, B_k)} - L\right|\right) \geq \varepsilon\right\}\right| \geq \delta\right\} \in I, \tag{25}$$

denoted by $(A_k) \overset{I-W\mathcal{S}_\phi^L(M)}{\sim} (B_k)$, and simply Wijsman Orlicz I-asymptotically ϕ-statistical equivalent if $L = 1$.

Theorem 16. *Let (X, ρ) be a metric space. For any non-empty closed subsets $A_k, B_k \subseteq X$ such that $d(x, A_k) > 0$ and $d(x, B_k) > 0$ for each $x \in X$ and M be an Orlicz function. Then we have*

(a) $(A_k) \overset{I-W[C_1]^L(M)}{\sim} (B_k) \Rightarrow (A_k) \overset{I-W\mathcal{S}^L}{\sim} (B_k)$;

(b) *if M satisfies the Δ_2-condition and $(A_k) \in L_\infty(M)$ such that $(A_k) \overset{I-W\mathcal{S}^L}{\sim} (B_k)$, then $(A_k) \overset{I-W[C_1]^L(M)}{\sim} (B_k)$;*

(c) *if M satisfies the Δ_2-condition, then $I - W[C_1]^L(M) \cap L_\infty(M) = I-W\mathcal{S}^L \cap L_\infty(M)$, where $L_\infty(M) = \{(A_k) : M(d(x, A_k)) \in L_\infty, x \in X\}$.*

Proof. (a) Suppose that $(A_k) \overset{I-W[C_1]^L(M)}{\sim} (B_k)$. Let $\varepsilon > 0$ be given. Then we can write

$$\begin{aligned} &\frac{1}{n}\sum_{k=1}^{n} M\left(\left|\frac{d(x, A_k)}{d(x, B_k)} - L\right|\right) \\ &\geq \frac{1}{n}\sum_{\substack{k=1 \\ |(d(x,A_k)/d(x,B_k))-L|\geq\varepsilon}}^{n} M\left(\left|\frac{d(x, A_k)}{d(x, B_k)} - L\right|\right) \\ &\geq \frac{M(\varepsilon)}{n}\left|\left\{k \leq n : \left|\frac{d(x, A_k)}{d(x, B_k)} - L\right| \geq \varepsilon\right\}\right|. \end{aligned} \tag{26}$$

Consequently, for any $\delta > 0$, we have

$$\left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \le n : \left|\frac{d(x, A_k)}{d(x, B_k)} - L\right| \ge \varepsilon\right\}\right| \ge \frac{\delta}{M(\varepsilon)}\right\} \subseteq \left\{n \in \mathbb{N} : \frac{1}{n}\sum_{k=1}^{n} M\left(\left|\frac{d(x, A_k)}{d(x, B_k)} - L\right|\right) \ge \delta\right\} \in I. \tag{27}$$

Hence, $(u_k) \overset{I-W\mathcal{S}^L}{\sim} (v_k)$.

(b) Suppose that M is bounded and $(A_k) \overset{I-W\mathcal{S}^L}{\sim} (B_k)$. Since M is bounded then there exists a real number $K > 0$ such that $\sup_t M(t) \le K$. Moreover, for any $\varepsilon > 0$ we can write

$$\begin{aligned}
&\frac{1}{n}\sum_{k=1}^{n} M\left(\left|\frac{d(x, A_k)}{d(x, B_k)} - L\right|\right) \\
&= \frac{1}{n}\left[\sum_{\substack{k=1 \\ |(d(x,A_k)/d(x,B_k))-L| \ge \varepsilon}}^{n} M\left(\left|\frac{d(x, A_k)}{d(x, B_k)} - L\right|\right) + \sum_{\substack{k=1 \\ |(d(x,A_k)/d(x,B_k))-L| < \varepsilon}}^{n} M\left(\left|\frac{d(x, A_k)}{d(x, B_k)} - L\right|\right)\right] \\
&\le \frac{K}{n}\left|\left\{k \le n : \left|\frac{d(x, A_k)}{d(x, B_k)} - L\right| \ge \varepsilon\right\}\right| + M(\varepsilon).
\end{aligned} \tag{28}$$

Now, for any $\delta > 0$, we get

$$\begin{aligned}
&\left\{n \in \mathbb{N} : \frac{1}{n}\sum_{k=1}^{n} M\left(\left|\frac{d(x, A_k)}{d(x, B_k)} - L\right|\right) \ge \delta\right\} \\
&\subseteq \left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \le n : \left|\frac{d(x, A_k)}{d(x, B_k)} - L\right| \ge \varepsilon\right\}\right| \ge \frac{\delta}{K}\right\} \in I.
\end{aligned} \tag{29}$$

Hence, $(A_k) \overset{I-W[C_1]^L(M)}{\sim} (B_k)$.

(c) The proof of this part follows from parts (a) and (b). □

Theorem 17. *Let (X, ρ) be a metric space. For any non-empty closed subsets $A_k, B_k \subseteq X$ such that $d(x, A_k) > 0$ and $d(x, B_k) > 0$ for each $x \in X$ and (ϕ_s) be a nondecreasing sequence of positive real numbers such that $\phi_s \to \infty$ as $s \to \infty$ and $\phi_s \le s$ for every $s \in \mathbb{N}$. Then, $(A_k) \overset{I-W\mathcal{S}^L}{\sim} (B_k) \Rightarrow (A_k) \overset{I-W\mathcal{S}^L_\phi}{\sim} (B_k)$.*

Proof. By the definition of the sequences ϕ_s, it follows that $\inf_s(s/(s - \phi_s)) \ge 1$. Then there exists $a > 0$ such that

$$\frac{s}{\phi_s} \le \frac{1 + a}{a}. \tag{30}$$

Suppose that $(A_k) \overset{I-W\mathcal{S}^L}{\sim} (B_k)$; then, for every $\varepsilon > 0$ and sufficiently large s, we have

$$\begin{aligned}
&\frac{1}{\phi_s}\left|\left\{k \in \sigma, \sigma \in P_s : \left|\frac{d(x, A_k)}{d(x, B_k)} - L\right| \ge \varepsilon\right\}\right| \\
&= \frac{1}{s} \cdot \frac{s}{\phi_s}\left|\left\{k \le s : \left|\frac{d(x, A_k)}{d(x, B_k)} - L\right| \ge \varepsilon\right\}\right| \\
&\quad - \frac{1}{\phi_s}\left|\left\{k \in \{1, 2, \ldots, s\} - \sigma, \sigma \in P_s : \left|\frac{d(x, A_k)}{d(x, B_k)} - L\right| \ge \varepsilon\right\}\right| \\
&\le \frac{1 + a}{a}\frac{1}{s}\left|\left\{k \le s : \left|\frac{d(x, A_k)}{d(x, B_k)} - L\right| \ge \varepsilon\right\}\right| \\
&\quad - \frac{1}{\phi_s}\left|\left\{k_0 \in \{1, 2, \ldots, s\} - \sigma, \sigma \in P_s : \left|\frac{d(x, A_{k_0})}{d(x, B_{k_0})} - L\right| \ge \varepsilon\right\}\right| \\
&\le \frac{1 + a}{a}\frac{1}{s}\left|\left\{k \le s : \left|\frac{d(x, A_k)}{d(x, B_k)} - L\right| \ge \varepsilon\right\}\right|.
\end{aligned} \tag{31}$$

Consequently, for any $\eta > 0$, we have

$$\begin{aligned}
&\left\{s \in \mathbb{N} : \frac{1}{\phi_s} \times \left|\left\{k \in \sigma, \sigma \in P_s : \left|\frac{d(x, A_k)}{d(x, B_k)} - L\right| \ge \varepsilon\right\}\right| \ge \eta\right\} \\
&\subseteq \left\{s \in \mathbb{N} : \frac{1}{s}\left|\left\{k \le s : \left|\frac{d(x, A_k)}{d(x, B_k)} - L\right| \ge \varepsilon\right\}\right| \ge \frac{a\eta}{1 + a}\right\} \in I.
\end{aligned} \tag{32}$$

This completes the proof of the theorem. □

Theorem 18. *Let (X, ρ) be a metric space. For any non-empty closed subsets $A_k, B_k \subseteq X$ such that $d(x, A_k) > 0$ and $d(x, B_k) > 0$ for each $x \in X$, and let M be an Orlicz function which satisfies the Δ_2-conditions. Then $(A_k) \overset{I-W\mathcal{S}^L}{\sim} (B_k) \Rightarrow (A_k) \overset{I-W\mathcal{S}^L_\phi(M)}{\sim} (B_k)$.*

Proof. By the definition of the sequences ϕ_s, it follows that $\inf_s(s/(s - \phi_s)) \ge 1$. Then there exists $a > 0$ such that

$$\frac{s}{\phi_s} \le \frac{1 + a}{a}. \tag{33}$$

Suppose that $(A_k) \overset{W\mathcal{S}^L}{\sim} (B_k)$, then for every $\varepsilon > 0$ and sufficiently large s we have

$$\begin{aligned}
&\frac{1}{\phi_s}\left|\left\{k \in \sigma, \sigma \in P_s : M\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right) \geq \varepsilon\right\}\right| \\
&= \frac{1}{s}\cdot\frac{s}{\phi_s}\left|\left\{k \leq s : M\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right) \geq \varepsilon\right\}\right| \\
&\quad - \frac{1}{\phi_s}\left|\left\{k \in \{1,2,\ldots,s\} - \sigma,\right.\right. \\
&\qquad\qquad \left.\left. \sigma \in P_s : M\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right) \geq \varepsilon\right\}\right| \\
&\leq \frac{1+a}{a}\frac{1}{s}\left|\left\{k \leq s : M\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right) \geq \varepsilon\right\}\right| \\
&\quad - \frac{1}{\phi_s}\left|\left\{k_0 \in \{1,2,\ldots,s\} - \sigma,\right.\right. \\
&\qquad\qquad \left.\left. \sigma \in P_s : M\left(\left|\frac{d(x,A_{k_0})}{s(x,B_{k_0})} - L\right|\right) \geq \varepsilon\right\}\right| \\
&\leq \frac{1+a}{a}\frac{1}{s}\left|\left\{k \leq s : M\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right) \geq \varepsilon\right\}\right|.
\end{aligned} \tag{34}$$

Since M satisfies the Δ_2-condition, it follows that

$$M\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right) \leq K\cdot\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \tag{35}$$

for some constant $K > 0$ in both cases where $|(d(x,A_k)/d(x,B_k)) - L| \leq 1$ and $|(d(x,A_k)/d(x,B_k)) - L| \geq 1$.

In the first case it follows from the definition of Orlicz function, and for the second case we have

$$\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| = 2\cdot L^{(1)} = 2^2\cdot L^{(2)} = \cdots = 2^s\cdot L^{(s)} \tag{36}$$

such that $L^{(s)} \leq 1$. Using the Δ_2-condition of Orlicz functions we get the following estimation

$$\begin{aligned}
M\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right) &\leq T\cdot L^{(s)}\cdot M(1) \\
&= K\cdot\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|,
\end{aligned} \tag{37}$$

where K and T are constants. The proof of the theorem follows from the relations (34) and (37). □

Theorem 19. *Let (X,ρ) be a metric space. For any non-empty closed subsets $A_k, B_k \subseteq X$ such that $d(x,A_k) > 0$ and $d(x,B_k) > 0$ for each $x \in X$. Let M be an Orlicz function and $k \in \mathbb{Z}$ such that $\phi_s \leq [\phi_s] + k$, $\sup_s(([\phi_s]+k)/\phi_{s-1}) < \infty$. Then $(A_k) \overset{I-W\mathcal{S}_\phi^L(M)}{\sim} (B_k) \Rightarrow (A_k) \overset{I-W\mathcal{S}^L}{\sim} (B_k)$.*

Proof. If $\sup_s(([\phi_s]+k)/\phi_{s-1}) < \infty$, then there exists $K > 0$ such that $(([\phi_s]+k)/\phi_{s-1}) < K$ for all $s \geq 1$. Let n be an integer such that $\phi_{s-1} < n \leq \phi_s$. Then for every $\varepsilon > 0$, we have

$$\begin{aligned}
&\frac{1}{n}\left|\left\{k \leq n : \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \geq \varepsilon\right\}\right| \\
&\leq \frac{1}{n}\left|\left\{k \leq n : M\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right) \geq M(\varepsilon)\right\}\right| \\
&\leq \frac{1}{[\phi_s]+k}\cdot\frac{[\phi_s]+k}{\phi_{s-1}} \\
&\quad\times\left|\left\{k \leq \phi_s : M\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right) \geq M(\varepsilon)\right\}\right| \\
&\leq \frac{1}{[\phi_s]+k}\cdot\frac{[\phi_s]+k}{\phi_{s-1}} \\
&\quad\times\left|\left\{k \in \sigma, \sigma \in P_{[\phi_s]+k} : M\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right) \geq M(\varepsilon)\right\}\right| \\
&\leq \frac{K}{[\phi_s]+k} \\
&\quad\times\left|\left\{k \in \sigma, \sigma \in P_{[\phi_s]+k} : M\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right) \geq M(\varepsilon)\right\}\right| \\
&\leq \frac{K}{\phi_s}\left|\left\{k \in \sigma, \sigma \in P_s : M\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right) \geq M(\varepsilon)\right\}\right|.
\end{aligned} \tag{38}$$

Consequently for any $\delta > 0$, we have

$$\begin{aligned}
&\left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leq n : \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \geq \varepsilon\right\}\right| \geq \delta\right\} \\
&\subseteq \left\{s \in \mathbb{N} : \frac{1}{\phi_s}\left|\left\{k \in \sigma, \sigma \in P_s : M\right.\right.\right. \\
&\qquad\qquad \times\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right) \\
&\qquad\qquad \left.\left.\left. \geq M(\varepsilon)\right\}\right| \geq \frac{\delta}{K}\right\} \in I.
\end{aligned} \tag{39}$$

This established the result. □

Theorem 20. *Let (X,ρ) be a metric space. For any non-empty closed subsets $A_k, B_k \subseteq X$ such that $d(x,A_k) > 0$ and $d(x,B_k) > 0$ for each $x \in X$. Let M be an Orlicz function. Then we have*

(a) $(A_k) \overset{I-W[C_1]^L(M)}{\sim} (B_k) \Rightarrow (A_k) \overset{I-W[\phi]^L(M)}{\sim} (B_k)$;

(b) $\sup_s(\phi_s/\phi_{s-1}) < \infty$ *for every* $s \in \mathbb{N}$, *then,* $(A_k) \overset{I-W[\phi]^L(M)}{\sim} (B_k) \Rightarrow (A_k) \overset{I-W[C_1]^L(M)}{\sim} (B_k)$.

Proof. (a) From the definition of the sequence (ϕ_s) it follows that $\inf_s(s/(s-\phi_s)) \geq 1$. Then there exists $a > 0$ such that

$$\frac{s}{\phi_s} \leq \frac{1+a}{a}. \tag{40}$$

Then we get the following relation:

$$\begin{aligned}
&\frac{1}{\phi_s}\sum_{k\in\sigma,\sigma\in P_s} M\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right) \\
&= \frac{s}{\phi_s}\cdot\frac{1}{s}\sum_{k=1}^{n} M\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right) \\
&\quad - \frac{1}{\phi_s}\sum_{k\in\{1,2,\ldots,s\}-\sigma,\sigma\in P_s} M\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right) \\
&\leq \frac{1+a}{a}\frac{1}{s}\sum_{k=1}^{s} M\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right) \\
&\quad - \frac{1}{\phi_s}\sum_{k_0\in\{1,2,\ldots,s\}-\sigma,\sigma\in P_s} M\left(\left|\frac{d(x,A_{k_0})}{d(x,B_{k_0})} - L\right|\right) \\
&\leq \frac{1+a}{a}\frac{1}{s}\sum_{k=1}^{s} M\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right).
\end{aligned} \tag{41}$$

Since $(A_k) \overset{I-W[C_1]^L(M)}{\sim} (B_k)$ and M are continuous, then for any $\delta > 0$, from the last relation we get

$$\begin{aligned}
&\left\{s\in\mathbb{N} : \frac{1}{s}\sum_{k=1}^{s} M\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right) \geq \frac{a\delta}{1+a}\right\} \\
&\subseteq \left\{s\in\mathbb{N} : \frac{1}{\phi_s}\sum_{k\in\sigma,\sigma\in P_s} M\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right) \geq \delta\right\} \in I.
\end{aligned} \tag{42}$$

Hence, $(A_k) \overset{I-W[\phi]^L(M)}{\sim} (B_k)$.

(b) For this part we assume that the sequence (ϕ_s) satisfies the condition that for any set $Q \in \mathcal{F}, \bigcup\{n : \phi_{s-1} < n \leq [\phi_s], s \in Q\} \in \mathcal{F}$. Suppose that $\sup_s(\phi_s/\phi_{s-1}) < \infty$ then there exists $K > 0$ such that $\phi_s/\phi_{s-1} < K$ for all $s \geq 1$. Suppose that $(A_k) \overset{I-W[\phi]^L(M)}{\sim} (B_k)$. Then for every $\varepsilon > 0, \delta > 0$ we put

$$\begin{aligned}
Q &= \left\{s\in\mathbb{N} : \frac{1}{\phi_s}\sum_{k\in\sigma,\sigma\in P_s} M\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right) < \varepsilon\right\}, \\
U &= \left\{n\in\mathbb{N} : \frac{1}{n}\sum_{k=1}^{n} M\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right) < \delta\right\}.
\end{aligned} \tag{43}$$

From our assumption it is clear that $Q \in \mathcal{F}$. Further, observe that

$$V_s = \frac{1}{\phi_s}\sum_{k\in\sigma,\sigma\in P_s} M\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right) < \varepsilon, \quad \forall s \in Q. \tag{44}$$

Let n be any integer with $\phi_{s-1} < n \leq [\phi_s]$ for some $s \in Q$. Then we have

$$\begin{aligned}
&\frac{1}{n}\sum_{k=1}^{n} M\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right) \\
&\leq \frac{1}{\phi_{s-1}}\sum_{k=1}^{[\phi_s]} M\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right) \\
&= \frac{1}{\phi_{s-1}}\left(\sum_{k=1}^{[\phi_1]} M\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right)\right. \\
&\qquad + \sum_{[\phi_1]}^{[\phi_2]} M\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right) \\
&\qquad \left. + \cdots + \sum_{[\phi_{s-1}]}^{[\phi_s]} M\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right)\right) \\
&\leq \frac{\phi_1}{\phi_{s-1}}\left(\frac{1}{\phi_1}\sum_{k\in\sigma,\sigma\in P^{(1)}} M\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right)\right) \\
&\quad + \frac{\phi_2}{\phi_{s-1}}\left(\frac{1}{\phi_2}\sum_{k\in\sigma,\sigma\in P^{(2)}} M\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right)\right) \\
&\quad + \cdots + \frac{\phi_R}{\phi_{s-1}}\left(\frac{1}{\phi_R}\sum_{k\in\sigma,\sigma\in P^{(R)}} M\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right)\right) \\
&\quad + \cdots + \frac{\phi_s}{\phi_{s-1}}\left(\frac{1}{\phi_s}\sum_{k\in\sigma,\sigma\in P^{(s)}} M\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right)\right) \\
&\leq \sup_{s\in Q} V_s \frac{\phi_s}{\phi_{s-1}} < K\varepsilon,
\end{aligned} \tag{45}$$

where $P^{(t)}$ are sets of integer which have more than $[\phi_t]$ elements for $t \in \{1, 2, \ldots, s\}$. Choosing $\delta = \varepsilon/K$ and in view of the fact that $\bigcup\{n : \phi_{s-1} < n \leq [\phi_s], s \in Q\} \subset U$, where $Q \in \mathcal{F}$. It follows from our assumption that the set $U \in \mathcal{F}$. This completes the proof of the theorem. □

Theorem 21. *Let (X, ρ) be a metric space. For any non-empty closed subsets $A_k, B_k \subseteq X$ such that $d(x, A_k) > 0$ and $d(x, B_k) > 0$ for each $x \in X$. Let M be an Orlicz function. Then we have*

(a) $(A_k) \overset{I-W[\phi]^L(M)}{\sim} (B_k) \Rightarrow (A_k) \overset{I-W\mathcal{S}_\phi^L}{\sim} (B_k)$;

(b) *if M satisfies the Δ_2-condition and $(A_k), (B_k) \in L_\infty(M)$ such that $(A_k) \overset{I-W\mathcal{S}_\phi^L}{\sim} (B_k)$ then $(A_k) \overset{I-W[\phi]^L(M)}{\sim} (B_k)$;*

(c) *if M satisfies the Δ_2-condition, then $I - W[\phi]^L(M) \cap L_\infty(M) = I - W\mathcal{S}_\phi^L \cap L_\infty(M)$.*

Proof. The proof of this theorem follows from the same techniques used in the proofs of Theorems 16 and 20. It is omitted here. □

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