Research Article

Sensitivity Analysis for Nonlinear Set-Valued Variational Equations in Banach Framework

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The main purpose of this paper is to study the sensitivity analysis for nonlinear set-valued variational equations based on (A, η) -resolvent operator technique. The obtained results encompass a broad model of results.

1. Introduction

The set-valued inclusions problem, which was initiated by Di Bella [1] and Huang et al. [2, 3], is a useful extension of the mathematical analysis and variational equation and is important context in the set-valued equations. Verma [4], Huang [5], Fang and Huang [6], Lan et al. [7], Khan and Salahuddin [8, 9], Li et al. [10], and Yen and Lee [11] introduced the concepts of A -monotone, H -monotone, (A, η) -monotone operator, (A, η) -accretive mappings, and resolvent operator associated with them, respectively. Dafermos [12] studied the sensitivity property of solutions of particular kinds of variational inequality on parameter which takes values on an open subset of Euclidean space R^K . Tobin [13], Verma [14, 15], Lee and Salahuddin [3], Kypris [16], Moudafi [17], Noor [18], Robinson [19], Yen and Lee [11], and Hussain et al. [20] studied the sensitivity analysis of various types of variational inequalities.

In this paper, we present the sensitivity analysis for (A, η) -accretive variational equations based on the (A, η) -resolvent operator technique. The obtained results generalize a wide range of results on the sensitivity analysis for nonlinear set-valued variational equations in Banach spaces.

2. Preliminaries

Let X be a real Banach space with dual space X^* , and let $\langle \cdot, \cdot \rangle$ be the dual pair between X and X^* , 2^X denote the family

of all the nonempty subset of X and $CB(X)$ the family of all nonempty closed bounded subsets of X . The generalized duality mapping $J_q : X \rightarrow 2^{X^*}$ is defined by

$$J_q(x) = \{f^* \in X^* : \langle u, f^* \rangle = \|u\|^q, \|f^*\| = \|u\|^{q-1}\}, \quad \forall u \in X, \quad (1)$$

where $q > 1$ is constant.

The modulus of smoothness of X is the function $\rho_X : [0, \infty) \rightarrow [0, \infty)$ defined by

$$\rho_X(t) = \sup \left\{ \frac{1}{2} (\|u + v\| + \|u - v\|) - 1 : \|u\| \leq 1, \|v\| < t \right\}. \quad (2)$$

A Banach space X is called uniformly smooth if

$$\lim_{t \rightarrow 0} \frac{\rho_X(t)}{t} = 0. \quad (3)$$

X is called q -uniformly smooth if there exists a constant $c > 0$ such that

$$\rho_X(t) \leq ct^q \quad (q > 1). \quad (4)$$

Let $\eta : X \times X \rightarrow X$ be a single-valued mapping. The map η is called τ -Lipschitz continuous if there exists a constant $\tau > 0$ such that

$$\|\eta(u, v)\| \leq \tau \|u - v\|, \quad \forall u, v \in X. \quad (5)$$

Definition 1. Let $\eta : X \times X \rightarrow X$ be a single-valued mapping, and let $M : X \rightarrow 2^X$ be a mapping on X . A set-valued mapping M is said to be

(i) accretive if

$$\langle u^* - v^*, J_q(u - v) \rangle \geq 0, \quad \forall u, v \in X, \quad u^* \in M(u), \quad v^* \in M(v); \quad (6)$$

(ii) η -accretive if

$$\langle u^* - v^*, J_q(\eta(u, v)) \rangle \geq 0, \quad \forall u, v \in X, \quad u^* \in M(u), \quad v^* \in M(v); \quad (7)$$

(iii) (r, η) -strongly accretive if

$$\langle u^* - v^*, J_q(\eta(u, v)) \rangle \geq r \|u - v\|^q, \quad \forall (u, u^*), (v, v^*) \in \text{Graph}(M); \quad (8)$$

(iv) η -pseudomonotone if

$$\langle v^*, J_q(\eta(u, v)) \rangle \geq 0, \quad (9)$$

implying

$$\langle u^*, J_q(\eta(u, v)) \rangle \geq 0, \quad \forall (u, u^*), (v, v^*) \in \text{Graph}(M); \quad (10)$$

(v) (m, η) -relaxed accretive if there exists a positive constant m such that

$$\langle u^* - v^*, J_q(\eta(u, v)) \rangle \geq (-m) \|u - v\|^q, \quad \forall (u, u^*), (v, v^*) \in \text{Graph}(M). \quad (11)$$

Definition 2. A single-valued mapping $A : X \rightarrow X$ is said to be

(i) accretive if

$$\langle A(u) - A(v), J_q(u - v) \rangle \geq 0, \quad \forall u, v \in X; \quad (12)$$

(ii) strictly accretive if A is accretive and

$$\langle A(u) - A(v), J_q(u - v) \rangle = 0, \quad \text{iff } u = v, \quad \forall u, v \in X; \quad (13)$$

(iii) (r, η) -strongly accretive if there exists a constant $r > 0$ such that

$$\langle A(u) - A(v), J_q(\eta(u, v)) \rangle \geq r \|u - v\|^q, \quad \forall u, v \in X; \quad (14)$$

(iv) α -Lipschitz continuous if there exists a constant $\alpha > 0$ such that

$$\|A(u) - A(v)\| \geq \alpha \|u - v\|, \quad \forall u, v \in X. \quad (15)$$

Definition 3. A single-valued mapping $N : X \times X \rightarrow X$ is said to be

(i) (μ, v) -Lipschitz continuous if there exist constants $\mu, v > 0$ such that

$$\|N(x_1, y_1) - N(x_2, y_2)\| \leq \mu \|x_1 - x_2\| + v \|y_1 - y_2\|, \quad \forall x_i, y_i \in X, \quad i = 1, 2; \quad (16)$$

(ii) (φ, κ) - Q -relaxed cocoercive with respect to AQ in the second argument if there exists constant $\varphi, \kappa > 0$ and for all $x_i \in X, y_i \in Q(u_i), (i = 1, 2)$ such that

$$\langle N(y_1, \cdot) - N(y_2, \cdot), J_q(A(y_1) - A(y_2)) \rangle \geq \varphi \|N(y_1, \cdot) - N(y_2, \cdot)\|^q + \kappa \|x_1 - x_2\|^q, \quad (17)$$

where $A, Q : X \rightarrow X$ are single-valued mappings.

Definition 4. A mapping $M : X \rightarrow 2^X$ is said to be maximal (m, η) -relaxed accretive if

(i) M is (M, η) -accretive;

(ii) for $(u, u^*) \in X \times X$ and

$$\langle u^* - v^*, J_q(\eta(u, v)) \rangle \geq (-m) \|u - v\|^q, \quad \forall (v, v^*) \in \text{Graph}(M), \quad (18)$$

one has $u^* \in M(u)$.

Definition 5. Let $A : X \rightarrow X$ and $\eta : X \times X \rightarrow X$ be two single-valued mappings; the mapping $M : X \rightarrow 2^X$ is said to be (A, η) -accretive if

(i) M is (M, η) -relaxed accretive;

(ii) $R(A + \rho M) = X$ for $\rho > 0$.

Note that alternatively, the mapping $M : X \rightarrow 2^X$ is said to be (A, η) -accretive if

(i) M is (M, η) -relaxed accretive;

(ii) $A + \rho M$ is η -pseudoaccretive for $\rho > 0$.

The following propositions will be needed in the sequel. For more details one can refer to [7, 10, 15].

Proposition 6. Let $A : X \rightarrow X$ be an (r, η) -strongly accretive single-valued mapping, and let $M : X \rightarrow 2^X$ be an (A, η) -accretive mapping. Let $\eta : X \times X \rightarrow X$ be τ -Lipschitz continuous single-valued mapping. Then, M is maximal (m, η) -relaxed accretive, and $(A + \rho M)X = X$ for $0 < \rho < r/m$.

Proposition 7. Let $A : X \rightarrow X$ be an (r, η) -strongly accretive single-valued mapping, and let $M : X \rightarrow 2^X$ be an (A, η) -accretive mapping. In addition, let $\eta : X \times X \rightarrow X$ be τ -Lipschitz continuous. Then $(A + \rho M)$ is maximal η -accretive for $0 < \rho < r/m$.

Proposition 8. Let $A : X \rightarrow X$ be an (r, η) -strongly accretive mapping, and let $M : X \rightarrow 2^X$ be an (A, η) -accretive mapping. If, in addition, $\eta : X \times X \rightarrow X$ is τ -Lipschitz continuous, then, the operator $(A + \rho M)^{-1}$ is single-valued for $0 < \rho < r/m$.

Definition 9. Let $\eta : X \times X \rightarrow X$ be a single-valued mapping. Let $A : X \rightarrow X$ be an (r, η) -strongly accretive mapping, and let $M : X \rightarrow 2^X$ be an (A, η) -accretive mapping. Then, the generalized resolvent operator $R_M^{\eta, \rho, A} : X \rightarrow X$ is defined by

$$R_M^{\eta, \rho, A}(u) = (A + \rho M)^{-1}(u), \quad (19)$$

where $\rho > 0$ is a constant.

Definition 10. The mapping $f : X \times X \times L \rightarrow X$ is said to be β -strongly accretive with respect to A in the first argument if there exists a positive constant β such that

$$\begin{aligned} \langle f(x, \lambda) - f(y, \lambda), J_q(x - y) \rangle \\ \geq \beta \|x - y\|^q, \quad \forall (x, y, u, \lambda) \in X \times X \times X \times L. \end{aligned} \quad (20)$$

3. Sensitivity Analysis

Let Q , let T , and let $V : X \times L \rightarrow 2^X$ be three set-valued mappings, and let a nonlinear mapping $N : X \times X \times L \rightarrow X$ be an AQ-accretive mapping with respect to first argument, and let $M : X \times X \times L \rightarrow 2^X$ be the mapping, where L is a nonempty open subset of X . Let $f, g : X \times L \rightarrow X$ be the single-valued mappings. Furthermore, let $\eta : X \times X \rightarrow X$ be a nonlinear mapping. Then, the problem of finding an element $u \in X$, $x \in Q(u, \lambda)$, $y \in T(u, \lambda)$, $w \in V(u, \lambda)$ such that

$$u \in f(y, \lambda) + N(x, g(u, \lambda), \lambda) + M(u, w, \lambda), \quad (21)$$

where $A \in L$ is the set-valued parameter is called a generalized set-valued variational inclusions.

The solvability of problem (21) depends on the equivalence between (21) and the problem of finding the fixed point of the associated resolvent operator.

Note that if M is (A, η) -accretive, then the corresponding resolvent operator defined by

$$R_{M(\cdot, v, \lambda)}^{\eta, \rho, A}(u) = (A + \rho M(\cdot, v, \lambda))^{-1}(u), \quad \forall u \in X, \quad (22)$$

where $\rho > 0$ and A is an (r, η) -strongly accretive mapping.

Lemma 11 (see [7, 10, 15]). Let X be a real Banach spaces, and let $\eta : X \times X \rightarrow X$ be a τ -Lipschitz continuous nonlinear mapping. Let $A : X \rightarrow X$ be (r, η) -strongly accretive mapping, and let $M : X \times X \times L \rightarrow 2^X$ be (A, η) -accretive in the first variable with m . Then, the resolvent operator associated with $M(\cdot, v, \lambda)$ for a fixed $v \in X$, defined by

$$R_{M(\cdot, v, \lambda)}^{\eta, \rho, A}(u) = (A + \rho M(\cdot, v, \lambda))^{-1}(u), \quad \forall u \in X, \quad (23)$$

is $(\tau^{q-1}/(r - \rho m))$ -Lipschitz continuous, where $\rho \in (0, r/m)$; that is,

$$\begin{aligned} \|R_{M(\cdot, w, \lambda)}^{\eta, \rho, A}(u) - R_{M(\cdot, w, \lambda)}^{\eta, \rho, A}(v)\| \\ \leq \frac{\tau^{q-1}}{r - \rho m} \|u - v\|, \quad \forall u, v, w \in X. \end{aligned} \quad (24)$$

Lemma 12 (see [7, 10, 15]). Let X be a real uniformly smooth Banach spaces, let $A : X \rightarrow X$ be the (r, η) -strongly accretive mapping, and let $M : X \times X \times L \rightarrow 2^X$ be an (A, η) -accretive in the first variable. Let $\eta : X \times X \rightarrow X$ be a τ -Lipschitz continuous nonlinear mapping; then the following statements are mutually equivalent:

(i) an element $u \in X$, $x \in Q(u, \lambda)$, $y \in T(u, \lambda)$, $w \in V(u, \lambda)$ is a solution to (21);

(ii) the mapping $G : X \times L \rightarrow X$ defined by

$$\begin{aligned} G(u, \lambda) \\ = R_{M(\cdot, w, \lambda)}^{\eta, \rho, A}(A(u) - \rho N(x, g(u, \lambda), \lambda) + \rho u - \rho f(y, \lambda)) \end{aligned} \quad (25)$$

has a fixed point.

Lemma 13 (see [21]). Let X be a real uniformly smooth Banach spaces. Then, X is q -uniformly smooth if and only if there exists a constant $c_q > 0$ such that for all $u, v \in X$

$$\|u + v\|^q \leq \|u\|^q + q \langle v, J_q(u) \rangle + c_q \|v\|^q. \quad (26)$$

Theorem 14. Let X be a q -uniformly smooth Banach spaces, and let $\eta : X \times X \rightarrow X$ be a τ -Lipschitz continuous nonlinear mapping. Let $A : X \rightarrow X$ be the (r, η) -strongly accretive, and let s -Lipschitz continuous mapping and $M : X \times X \times L \rightarrow 2^X$ be the (A, η) -accretive in the first variable with m . Let $T, Q, V : X \times L \rightarrow 2^X$ be the set-valued mappings and $\mathcal{H}$ -Lipschitz continuous with γ, σ , and ζ constants, respectively. Let $g, f : X \times L \rightarrow X$ be the π -Lipschitz continuous and χ -Lipschitz continuous mappings. Let $N : X \times X \times L \rightarrow X$ be the μ -Lipschitz continuous with first variable and v -Lipschitz continuous in the second variable. Let N be the (φ, κ) -Q-relaxed cocoercive with respect to AQ in the first argument. Let f be the strongly accretive mapping. If the following condition holds:

$$\|R_{M(\cdot, w, \lambda)}^{\eta, \rho, A}(u) - R_{M(\cdot, w, \lambda)}^{\eta, \rho, A}(v)\| \leq \frac{\tau^{q-1}}{r - \rho m} \|u - v\|, \quad (27)$$

$$\|R_{M(\cdot, w_1, \lambda)}^{\eta, \rho, A}(u) - R_{M(\cdot, w_2, \lambda)}^{\eta, \rho, A}(u)\| \leq \delta \|w_1 - w_2\|, \quad (28)$$

$$\begin{aligned} \tau^q \left[(s^q + c_q \rho^q \mu^q \sigma^q + q \rho \psi \mu^q \sigma^q - q \rho \kappa)^{1/q} \right. \\ \left. + \rho \left(v \pi + (1 + c_q \chi^q \gamma^q - q \beta)^{1/q} \right) \right] \\ < (r - \rho m) (1 - \delta \zeta), \end{aligned} \quad (29)$$

where $c_q > 0$ is the same as in Lemma 13 and $\rho \in (0, r/m)$. Consequently, for each $\lambda \in L$, the mapping $G(u, \lambda)$ in the light (28) has a unique fixed point $z(\lambda)$. Hence, in light of Lemma 12, $z(\lambda)$ is a unique solution to (21). Thus, one has

$$G(z(\lambda), \lambda) = z(\lambda). \quad (30)$$

Proof. For any elements $(u, v, \lambda) \in X \times X \times L$ and $x_1 \in Q(u, \lambda)$, $x_2 \in Q(v, \lambda)$, $y_1 \in T(u, \lambda)$, $y_2 \in T(v, \lambda)$, $w_1 \in V(u, \lambda)$, and $w_2 \in V(v, \lambda)$, we have

$$\begin{aligned} & \|G(u, \lambda) - G(v, \lambda)\| \\ &= \left\| R_{M(\cdot, w_1, \lambda)}^{\eta, \rho, A} (A(u) - \rho N(x_1, g(u, \lambda), \lambda) \right. \\ & \quad \left. + \rho u - \rho f(y_1, \lambda)) \right. \\ & \quad \left. - R_{M(\cdot, w_2, \lambda)}^{\eta, \rho, A} (A(v) - \rho N(x_2, g(v, \lambda), \lambda) \right. \\ & \quad \left. + \rho v - \rho f(y_2, \lambda)) \right\| \\ &\leq \left\| R_{M(\cdot, w_1, \lambda)}^{\eta, \rho, A} [A(u) - \rho N(x_1, g(u, \lambda), \lambda) \right. \\ & \quad \left. + \rho u - \rho f(y_1, \lambda)] \right. \\ & \quad \left. - R_{M(\cdot, w_2, \lambda)}^{\eta, \rho, A} [A(v) - \rho N(x_2, g(v, \lambda), \lambda) \right. \\ & \quad \left. + \rho v - \rho f(y_2, \lambda)] \right\| \\ & \quad + \left\| R_{M(\cdot, w_1, \lambda)}^{\eta, \rho, A} \right. \\ & \quad \times [A(v) - \rho N(x_2, g(v, \lambda), \lambda) + \rho v - \rho f(y_2, \lambda)] \\ & \quad \left. - R_{M(\cdot, w_2, \lambda)}^{\eta, \rho, A} \right. \\ & \quad \times [A(v) - \rho N(x_2, g(v, \lambda), \lambda) + \rho v - \rho f(y_2, \lambda)] \left\| \\ &\leq \frac{\tau^{q-1}}{r - \rho m} \left[\|A(u) - A(v) \right. \right. \\ & \quad \left. - \rho (N(x_1, g(u, \lambda), \lambda) - N(x_2, g(v, \lambda), \lambda)) \right. \\ & \quad \left. + \rho (u - v - (f(y_1, \lambda) - f(y_2, \lambda))) \right\| \\ & \quad + \delta \|w_1 - w_2\| \\ &\leq \frac{\tau^{q-1}}{r - \rho m} \left[\|A(u) - A(v) \right. \\ & \quad - \rho (N(x_1, g(u, \lambda), \lambda) - N(x_2, g(u, \lambda), \lambda)) \| \\ & \quad + \rho \|N(x_2, g(u, \lambda), \lambda) - N(x_2, g(v, \lambda), \lambda)\| \\ & \quad + \rho \|u - v - (f(y_1, \lambda) - f(y_2, \lambda))\| \\ & \quad \left. + \delta \|w_1 - w_2\| \right]. \end{aligned} \quad (31)$$

Since T , V , and Q are $\mathcal{H}$ -Lipschitz continuous and N is Lipschitz continuous with respect to first and second arguments, f , g are Lipschitz continuous, we have

$$\begin{aligned} & \|N(x_1, g(u, \lambda), \lambda) - N(x_2, g(u, \lambda), \lambda)\| \\ & \leq \mu \|x_1 - x_2\| \\ & \leq \mu \mathcal{H}(Q(u, \lambda), Q(v, \lambda)) \\ & \leq \mu \sigma \|u - v\|, \\ & \|N(x_2, g(u, \lambda), \lambda) - N(x_2, g(v, \lambda), \lambda)\| \\ & \leq v \|g(u, \lambda) - g(v, \lambda)\| \\ & \leq v \pi \|u - v\|, \\ & \|f(y_1, \lambda) - f(y_2, \lambda)\| \leq \chi \|y_1 - y_2\| \\ & \leq \chi \mathcal{H}(T(u, \lambda), T(v, \lambda)) \\ & \leq \chi \gamma \|u - v\|, \\ & \|w_1 - w_2\| \leq \mathcal{H}(V(u, \lambda), V(v, \lambda)) \leq \zeta \|u - v\|. \end{aligned} \quad (32)$$

Since $N(\cdot, \cdot)$ is (φ, κ) - Q -relaxed cocoercive with respect to AQ in the first argument and Q is $\mathcal{H}$ -Lipschitz continuous mapping, so we obtain

$$\begin{aligned} & \|A(u) - A(v) \\ & \quad - \rho (N(x_1, g(u, \lambda), \lambda) - N(x_2, g(u, \lambda), \lambda))\|^q \\ & \leq \|A(u) - A(v)\|^q \\ & \quad - q\rho \langle N(x_1, g(u, \lambda), \lambda) \\ & \quad \quad - N(x_2, g(u, \lambda), \lambda), J_q(A(u) - A(v)) \rangle \\ & \quad + c_q \rho^q \|N(x_1, g(u, \lambda), \lambda) - N(x_2, g(u, \lambda), \lambda)\|^q \\ & \leq [s^q - q\rho\kappa + q\rho\psi\mu^q\sigma^q + c_q\rho^q\mu^q\sigma^q] \|u - v\|^q. \end{aligned} \quad (33)$$

Since f is β -strongly accretive and T is the $\mathcal{H}$ -Lipschitz continuous, we have

$$\begin{aligned} & \|u - v - (f(y_1, \lambda) - f(y_2, \lambda))\|^q \\ & \leq \|u - v\|^q - q \langle f(y_1, \lambda) - f(y_2, \lambda), J_q(u - v) \rangle \\ & \quad + c_q \|f(y_1, \lambda) - f(y_2, \lambda)\|^q \\ & \leq [1 + c_q \chi^q \gamma^q - q\beta] \|u - v\|^q. \end{aligned} \quad (34)$$

Combining (31)–(34), we can get

$$\|G(u, \lambda) - G(v, \lambda)\| \leq \theta \|u - v\|, \quad (35)$$

where

$$\theta = \frac{\tau^{q-1}}{(r - \rho m)} \left[(s^q - q\rho\kappa + c_q \rho^q \mu^q \sigma^q + q\rho\psi\mu^q \sigma^q)^{1/q} + \rho v\pi + \rho(1 + c_q \chi^q \gamma^q - q\beta)^{1/q} \right] + \delta\zeta. \quad (36)$$

It follows from (29) such that $\theta < 1$; it concludes the proof. $\square$

Theorem 15. Let X be a real q -uniformly smooth Banach spaces, and let $\eta : X \times X \rightarrow X$ be a τ -Lipschitz continuous nonlinear mapping. Let $A : X \rightarrow X$ be a (r, η) -strongly accretive and s -Lipschitz continuous mapping, and let $M : X \times X \times L \rightarrow 2^X$ be (A, η) -accretive in the first variable with m . Let T, Q , and $V : X \times L \rightarrow 2^X$ be the set-valued mappings and $\mathcal{H}$ -Lipschitz continuous with respect to γ, σ , and ζ constants, respectively. Let $g, f : X \times L \rightarrow X$ be the π -Lipschitz continuous and χ -Lipschitz continuous mappings, respectively. Let $N : X \times X \times L \rightarrow X$ be the μ -Lipschitz continuous with first variable and v -Lipschitz continuous with second variable. Let N be the $(\varphi, \kappa) - Q$ -relaxed cocoercive with respect to AQ in the first variable. Let f be the β -strongly accretive mapping if the following condition holds:

$$\begin{aligned} \|R_{M(\cdot, w, \lambda)}^{\eta, \rho, A}(u) - R_{M(\cdot, w, \lambda)}^{\eta, \rho, A}(v)\| &\leq \frac{\tau^{q-1}}{r - \rho m} \|u - v\|, \\ \|R_{M(\cdot, w_1, \lambda)}^{\eta, \rho, A}(u) - R_{M(\cdot, w_2, \lambda)}^{\eta, \rho, A}(u)\| &\leq \delta \|w_1 - w_2\|, \\ \tau^q \left[(s^q + c_q \rho^q \mu^q \sigma^q + q\rho\psi\mu^q \sigma^q - q\rho\kappa)^{1/q} \right. \\ &\quad \left. + \rho \left(v\pi + (1 + c_q \chi^q \gamma^q - q\beta)^{1/q} \right) \right] \\ &< (r - \rho m)(1 - \delta\zeta), \end{aligned} \quad (37)$$

where $c_q > 0$ is the same as in Lemma 13 and $\rho \in (0, r/m)$.

If the mapping $\lambda \rightarrow N(x_1, g(u, \lambda), \lambda), \lambda \rightarrow f(y, \lambda)$, and $\lambda \rightarrow R_{M(\cdot, w, \lambda)}^{\eta, \rho, A}, \lambda \rightarrow T(u, \lambda), \lambda \rightarrow V(u, \lambda), \lambda \rightarrow g(u, \lambda)$, and $\lambda \rightarrow Q(u, \lambda)$ are continuous (or Lipschitz continuous) from L to X , then the solution $z(\lambda)$ of (21) is continuous (or Lipschitz continuous) from L to X .

Proof. From the hypothesis of theorem, for any $\lambda, \lambda^* \in L$, we have

$$\begin{aligned} \|z(\lambda) - z(\lambda^*)\| &= \|G(z(\lambda), \lambda) - G(z(\lambda^*), \lambda^*)\| \\ &\leq \|G(z(\lambda), \lambda) - G(z(\lambda^*), \lambda)\| \\ &\quad + \|G(z(\lambda^*), \lambda) - G(z(\lambda^*), \lambda^*)\| \\ &\leq \theta \|z(\lambda) - z(\lambda^*)\| \\ &\quad + \|G(z(\lambda^*), \lambda) - G(z(\lambda^*), \lambda^*)\|. \end{aligned} \quad (38)$$

It follows that

$$\begin{aligned} &\|G(z(\lambda^*), \lambda) - G(z(\lambda^*), \lambda^*)\| \\ &= \|R_{M(\cdot, w(z(\lambda^*), \lambda), \lambda)}^{\eta, \rho, A} \\ &\quad \times [A(z(\lambda^*)) - \rho N(x(z(\lambda^*), \lambda), g(z(\lambda^*), \lambda), \lambda)) \\ &\quad \times g((z(\lambda^*), \lambda), \lambda) + \rho z(\lambda^*) - \rho f(y(z(\lambda^*), \lambda), \lambda)] \\ &\quad - R_{M(\cdot, w(z(\lambda^*), \lambda^*), \lambda^*)}^{\eta, \rho, A} \\ &\quad \times [A(z(\lambda^*)) - \rho N(x(z(\lambda^*), \lambda^*), g(z(\lambda^*), \lambda^*), \lambda^*) \\ &\quad + \rho z(\lambda^*) - \rho f(y(z(\lambda^*), \lambda^*), \lambda^*)]\| \\ &\leq \|R_{M(\cdot, w(z(\lambda^*), \lambda), \lambda)}^{\eta, \rho, A} \\ &\quad \times [A(z(\lambda^*)) - \rho N(x(z(\lambda^*), \lambda), g(z(\lambda^*), \lambda), \lambda) \\ &\quad + \rho z(\lambda^*) - \rho f(y(z(\lambda^*), \lambda), \lambda)] \\ &\quad - R_{M(\cdot, w(z(\lambda^*), \lambda), \lambda)}^{\eta, \rho, A} \\ &\quad \times [A(z(\lambda^*)) - \rho N(x(z(\lambda^*), \lambda^*), g(z(\lambda^*), \lambda^*), \lambda^*) \\ &\quad + \rho z(\lambda^*) - \rho f(y(z(\lambda^*), \lambda^*), \lambda^*)]\| \\ &\quad + \|R_{M(\cdot, w(z(\lambda^*), \lambda), \lambda)}^{\eta, \rho, A} \\ &\quad \times [A(z(\lambda^*)) - \rho N(x(z(\lambda^*), \lambda^*), g(z(\lambda^*), \lambda^*), \lambda^*) \\ &\quad + \rho z(\lambda^*) - \rho f(y(z(\lambda^*), \lambda^*), \lambda^*)] \\ &\quad - R_{M(\cdot, w(z(\lambda^*), \lambda^*), \lambda^*)}^{\eta, \rho, A} \\ &\quad \times [A(z(\lambda^*)) - \rho N(x(z(\lambda^*), \lambda^*), g(z(\lambda^*), \lambda^*), \lambda^*) \\ &\quad + \rho z(\lambda^*) - \rho f(y(z(\lambda^*), \lambda^*), \lambda^*)]\| \\ &\leq \frac{\rho\tau^{q-1}}{r - \rho m} [\|N(x(z(\lambda^*), \lambda), g(z(\lambda^*), \lambda), \lambda) \\ &\quad - N(x(z(\lambda^*), \lambda^*), g(z(\lambda^*), \lambda^*), \lambda^*) \\ &\quad - (f(y(z(\lambda^*), \lambda), \lambda) - f(y(z(\lambda^*), \lambda^*), \lambda^*))\|] \\ &\quad + \|R_{M(\cdot, w(z(\lambda^*), \lambda), \lambda)}^{\eta, \rho, A} \\ &\quad \times [A(z(\lambda^*)) - \rho N(x(z(\lambda^*), \lambda^*), g(z(\lambda^*), \lambda^*), \lambda^*) \\ &\quad + \rho z(\lambda^*) - \rho f(y(z(\lambda^*), \lambda^*), \lambda^*)] \\ &\quad - R_{M(\cdot, w(z(\lambda^*), \lambda^*), \lambda^*)}^{\eta, \rho, A} \\ &\quad \times [A(z(\lambda^*)) - \rho N(x(z(\lambda^*), \lambda^*), g(z(\lambda^*), \lambda^*), \lambda^*) \\ &\quad + \rho z(\lambda^*) - \rho f(y(z(\lambda^*), \lambda^*), \lambda^*)]\|. \end{aligned} \quad (39)$$

Hence, we get

$$\begin{aligned}
& \|z(\lambda) - z(\lambda^*)\| \\
& \leq \frac{\rho\tau^{q-1}}{(r - \rho m)(1 - \theta)} \\
& \quad \times [\|N(x(z(\lambda^*), \lambda), g(z(\lambda^*), \lambda), \lambda) \\
& \quad \quad - N(x(z(\lambda^*), \lambda^*), g(z(\lambda^*), \lambda^*), \lambda^*) \\
& \quad \quad - (f(y(z(\lambda^*), \lambda), \lambda)) \\
& \quad \quad - f(y(z(\lambda^*), \lambda^*), \lambda^*)\|] \\
& + \frac{1}{1 - \theta} \|R_{M(\cdot, w(z(\lambda^*), \lambda), \lambda)}^{\eta, \rho, A} \\
& \quad \times [A(z(\lambda^*)) \\
& \quad \quad - \rho N(x(z(\lambda^*), \lambda^*), g(z(\lambda^*), \lambda^*), \lambda^*) + \rho z(\lambda^*) \\
& \quad \quad - \rho f(y(z(\lambda^*), \lambda^*), \lambda^*)] \\
& \quad - R_{M(\cdot, w(z(\lambda^*), \lambda^*), \lambda^*)}^{\eta, \rho, A} \\
& \quad \times [A(z(\lambda^*)) \\
& \quad \quad - \rho N(x(z(\lambda^*), \lambda^*), g(z(\lambda^*), \lambda^*), \lambda^*) \\
& \quad \quad + \rho z(\lambda^*) - \rho f(y(z(\lambda^*), \lambda^*), \lambda^*)]\|.
\end{aligned} \tag{40}$$

This completes the proof. $\square$

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