Research Article

Higher Order Commutators of Fractional Integral Operator on the Homogeneous Herz Spaces with Variable Exponent

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By decomposing functions, we establish estimates for higher order commutators generated by fractional integral with BMO functions or the Lipschitz functions on the homogeneous Herz spaces with variable exponent. These estimates extend some known results in the literatures.

1. Introduction

Let b be a locally integrable function, $0 < \beta < n$, and $m \in \mathbb{N}$; the higher order commutators of fractional integral operator $I_{\beta,b}^m$ are defined by

$$I_{\beta,b}^m f(x) = \int_{\mathbb{R}^n} \frac{[b(x) - b(y)]^m}{|x - y|^{n-\beta}} f(y) dy. \quad (1)$$

Obviously, $I_{\beta,b}^0 = I_\beta$ and $I_{\beta,b}^1 = [b, I_\beta]$. The famous Hardy-Littlewood-Sobolev theorem tells us that the fractional integral operator I_β is a bounded operator from the usual Lebesgue spaces $L^{p_1}(\mathbb{R}^n)$ to $L^{p_2}(\mathbb{R}^n)$ when $0 < p_1 < p_2 < \infty$ and $1/p_1 - 1/p_2 = \beta/n$. Also, many generalized results about I_β and the commutator $[b, I_\beta]$ on some function spaces have been studied; see [1–3] for details.

It is well known that the main motivation for studying the spaces with variable exponent arrived in the nonlinear elasticity theory and differential equations with nonstandard growth. Since the fundamental paper [4] by Kováčik and Rákosník appeared in 1991, the Lebesgue spaces with variable exponent $L^{p(\cdot)}(\mathbb{R}^n)$ have been extensively investigated. In the recent twenty years, boundedness of some important operators, for example, the Calderón-Zygmund operators, fractional integrals, and commutators, on $L^{p(\cdot)}(\mathbb{R}^n)$ has been obtained; see [5–7]. Recently, Diening [8] extended the

($L^{p_1}(\mathbb{R}^n), L^{p_2}(\mathbb{R}^n)$) boundedness of I_β to the Lebesgue spaces with variable exponent. Izuki [7] first introduced the Herz spaces with variable exponent $\dot{K}_{p(\cdot)}^{\alpha,q}(\mathbb{R}^n)$, which is a generalized space of the Herz space $\dot{K}_p^{\alpha,q}(\mathbb{R}^n)$; see [9, 10], and in case of $b \in \text{BMO}(\mathbb{R}^n)$, he obtained the boundedness properties of the commutator $[b, I_\beta]$. The paper [11] by Lu et al. indicates that the commutator $[b, I_\beta]$ with $b \in \text{BMO}(\mathbb{R}^n)$ and with $b \in \text{Lip}_\alpha(\mathbb{R}^n)$ ($0 < \alpha \leq 1$) has many different properties. In 2012, Zhou [12] studied the boundedness of I_β on the Herz spaces with variable exponent and proved that the boundedness properties of the commutator $[b, I_\beta]$ also hold in case of $b \in \text{Lip}_\alpha(\mathbb{R}^n)$ ($0 < \alpha \leq 1$). The higher order commutators $I_{\beta,b}^m$ are recently considered by Wang et al. in the paper [13, 14]; they established the BMO and the Lipschitz estimates for $I_{\beta,b}^m$ on the Lebesgue spaces with variable exponent $L^{p(\cdot)}(\mathbb{R}^n)$. Motivated by [7, 12–14], in this note, we establish the boundedness of the higher order commutators $I_{\beta,b}^m$ on the Herz spaces with variable exponent.

For brevity, $|E|$ denotes the Lebesgue measure for a measurable set $E \subset \mathbb{R}^n$, and f_E denotes the mean value of f on E ($f_E = (1/|E|) \int_E f(x) dx$). The exponent $p'(\cdot)$ means the conjugate of $p(\cdot)$, that is, $1/p(\cdot) + 1/p'(\cdot) = 1$. C denotes a positive constant, which may have different values even in the same line. Let us first recall some definitions and notations.

Definition 1. For $0 < \gamma \leq 1$, the Lipschitz space $\text{Lip}_\gamma(\mathbb{R}^n)$ is the space of functions f satisfying

$$\|f\|_{\text{Lip}_\gamma} = \sup_{x, y \in \mathbb{R}^n, x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\gamma} < \infty. \quad (2)$$

Definition 2. For $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, the bounded mean oscillation space $\text{BMO}(\mathbb{R}^n)$ is the space of functions f satisfying

$$\|f\|_{\text{BMO}} = \sup_B \frac{1}{|B|} \int_B |f(x) - f_B| dx < \infty, \quad (3)$$

where the supremum is taken over all balls B in $\mathbb{R}^n$.

Definition 3. Let $p(\cdot) : E \rightarrow [1, \infty)$ be a measurable function.

- (1) The Lebesgue space with variable exponent $L^{p(\cdot)}(E)$ is defined by

$$L^{p(\cdot)}(E) = \left\{ f \text{ is measurable} : \int_E \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} dx < \infty \text{ for some constant } \lambda > 0 \right\}. \quad (4)$$

- (2) The space with variable exponent $L^{p(\cdot)}_{\text{loc}}(E)$ is defined by

$$L^{p(\cdot)}_{\text{loc}}(E) = \{f : f \in L^{p(\cdot)}(K) \text{ for all compact subsets } K \subset E\}. \quad (5)$$

The Lebesgue space $L^{p(\cdot)}(E)$ is a Banach space with the Luxemburg norm

$$\|f\|_{L^{p(\cdot)}(E)} = \inf \left\{ \lambda > 0 : \int_E \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} dx \leq 1 \right\}. \quad (6)$$

We denote

$$\begin{aligned} p_- &= \text{ess inf } \{p(x) : x \in E\}, \\ p_+ &= \text{ess sup } \{p(x) : x \in E\}, \\ \mathcal{P}(E) &= \{p(\cdot) : p_- > 1, p_+ < \infty\}, \\ \mathcal{B}(E) &= \{p(\cdot) : p(\cdot) \in \mathcal{P}(E), \\ &\quad M \text{ is bounded on } L^{p(\cdot)}(E)\}, \end{aligned} \quad (7)$$

where the Hardy-Littlewood maximal operator M is defined by

$$Mf(x) = \sup_{r>0} r^{-n} \int_{B(x,r) \cap E} |f(y)| dy, \quad (8)$$

where $B(x, r) = \{y \in \mathbb{R}^n : |x - y| < r\}$.

Proposition 4 (see [15]). If $p(\cdot) \in \mathcal{P}(E)$ satisfies

$$\begin{aligned} |p(x) - p(y)| &\leq \frac{-C}{\log(|x - y|)}, \quad |x - y| \leq \frac{1}{2}, \\ |p(x) - p(y)| &\leq \frac{C}{\log(e + |x|)}, \quad |y| \leq |x|, \end{aligned} \quad (9)$$

then one has $p(\cdot) \in \mathcal{B}(E)$.

Let $B_k = \{x \in \mathbb{R}^n : |x| \leq 2^k\}$, $R_k = B_k \setminus B_{k-1}$, and $\chi_k = \chi_{R_k}$ be the characteristic function of the set R_k for $k \in \mathbb{Z}$. For $m \in \mathbb{N}$, we denote $\widetilde{\chi}_m = \chi_{R_m}$ if $m \geq 1$, and $\widetilde{\chi}_0 = \chi_{B_0}$.

Definition 5 (see [7]). For $\alpha \in \mathbb{R}$, $0 < q \leq \infty$ and $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$.

- (1) The homogeneous Herz spaces $\dot{K}^{\alpha, q}_{p(\cdot)}(\mathbb{R}^n)$ are defined by

$$\dot{K}^{\alpha, q}_{p(\cdot)}(\mathbb{R}^n) = \left\{ f \in L^{p(\cdot)}_{\text{loc}}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{\dot{K}^{\alpha, q}_{p(\cdot)}(\mathbb{R}^n)} < \infty \right\}, \quad (10)$$

where

$$\|f\|_{\dot{K}^{\alpha, q}_{p(\cdot)}(\mathbb{R}^n)} = \left\| \left\{ 2^{\alpha k} \|f \chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)} \right\}_{k=-\infty}^{\infty} \right\|_{\ell^q(\mathbb{Z})}. \quad (11)$$

- (2) The nonhomogeneous Herz spaces $K^{\alpha, q}_{p(\cdot)}(\mathbb{R}^n)$ are defined by

$$K^{\alpha, q}_{p(\cdot)}(\mathbb{R}^n) = \left\{ f \in L^{p(\cdot)}_{\text{loc}}(\mathbb{R}^n) : \|f\|_{K^{\alpha, q}_{p(\cdot)}(\mathbb{R}^n)} < \infty \right\}, \quad (12)$$

where

$$\|f\|_{K^{\alpha, q}_{p(\cdot)}(\mathbb{R}^n)} = \left\| \left\{ 2^{\alpha m} \|f \widetilde{\chi}_m\|_{L^{p(\cdot)}(\mathbb{R}^n)} \right\}_{m=0}^{\infty} \right\|_{\ell^q(\mathbb{N})}. \quad (13)$$

In this note, we obtain the following results.

Theorem 6. Suppose that $b \in \text{Lip}_{\beta_1}(\mathbb{R}^n)$ ($0 < \beta_1 < 1$), $p_2(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfies conditions (9) in Proposition 4. If $0 < r < \min \{1/(p_1)_+, 1/(p'_2)_+\}$, $0 < \beta + m\beta_1 < nr$, $0 < \alpha < nr - \beta - m\beta_1$, $0 < q_1 \leq q_2 < \infty$, and $1/p_1(x) - 1/p_2(x) = (\beta + m\beta_1)/n$, then the higher order commutators $I^m_{\beta, b}$ are bounded from $\dot{K}^{\alpha, q_1}_{p_1(\cdot)}(\mathbb{R}^n)$ to $\dot{K}^{\alpha, q_2}_{p_2(\cdot)}(\mathbb{R}^n)$.

Theorem 7. Suppose that $b \in \text{BMO}(\mathbb{R}^n)$, $p_2(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfies conditions (9) in Proposition 4. If $0 < r < \min \{1/(p_1)_+, 1/(p'_2)_+\}$, $0 < \beta < nr$, $0 < \alpha < nr - \beta$, $0 < q_1 \leq q_2 < \infty$, and $1/p_1(x) - 1/p_2(x) = \beta/n$, then the higher order commutators $I^m_{\beta, b}$ are bounded from $\dot{K}^{\alpha, q_1}_{p_1(\cdot)}(\mathbb{R}^n)$ to $\dot{K}^{\alpha, q_2}_{p_2(\cdot)}(\mathbb{R}^n)$.

Remark A. The previous main results generalize the $(L^{p(\cdot)}(\mathbb{R}^n), L^{q(\cdot)}(\mathbb{R}^n))$ boundedness of the higher order commutators $I^m_{\beta, b}$ in [13] to the case of the Herz spaces with variable exponent. If $m = 1$, our conclusions coincide with the corresponding results in [7, 12]. Moreover, the same boundedness also holds for the nonhomogeneous case.

2. Proof of Theorems 6 and 7

To prove our main results, we need the following lemmas.

Lemma 8 (see [4]). Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$; if $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and $g \in L^{p'(\cdot)}(\mathbb{R}^n)$, then

$$\int_{\mathbb{R}^n} |f(x)g(x)| dx \leq r_p \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|g\|_{L^{p'(\cdot)}(\mathbb{R}^n)}, \quad (14)$$

where $r_p = 1 + 1/p_- - 1/p_+$.

Lemma 9 (see [7]). Let $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$; then for all balls B in $\mathbb{R}^n$,

$$\frac{1}{|B|} \|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \leq C. \quad (15)$$

Lemma 10 (see [7]). Let $p'_2(\cdot) \in \mathcal{B}(\mathbb{R}^n)$; then for all balls B in $\mathbb{R}^n$ and all measurable subsets $S \subset B$, one can take a constant $0 < r < 1/(p'_2)_+$, so that

$$\frac{\|\chi_S\|_{L^{p'_2(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p'_2(\cdot)}(\mathbb{R}^n)}} \leq C \left(\frac{|S|}{|B|} \right)^r. \quad (16)$$

Lemma 11 (see [8]). Suppose that $p_1(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfies conditions (9) in Proposition 4, $0 < \beta < n/(p_1)_+$ and $1/p_1(x) - 1/p_2(x) = \beta/n$; then

$$\|I_\beta(f)\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \leq C \|f\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}. \quad (17)$$

Lemma 12 (see [13]). Suppose that $p_1(\cdot), p_2(\cdot) \in \mathcal{P}(\mathbb{R}^n)$.

- (1) Let $0 < \beta < n/(p_1)_+$, $b \in \text{BMO}(\mathbb{R}^n)$. If $p_2(\cdot)$ satisfies conditions (9) in Proposition 4 and $1/p_1(x) - 1/p_2(x) = \beta/n$, then

$$\|I_{\beta,b}^m(f)\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \leq C \|b\|_{\text{BMO}}^m \|f\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}. \quad (18)$$

- (2) Let $0 < \beta + m\beta_1 < n/(p_1)_+$, $b \in \text{Lip}_{\beta_1}(\mathbb{R}^n)$ ($0 < \beta_1 < 1$). If $p_2(\cdot)$ satisfies conditions (9) in Proposition 4 and $1/p_1(x) - 1/p_2(x) = (\beta + m\beta_1)/n$, then

$$\|I_{\beta,b}^m(f)\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \leq C \|b\|_{\text{Lip}_{\beta_1}}^m \|f\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}. \quad (19)$$

Lemma 13 (see [16]). Let $b \in \text{BMO}(\mathbb{R}^n)$, $k > j$ ($k, j \in \mathbb{N}$); one has

- (1) $C^{-1} \|b\|_{\text{BMO}}^m \leq \sup_{B \subset \mathbb{R}^n} (1/\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}) \|(b - b_B)^m \chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C \|b\|_{\text{BMO}}^m$;
 (2) $\|(b - b_{B_j})^m \chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C(k - j)^m \|b\|_{\text{BMO}}^m \times \|\chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)}$.

Proof of Theorem 6. Let $f \in \dot{K}_{p_1(\cdot)}^{\alpha, q_1}(\mathbb{R}^n)$; we can write

$$f(x) = \sum_{j=-\infty}^{\infty} f(x) \chi_j(x) = \sum_{j=-\infty}^{\infty} f_j(x). \quad (20)$$

For $0 < q_1/q_2 \leq 1$, applying the inequality

$$\left(\sum_{i=1}^{\infty} a_i \right)^{q_1/q_2} \leq \sum_{i=1}^{\infty} a_i^{q_1/q_2} \quad (a_i > 0, i = 1, 2, \dots), \quad (21)$$

we obtain

$$\begin{aligned} & \|I_{\beta,b}^m(f)\|_{\dot{K}_{p_2(\cdot)}^{\alpha, q_2}(\mathbb{R}^n)}^{q_1} \\ &= C \left(\sum_{k=-\infty}^{\infty} 2^{\alpha q_2 k} \|I_{\beta,b}^m(f) \chi_k\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}^{q_2} \right)^{q_1/q_2} \\ &\leq C \sum_{k=-\infty}^{\infty} 2^{\alpha q_1 k} \left(\sum_{j=-\infty}^{k-2} \|I_{\beta,b}^m(f_j) \chi_k\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \right)^{q_1} \\ &\quad + C \sum_{k=-\infty}^{\infty} 2^{\alpha q_1 k} \left(\sum_{j=k-1}^{\infty} \|I_{\beta,b}^m(f_j) \chi_k\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \right)^{q_1} \\ &= U_1 + U_2. \end{aligned} \quad (22)$$

We first estimate U_1 . Noting that if $x \in R_k, y \in R_j$, and $j \leq k-2$, then $|x - y| \sim |x| \sim 2^k$, we get

$$\begin{aligned} |I_{\beta,b}^m(f_j) \chi_k| &\leq \int_{R_j} \frac{|b(x) - b(y)|^m}{|x - y|^{n-\beta}} |f_j(y)| dy \cdot \chi_k(x) \\ &\leq C 2^{k(\beta-n)} \int_{R_j} |b(x) - b(y)|^m |f_j(y)| dy \cdot \chi_k(x) \\ &\leq C 2^{k(\beta+m\beta_1-n)} \|b\|_{\text{Lip}_{\beta_1}}^m \int_{R_j} |f_j(y)| dy \cdot \chi_k(x). \end{aligned} \quad (23)$$

By Hölder's inequality, Lemmas 9 and 10, we have

$$\begin{aligned} & \|I_{\beta,b}^m(f_j) \chi_k\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \\ &\leq C 2^{k(\beta+m\beta_1-n)} \|b\|_{\text{Lip}_{\beta_1}}^m \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \\ &\quad \times \|\chi_{B_k}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \|\chi_{B_j}\|_{L^{p'_1(\cdot)}(\mathbb{R}^n)} \\ &\leq C 2^{k(\beta+m\beta_1)} \|b\|_{\text{Lip}_{\beta_1}}^m \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \\ &\quad \times \|\chi_{B_j}\|_{L^{p'_1(\cdot)}(\mathbb{R}^n)} \|\chi_{B_k}\|_{L^{p'_2(\cdot)}(\mathbb{R}^n)}^{-1} \\ &\leq C 2^{k(\beta+m\beta_1)} \|b\|_{\text{Lip}_{\beta_1}}^m \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \\ &\quad \times \|\chi_{B_j}\|_{L^{p'_1(\cdot)}(\mathbb{R}^n)} \|\chi_{B_j}\|_{L^{p'_2(\cdot)}(\mathbb{R}^n)}^{-1} \frac{\|\chi_{B_j}\|_{L^{p'_2(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_k}\|_{L^{p'_2(\cdot)}(\mathbb{R}^n)}} \\ &\leq C 2^{k(\beta+m\beta_1)} 2^{nr(j-k)} \|b\|_{\text{Lip}_{\beta_1}}^m \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \\ &\quad \times \|\chi_{B_j}\|_{L^{p'_1(\cdot)}(\mathbb{R}^n)} \|\chi_{B_j}\|_{L^{p'_2(\cdot)}(\mathbb{R}^n)}^{-1}. \end{aligned} \quad (24)$$

Note that

$$\begin{aligned} I_{\beta+m\beta_1}(\chi_{B_j})(x) &\geq I_{\beta+m\beta_1}(\chi_{B_j})(x) \cdot \chi_{B_j}(x) \\ &= \int_{B_j} \frac{dy}{|x-y|^{n-\beta-m\beta_1}} \cdot \chi_{B_j}(x) \\ &\geq C 2^{j(\beta+m\beta_1)} \cdot \chi_{B_j}(x). \end{aligned} \quad (25)$$

By Lemmas 8 and 11, we obtain

$$\begin{aligned} \|\chi_{B_j}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}^{-1} &\leq C 2^{-nj} \|\chi_{B_j}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \\ &\leq C 2^{-nj} 2^{-j(\beta+m\beta_1)} \|I_{\beta+m\beta_1}(\chi_{B_j})\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \\ &\leq C 2^{-j(\beta+m\beta_1)} 2^{-nj} \|\chi_{B_j}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \\ &\leq C 2^{-j(\beta+m\beta_1)} \|\chi_{B_j}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}^{-1}. \end{aligned} \quad (26)$$

Combining (24) and (26), we have the estimate

$$\begin{aligned} \|I_{\beta,b}^m(f_j) \chi_k\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} &\leq C 2^{(k-j)(\beta+m\beta_1-nr)} \\ &\quad \times \|b\|_{\text{Lip}_{\beta_1}}^m \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}. \end{aligned} \quad (27)$$

Thus,

$$\begin{aligned} U_1 &\leq C \|b\|_{\text{Lip}_{\beta_1}}^{mq_1} \\ &\quad \times \sum_{k=-\infty}^{\infty} \left(\sum_{j=-\infty}^{k-2} 2^{\alpha j q_1} \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}^{q_1} 2^{(k-j)(\beta+m\beta_1-nr+\alpha)q_1} \right)^{q_1}. \end{aligned} \quad (28)$$

If $1 < q_1 < \infty$, noting that $\beta + m\beta_1 - nr + \alpha < 0$, by Hölder's inequality, we have

$$\begin{aligned} U_1 &\leq C \|b\|_{\text{Lip}_{\beta_1}}^{mq_1} \\ &\quad \times \sum_{k=-\infty}^{\infty} \left(\sum_{j=-\infty}^{k-2} 2^{\alpha j q_1} \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}^{q_1} 2^{(k-j)(\beta+m\beta_1-nr+\alpha)q_1/2} \right)^{q_1/q_1'} \\ &\quad \times \left(\sum_{j=-\infty}^{k-2} 2^{(k-j)(\beta+m\beta_1-nr+\alpha)q_1'/2} \right)^{q_1/q_1'} \\ &\leq C \|b\|_{\text{Lip}_{\beta_1}}^{mq_1} \sum_{j=-\infty}^{\infty} 2^{\alpha j q_1} \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}^{q_1} \sum_{k=j+2}^{\infty} 2^{(k-j)(\beta+m\beta_1-nr+\alpha)q_1/2} \\ &\leq C \|b\|_{\text{Lip}_{\beta_1}}^{mq_1} \|f\|_{\dot{K}_{p_1(\cdot)}^{\alpha, q_1}(\mathbb{R}^n)}^{q_1}. \end{aligned} \quad (29)$$

If $0 < q_1 \leq 1$, by inequality (21), we have

$$\begin{aligned} U_1 &\leq C \|b\|_{\text{Lip}_{\beta_1}}^{mq_1} \\ &\quad \times \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{k-2} 2^{\alpha j q_1} \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}^{q_1} 2^{(k-j)(\beta+m\beta_1-nr+\alpha)q_1} \\ &\leq C \|b\|_{\text{Lip}_{\beta_1}}^{mq_1} \sum_{j=-\infty}^{\infty} 2^{\alpha j q_1} \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}^{q_1} \sum_{k=j+2}^{\infty} 2^{(k-j)(\beta+m\beta_1-nr+\alpha)q_1} \\ &\leq C \|b\|_{\text{Lip}_{\beta_1}}^{mq_1} \|f\|_{\dot{K}_{p_1(\cdot)}^{\alpha, q_1}(\mathbb{R}^n)}^{q_1}. \end{aligned} \quad (30)$$

Next, we estimate U_2 . By Lemma 12(2), we obtain

$$U_2 \leq C \|b\|_{\text{Lip}_{\beta_1}}^{mq_1} \sum_{k=-\infty}^{\infty} \left(\sum_{j=k-1}^{\infty} 2^{\alpha j} \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}^{q_1} 2^{\alpha(k-j)} \right)^{q_1}. \quad (31)$$

If $1 < q_1 < \infty$, by Hölder's inequality, we have

$$\begin{aligned} U_2 &\leq C \|b\|_{\text{Lip}_{\beta_1}}^{mq_1} \sum_{k=-\infty}^{\infty} \left(\sum_{j=k-1}^{\infty} 2^{\alpha j q_1} \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}^{q_1} 2^{\alpha(k-j)q_1/2} \right) \\ &\quad \times \left(\sum_{j=k-1}^{\infty} 2^{\alpha(k-j)q_1'/2} \right)^{q_1/q_1'} \\ &\leq C \|b\|_{\text{Lip}_{\beta_1}}^{mq_1} \sum_{j=-\infty}^{\infty} 2^{\alpha j q_1} \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}^{q_1} \sum_{k=-\infty}^{j+1} 2^{\alpha(k-j)q_1/2} \\ &\leq C \|b\|_{\text{Lip}_{\beta_1}}^{mq_1} \|f\|_{\dot{K}_{p_1(\cdot)}^{\alpha, q_1}(\mathbb{R}^n)}^{q_1}. \end{aligned} \quad (32)$$

If $0 < q_1 \leq 1$, by inequality (21), we have

$$\begin{aligned} U_2 &\leq C \|b\|_{\text{Lip}_{\beta_1}}^{mq_1} \sum_{k=-\infty}^{\infty} \sum_{j=k-1}^{\infty} 2^{\alpha j q_1} \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}^{q_1} 2^{\alpha(k-j)q_1} \\ &\leq C \|b\|_{\text{Lip}_{\beta_1}}^{mq_1} \sum_{j=-\infty}^{\infty} 2^{\alpha j q_1} \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}^{q_1} \sum_{k=-\infty}^{j+1} 2^{\alpha(k-j)q_1} \\ &\leq C \|b\|_{\text{Lip}_{\beta_1}}^{mq_1} \|f\|_{\dot{K}_{p_1(\cdot)}^{\alpha, q_1}(\mathbb{R}^n)}^{q_1}. \end{aligned} \quad (33)$$

Combining the estimates for U_1 and U_2 , the proof of Theorem 6 is completed. $\square$

Proof of Theorem 7. Let $f \in \dot{K}_{p_1(\cdot)}^{\alpha, q_1}(\mathbb{R}^n)$; we can write

$$f(x) = \sum_{j=-\infty}^{\infty} f(x) \chi_j(x) = \sum_{j=-\infty}^{\infty} f_j(x). \quad (34)$$

By inequality (21), we obtain

$$\begin{aligned}
 & \|I_{\beta,b}^m(f)\|_{\dot{K}_{p_2(\cdot)}^{\alpha,q_2}(\mathbb{R}^n)}^{q_1} \\
 &= C \left(\sum_{k=-\infty}^{\infty} 2^{\alpha q_2 k} \|I_{\beta,b}^m(f) \chi_k\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}^{q_2} \right)^{q_1/q_2} \\
 &\leq C \sum_{k=-\infty}^{\infty} 2^{\alpha q_1 k} \left(\sum_{j=-\infty}^{k-2} \|I_{\beta,b}^m(f_j) \chi_k\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \right)^{q_1} \quad (35) \\
 &+ C \sum_{k=-\infty}^{\infty} 2^{\alpha q_1 k} \left(\sum_{j=k-1}^{\infty} \|I_{\beta,b}^m(f_j) \chi_k\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \right)^{q_1} \\
 &= V_1 + V_2.
 \end{aligned}$$

For V_1 , using Hölder's inequality and Lemma 8, we have

$$\begin{aligned}
 |I_{\beta,b}^m(f_j) \chi_k| &\leq C 2^{k(\beta-n)} \int_{R_j} |b(x) - b(y)|^m |f_j(y)| dy \cdot \chi_k(x) \\
 &\leq C 2^{k(\beta-n)} \sum_{i=0}^m C_m^i |b(x) - b_{B_j}|^{m-i} \\
 &\quad \times \int_{R_j} |b_{B_j} - b(y)|^i |f_j(y)| dy \\
 &\leq C 2^{k(\beta-n)} \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \\
 &\quad \times \sum_{i=0}^m C_m^i |b(x) - b_{B_j}|^{m-i} \|(b_{B_j} - b)^i \chi_j\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}. \quad (36)
 \end{aligned}$$

By Lemmas 9, 10, and 13, we have

$$\begin{aligned}
 & \|I_{\beta,b}^m(f_j) \chi_k\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \\
 &\leq C 2^{k(\beta-n)} \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \\
 &\quad \times \sum_{i=0}^m C_m^i \|(b(x) - b_{B_j})^{m-i} \chi_k\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \\
 &\quad \times \|(b_{B_j} - b)^i \chi_j\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)} \\
 &\leq C 2^{k(\beta-n)} \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \\
 &\quad \times \sum_{i=0}^m C_m^i (k-j)^{m-i} \|b\|_{\text{BMO}}^{m-i} \|\chi_{B_k}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \\
 &\quad \times \|b\|_{\text{BMO}}^i \|\chi_{B_j}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}
 \end{aligned}$$

$$\begin{aligned}
 &= C(k-j+1)^m \|b\|_{\text{BMO}}^m \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \\
 &\quad \times 2^{k(\beta-n)} \|\chi_{B_k}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \|\chi_{B_j}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)} \\
 &\leq C(k-j+1)^m \|b\|_{\text{BMO}}^m \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \\
 &\quad \times 2^{k\beta} \|\chi_{B_j}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)} \|\chi_{B_k}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}^{-1} \\
 &\leq C(k-j+1)^m \|b\|_{\text{BMO}}^m \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \\
 &\quad \times 2^{k\beta} \|\chi_{B_j}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)} \|\chi_{B_k}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}^{-1} \frac{\|\chi_{B_j}\|_{L^{p_2'(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_k}\|_{L^{p_2'(\cdot)}(\mathbb{R}^n)}} \\
 &\leq C(k-j+1)^m \|b\|_{\text{BMO}}^m \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \\
 &\quad \times 2^{k\beta} 2^{nr(j-k)} \|\chi_{B_j}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)} \|\chi_{B_j}\|_{L^{p_2'(\cdot)}(\mathbb{R}^n)}^{-1}. \quad (37)
 \end{aligned}$$

Note that

$$I_{\beta}(\chi_{B_j})(x) \geq \int_{B_j} \frac{dy}{|x-y|^{n-\beta}} \cdot \chi_{B_j}(x) \geq C 2^{j\beta} \cdot \chi_{B_j}(x). \quad (38)$$

By Lemmas 8 and 11, we obtain

$$\begin{aligned}
 \|\chi_{B_j}\|_{L^{p_2'(\cdot)}(\mathbb{R}^n)}^{-1} &\leq C 2^{-nj} \|\chi_{B_j}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \\
 &\leq C 2^{-nj} 2^{-j\beta} \|I_{\beta}(\chi_{B_j})\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \\
 &\leq C 2^{-j\beta} 2^{-nj} \|\chi_{B_j}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \\
 &\leq C 2^{-j\beta} \|\chi_{B_j}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}^{-1}. \quad (39)
 \end{aligned}$$

Combining (37) and (39), we have the estimate

$$\begin{aligned}
 \|I_{\beta,b}^m(f_j) \chi_k\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} &\leq C(k-j+1)^m \\
 &\quad \times \|b\|_{\text{BMO}}^m 2^{(k-j)(\beta-nr)} \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}. \quad (40)
 \end{aligned}$$

Thus,

$$\begin{aligned}
 V_1 &\leq C \|b\|_{\text{BMO}}^{mq_1} \sum_{k=-\infty}^{\infty} \left(\sum_{j=-\infty}^{k-2} 2^{\alpha j} \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \right. \\
 &\quad \left. \times (k-j+1)^m 2^{(k-j)(\beta-nr+\alpha)} \right)^{q_1}. \quad (41)
 \end{aligned}$$

In case of $1 < q_1 < \infty$, noting that $\beta - nr + \alpha < 0$, by Hölder's inequality, we have

$$\begin{aligned}
 V_1 &\leq C \|b\|_{\text{BMO}}^{mq_1} \\
 &\times \sum_{k=-\infty}^{\infty} \left(\sum_{j=-\infty}^{k-2} 2^{\alpha j q_1} \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}^{q_1} 2^{(k-j)(\beta-nr+\alpha)q_1/2} \right) \\
 &\times \left(\sum_{j=-\infty}^{k-2} (k-j+1)^{mq'_1} 2^{(k-j)(\beta-nr+\alpha)q'_1/2} \right)^{q_1/q'_1} \\
 &\leq C \|b\|_{\text{BMO}}^{mq_1} \sum_{j=-\infty}^{\infty} 2^{\alpha j q_1} \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}^{q_1} \sum_{k=j+2}^{\infty} 2^{(k-j)(\beta-nr+\alpha)q_1/2} \\
 &\leq C \|b\|_{\text{BMO}}^{mq_1} \|f\|_{\dot{K}_{p_1(\cdot)}^{\alpha, q_1}(\mathbb{R}^n)}^{q_1}.
 \end{aligned} \tag{42}$$

In case of $0 < q_1 \leq 1$, by inequality (21), we have

$$\begin{aligned}
 V_1 &\leq C \|b\|_{\text{BMO}}^{mq_1} \\
 &\times \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{k-2} 2^{\alpha j q_1} \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}^{q_1} \\
 &\times (k-j+1)^{mq_1} 2^{(k-j)(\beta-nr+\alpha)q_1} \\
 &\leq C \|b\|_{\text{BMO}}^{mq_1} \sum_{j=-\infty}^{\infty} 2^{\alpha j q_1} \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}^{q_1} \\
 &\times \sum_{k=j+2}^{\infty} (k-j+1)^{mq_1} 2^{(k-j)(\beta-nr+\alpha)q_1} \\
 &\leq C \|b\|_{\text{BMO}}^{mq_1} \|f\|_{\dot{K}_{p_1(\cdot)}^{\alpha, q_1}(\mathbb{R}^n)}^{q_1}.
 \end{aligned} \tag{43}$$

For V_2 , by Lemma 12(1), we obtain

$$V_2 \leq C \|b\|_{\text{BMO}}^{mq_1} \sum_{k=-\infty}^{\infty} \left(\sum_{j=k-1}^{\infty} 2^{\alpha j} \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}^{q_1} 2^{\alpha(k-j)} \right)^{q_1}. \tag{44}$$

If $1 < q_1 < \infty$, by Hölder's inequality, we have

$$\begin{aligned}
 V_2 &\leq C \|b\|_{\text{BMO}}^{mq_1} \sum_{k=-\infty}^{\infty} \left(\sum_{j=k-1}^{\infty} 2^{\alpha j q_1} \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}^{q_1} 2^{\alpha(k-j)q_1/2} \right) \\
 &\times \left(\sum_{j=k-1}^{\infty} 2^{\alpha(k-j)q'_1/2} \right)^{q_1/q'_1} \\
 &\leq C \|b\|_{\text{BMO}}^{mq_1} \sum_{j=-\infty}^{\infty} 2^{\alpha j q_1} \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}^{q_1} \sum_{k=-\infty}^{j+1} 2^{\alpha(k-j)q_1/2} \\
 &\leq C \|b\|_{\text{BMO}}^{mq_1} \|f\|_{\dot{K}_{p_1(\cdot)}^{\alpha, q_1}(\mathbb{R}^n)}^{q_1}.
 \end{aligned} \tag{45}$$

If $0 < q_1 \leq 1$, by inequality (21), we have

$$\begin{aligned}
 V_2 &\leq C \|b\|_{\text{BMO}}^{mq_1} \sum_{k=-\infty}^{\infty} \sum_{j=k-1}^{\infty} 2^{\alpha j q_1} \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}^{q_1} 2^{\alpha(k-j)q_1} \\
 &\leq C \|b\|_{\text{BMO}}^{mq_1} \sum_{j=-\infty}^{\infty} 2^{\alpha j q_1} \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}^{q_1} \sum_{k=-\infty}^{j+1} 2^{\alpha(k-j)q_1} \\
 &\leq C \|b\|_{\text{BMO}}^{mq_1} \|f\|_{\dot{K}_{p_1(\cdot)}^{\alpha, q_1}(\mathbb{R}^n)}^{q_1}.
 \end{aligned} \tag{46}$$

Combining the estimates for V_1 and V_2 , consequently, we have proved Theorem 7. $\square$

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