Research Article

Some New Characterizations and g -Minimality and Stability of g -Bases in the Hilbert Spaces

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The concept of g -basis in the Hilbert spaces is introduced by Guo (2012) who generalizes the Schauder basis in the Hilbert spaces. g -basis plays the similar role in g -frame theory to that the Schauder basis plays in frame theory. In this paper, we establish some important properties of g -bases in the Hilbert spaces. In particular, we obtain a simple condition under which some important properties established in Guo (2012) are still true. With these conditions, we also establish some new interesting properties of g -bases which are related to g -minimality. Finally, we obtain a perturbation result about g -bases.

1. Introduction

In 1946, Gabor [1] introduced a fundamental approach to signal decomposition in terms of elementary signals. In 1952, Duffin and Schaeffer [2] abstracted Gabor's method to define frames in the Hilbert spaces. Frame was reintroduced by Daubechies et al. [3] in 1986. Today, frame theory is a central tool in many areas such as characterizing function spaces and signal analysis. We refer to [4–10] for an introduction to frame theory and its applications. The following are the standard definitions on frames in the Hilbert spaces. A sequence $\{f_i\}_{i \in \mathcal{N}}$ of elements of a Hilbert space H is called a *frame* for H if there are constants $A, B > 0$ so that

$$A\|f\|^2 \leq \sum_{i \in \mathcal{N}} |\langle f, f_i \rangle|^2 \leq B\|f\|^2. \quad (1)$$

The numbers A, B are called the *lower* (resp., *upper*) frame bounds. The frame is a *tight frame* if $A = B$ and a *normalized tight frame* if $A = B = 1$.

In [11], Sun raised the concept of g -frame as follows, which generalizes the concept of frame extensively. A sequence $\{\Lambda_i \in B(H, H_i) : i \in \mathcal{N}\}$ is called a g -frame for H with respect to $\{H_i : i \in \mathcal{N}\}$, which is a sequence of closed

subspaces of a Hilbert space V , if there exist two positive constants A and B such that for any $f \in H$

$$A\|f\|^2 \leq \sum_{i \in \mathcal{N}} \|\Lambda_i f\|^2 \leq B\|f\|^2. \quad (2)$$

We simply call $\{\Lambda_i : i \in \mathcal{N}\}$ a g -frame for H whenever the space sequence $\{H_i : i \in \mathcal{N}\}$ is clear. The tight g -frame, normalized tight g -frame, g -Riesz basis are defined similarly. Recently, g -frames in the Hilbert spaces have been studied intensively; for more details see [12–16] and the references therein.

In the history of the Hilbert space theory, the concept of basis was introduced more earlier than the concept of frame. And frames are generalizations of bases. When we look back the development of g -frame theory, it is well known that g -frames are generalizations of frames. Then it is naturally to ask what are the generalizations of bases in the similar manner? It is the motivation of our paper [17]. In [17], we put up with the concept of g -basis in the Hilbert space. We also find that g -basis really fills in the gap in the g -frame theory. g -bases play the similar role in g -frame theory to that the bases play in the frame theory. But we also find that the properties of basis cannot be adjusted trivially so that they are hold by g -bases. For more details, please see [17]. In this paper, we will do some study on g -bases in the Hilbert spaces further. And some

important properties of g -bases are established. In Section 2, we will give the definitions and lemmas. In Section 3, we give some characterizations of g -bases. In Section 4, we study the property of g -minimal of g -bases. In Section 5, we discuss the stability of g -bases under perturbations.

Throughout this paper, we use $\mathcal{N}$ to denote the set of all natural numbers, $\mathbb{Z}$ to denote the set of all integer numbers, and $\mathbb{C}$ to denote the field of complex numbers. The sequence of $\{H_j : j \in \mathcal{N}\}$ always means a sequence of closed subspace of some Hilbert space V . And all Hilbert spaces are complex separable Hilbert spaces.

2. Definitions and Lemmas

In this section, we introduce the definitions and lemmas which will be needed in this paper.

Definition 1. $\{\Lambda_j \in B(H, H_j)\}_{j=1}^\infty$ is called g -complete with respect to $\{H_j\}$ if $\{f : \Lambda_j f = 0, \forall j\} = \{0\}$.

Definition 2. $\{\Lambda_j \in B(H, H_j)\}_{j=1}^\infty$ is called g -linearly independent with respect to $\{H_j\}$ if $\sum_{j=1}^\infty \Lambda_j^* g_j = 0$; then $g_j = 0$, where $g_j \in H_j$ ($j = 1, 2, \dots$).

Definition 3. $\{\Lambda_j \in B(H, H_j)\}_{j=1}^\infty$ is called g -minimal with respect to $\{H_j\}$ if, for any $m \in \mathcal{N}$ and any $g_m \in H_m$ such that $\Lambda_m^* g_m \neq 0$, we have $\Lambda_m^* g_m \notin \overline{\text{span}}_{i \neq m} \{\Lambda_i^*(H_i)\}$.

Definition 4. Let $M(H, K)$ denote the set of all maps from H to K . $\{T_i \in M(H_i, H) : i \in \mathcal{N}\}$ and $\{L_i \in M(H_i, H) : i \in \mathcal{N}\}$ are called biorthonormal if

$$\langle T_j g_j, L_i g_i \rangle = \delta_{j,i} \langle g_j, g_i \rangle, \quad \forall j, i \in \mathcal{N}, \quad g_j \in H_j, \quad g_i \in H_i. \quad (3)$$

$\{\Lambda_j \in B(H, H_j)\}_{j=1}^\infty$ and $\{\Gamma_j \in B(H, H_j)\}_{j=1}^\infty$ are called g -biorthonormal with respect to $\{H_j\}$, if $\{\Lambda_j^* : j \in \mathcal{N}\}$ and $\{\Gamma_j^* : j \in \mathcal{N}\}$ are biorthonormal; that is,

$$\langle \Lambda_j^* g_j, \Gamma_i^* g_i \rangle = \delta_{j,i} \langle g_j, g_i \rangle, \quad (4)$$

$$\forall i, j \in \mathcal{N}, \quad g_i \in H_i, \quad g_j \in H_j.$$

Definition 5. We call $\{\Lambda_j \in B(H, H_j)\}_{j=1}^\infty$ a g -basis for H with respect to $\{H_j : j \in \mathcal{N}\}$ if for any $x \in H$ there is a unique sequence $\{g_j : j \in \mathcal{N}\}$ with $g_j \in H_j$ such that $x = \sum_{j=1}^\infty \Lambda_j^* g_j$. The constant

$$C = \sup \left\{ \left\| \sum_{k=1}^m \Lambda_k^* g_k \right\| \mid \left\| \sum_{k=1}^n \Lambda_k^* g_k \right\| = 1, \quad m \leq n, \quad g_k \in H_k \right\} \quad (5)$$

is called the basis constant associated with $\{\Lambda_j : j \in \mathcal{N}\}$.

Remark 6. If $\{x_i\}_{i \in \mathcal{N}}$ is a Schauder basis of Hilbert space H , then it induces a g -basis $\{\Lambda_{x_i} : i \in \mathcal{N}\}$ of H with respect to the complex number field $\mathbb{C}$, where Λ_{x_i} is defined by $\Lambda_{x_i} f = \langle f, x_i \rangle$. In fact, it is easy to see that $\Lambda_{x_i}^* c = c \cdot x_i$ for any $c \in \mathbb{C}$,

so, for any $x \in H$, there exists a unique sequence of constants $\{a_n : n \in \mathcal{N}\}$ such that $x = \sum_{i \in \mathcal{N}} a_i x_i = \sum_{i \in \mathcal{N}} \Lambda_i^* a_i$.

Definition 7. If $\{\Lambda_j : j \in \mathcal{N}\}$ is a g -basis of H with respect to $\{H_j : j \in \mathcal{N}\}$, then, for any $x \in H$, there exists a unique sequence $\{g_j : j \in \mathcal{N}\}$ such that $g_j \in H_j$ and $x = \sum_{j=1}^\infty \Lambda_j^* g_j$. We define a map $\Gamma_j : H \rightarrow H_j$, by $\Lambda_j x = g_j$ for each j . Then $\{\Gamma_j : j \in \mathcal{N}\}$ is well defined. We call it the g -dual sequence of $\{\Lambda_j : j \in \mathcal{N}\}$; in case that $\{\Gamma_j : j \in \mathcal{N}\}$ is also a g -basis, we call it the dual g -basis of $\{\Lambda_j : j \in \mathcal{N}\}$.

Lemma 8 (see [13]). $\{\Lambda_j : j \in \mathcal{N}\}$ is g -complete if and only if $\overline{\text{span}} \{\Lambda_j^*(H_j) : j \in \mathcal{N}\} = H$.

The following result is about pseudoinverse, which plays an important role in some proofs.

Lemma 9 (see [5]). Suppose that $T : K \rightarrow H$ is a bounded surjective operator. Then there exists a bounded operator (called the pseudoinverse of T) $T^\dagger : H \rightarrow K$ for which

$$TT^\dagger f = f, \quad \forall f \in H. \quad (6)$$

The following is a simple property about g -basis, which gives a necessary condition for g -basis in terms of g -complete and g -linearly independent.

Lemma 10 (see [17]). If $\{\Lambda_j : j \in \mathcal{N}\}$ is a g -basis for H with respect to $\{H_j : j \in \mathcal{N}\}$, then $\{\Lambda_j : j \in \mathcal{N}\}$ is g -complete and g -linearly independent with respect to $\{H_j : j \in \mathcal{N}\}$.

3. Characterizations of g -Bases

In this section, we characterize g -bases under the condition that the sequence is bounded surjective operators.

Theorem 11. Suppose that $\Lambda_j \in B(H, H_j)$ and Λ_j is surjective for each $j \in \mathcal{N}$. Let $Y = \{\{g_j\}_{j=1}^\infty \mid g_j \in H_j, \sum_{j=1}^\infty \Lambda_j^* g_j \text{ is convergent}\}$. If, for any $\{g_j\}_{j=1}^\infty \in Y$, set $\|\{g_j\}\|_Y = \sup_N \|\sum_{j=1}^N \Lambda_j^* g_j\|$, then

- (1) Y is a Banach space,
- (2) when $\{\Lambda_j : j \in \mathcal{N}\}$ is a g -basis with respect to $\{H_j : j \in \mathcal{N}\}$ as well, $S : Y \rightarrow H$, $S(\{g_j\}_{j=1}^\infty) = \sum_{j=1}^\infty \Lambda_j^* g_j$ is a linear bounded and invertible operator; that is, S is a homeomorphism between H and Y .

Proof. (1) Let $\{g_j\}_{j=1}^\infty \in Y$; then $\sum_{j=1}^N \Lambda_j^* g_j$ is convergent as $N \rightarrow \infty$. Hence $\{\sum_{j=1}^N \Lambda_j^* g_j\}_{N=1}^\infty$ is a convergent sequence, so it is bounded. So $\|\{g_j\}\|_Y < \infty$. It is obvious that, for $a \in \mathbb{C}$, $\{g_j\}, \{h_j\} \in Y$, we have $\|\{g_j\} + \{h_j\}\|_Y \leq \|\{g_j\}\|_Y + \|\{h_j\}\|_Y$ and $\|a \cdot \{g_j\}\|_Y = |a| \cdot \|\{g_j\}\|_Y$. If $\|\{g_j\}\|_Y = 0$, then, for any N , $\|\sum_{j=1}^N \Lambda_j^* g_j\| = 0$. In particular, $\|\Lambda_1^* g_1\| = 0$. Since Λ_1 is surjective, there exists a bounded operator $\Lambda_1^\dagger \in B(H_1, H)$ such that $\Lambda_1 \Lambda_1^\dagger = I_{H_1}$ by Lemma 9. So $(\Lambda_1^\dagger)^* \Lambda_1^* = I_{H_1}$, which implies that $g_1 = (\Lambda_1^\dagger)^* \Lambda_1^* g_1$. It implies that

$\|g_1\| \leq \|(\Lambda_1^\dagger)^* \Lambda_1^* g_1\| = 0$. So $g_1 = 0$. Since $\|\sum_{j=1}^2 \Lambda_j^* g_j\| = \|\Lambda_2^* g_2\| = 0$, similarly, we have $g_2 = 0$. Keep doing in this way; then we get $g_j = 0$ for all $j = 1, \dots, N$. But N is arbitrary, so $\{g_j\} = \{0\}$. Thus $\|\cdot\|_Y$ is a norm on Y . Suppose $\{G^k\}_{k=1}^\infty \in Y$ is a Cauchy sequence, where $G^k = \{g_j^k\}_{j=1}^\infty$. Then

$$\lim_{k,l \rightarrow \infty} \|G^k - G^l\|_Y = \lim_{k,l \rightarrow \infty} \|\{g_j^k - g_j^l\}_Y\|_Y \quad (7)$$

$$= \lim_{k,l \rightarrow \infty} \sup_N \left\| \sum_{j=1}^N \Lambda_j^* (g_j^k - g_j^l) \right\| = 0. \quad (8)$$

For any fixed j , we have that

$$\begin{aligned} & \|\Lambda_j^* (g_j^k - g_j^l)\| \\ &= \left\| \sum_{t=1}^j \Lambda_t^* (g_t^k - g_t^l) - \sum_{t=1}^{j-1} \Lambda_t^* (g_t^k - g_t^l) \right\| \\ &\leq \left\| \sum_{t=1}^j \Lambda_t^* (g_t^k - g_t^l) \right\| + \left\| \sum_{t=1}^{j-1} \Lambda_t^* (g_t^k - g_t^l) \right\| \\ &\leq 2\|G^k - G^l\|_Y. \end{aligned} \quad (9)$$

Since Λ_j is surjective, $\Lambda_j \Lambda_j^\dagger = I_{H_j}$ by Lemma 9. So $(\Lambda_j^\dagger)^* \Lambda_j^* = I_{H_j}$. It implies that

$$\|g_j^k - g_j^l\| = \|(\Lambda_j^\dagger)^* \Lambda_j^* (g_j^k - g_j^l)\| \quad (10)$$

$$\begin{aligned} &\leq \|(\Lambda_j^\dagger)^*\| \cdot \|\Lambda_j^* (g_j^k - g_j^l)\| \\ &\leq 2 \cdot \|(\Lambda_j^\dagger)^*\| \cdot \|G^k - G^l\|_Y. \end{aligned} \quad (11)$$

So, for any fixed j , $\{g_j^k\}_{k=1}^\infty$ is a Cauchy sequence. Suppose $\lim_{k \rightarrow \infty} g_j^k = g_j$. From (8), we know that, for any $\varepsilon > 0$, there exists $L_0 > 0$, such that whenever $k, l \geq L_0$, we have

$$\sup_Q \left\| \sum_{j=1}^Q \Lambda_j^* (g_j^k - g_j^l) \right\| < \varepsilon. \quad (12)$$

Fix $l \geq L_0$, since $\lim_{k \rightarrow \infty} g_j^k = g_j$, so whenever $l \geq L_0$,

$$\sup_Q \left\| \sum_{j=1}^Q \Lambda_j^* (g_j - g_j^l) \right\| \leq \varepsilon. \quad (13)$$

Since $G^{L_0} = \{g_j^{L_0}\}_{j=1}^\infty \in Y$, $\sum_{j=1}^\infty \Lambda_j^* g_j^{L_0}$ is convergent. So there exists $K_0 > 0$, such that whenever $M > P \geq K_0$, we have

$\|\sum_{j=P+1}^M \Lambda_j^* g_j^{L_0}\| < \varepsilon$. So when $M > P > \max\{L_0, K_0\}$, we have

$$\begin{aligned} & \left\| \sum_{j=P+1}^M \Lambda_j^* g_j \right\| \\ &= \left\| \sum_{j=1}^M \Lambda_j^* (g_j - g_j^{L_0}) - \sum_{j=1}^P \Lambda_j^* (g_j - g_j^{L_0}) + \sum_{j=P+1}^M \Lambda_j^* g_j^{L_0} \right\| \\ &\leq \left\| \sum_{j=1}^M \Lambda_j^* (g_j - g_j^{L_0}) \right\| + \left\| \sum_{j=1}^P \Lambda_j^* (g_j - g_j^{L_0}) \right\| \\ &\quad + \left\| \sum_{j=P+1}^M \Lambda_j^* g_j^{L_0} \right\| \leq 3\varepsilon. \end{aligned} \quad (14)$$

So $\sum_{j=1}^\infty \Lambda_j^* g_j$ is convergent; thus $G = \{g_j\} \in Y$. Let $l \rightarrow \infty$ in inequality (13), we get that $\|G^l - G\|_Y \rightarrow 0$. Hence Y is a complete normed space; that is, Y is a Banach space.

(2) If $\{\Lambda_j : j \in \mathcal{N}\}$ is a g -basis, then it is g -complete and g -linearly independent with respect to $\{H_j : j \in \mathcal{N}\}$ by Lemma 10; then the operator $S : Y \rightarrow H$, $S(\{g_j\}_{j=1}^\infty) = \sum_{j=1}^\infty \Lambda_j^* g_j$ is not only well defined but also is one to one and onto. And, for any $\{g_j\}_{j=1}^\infty \in Y$, we have

$$\begin{aligned} \|S(\{g_j\}_{j=1}^\infty)\| &= \left\| \sum_{j=1}^\infty \Lambda_j^* g_j \right\| = \lim_{N \rightarrow \infty} \left\| \sum_{j=1}^N \Lambda_j^* g_j \right\| \\ &\leq \sup_Q \left\| \sum_{j=1}^Q \Lambda_j^* g_j \right\| = \|\{g_j\}\|_Y. \end{aligned} \quad (15)$$

So S is bounded operator. Since Y is a Banach space, by the open mapping theorem we get that S is a homeomorphism. $\square$

Theorem 12. Suppose, for each $j \in \mathcal{N}$, $\Lambda_j \in B(H, H_j)$ and Λ_j is surjective. $\{\Lambda_j : j \in \mathcal{N}\}$ is a g -basis of H with respect to $\{H_j : j \in \mathcal{N}\}$ and $\{\Gamma_j : j \in \mathcal{N}\}$ is its g -dual sequence. Then

- (1) for all $x \in H$, let $S_N x = \sum_{j=1}^N \Lambda_j^* \Gamma_j x$; then $\sup_N \|S_N x\| < \infty$,
- (2) $C = \sup_N \|S_N\| < \infty$,
- (3) $\|x\| = \sup_N \|S_N x\|$ is a norm on H and $\|\cdot\| \leq \|\cdot\| \leq C \|\cdot\|$.

Proof. (1) Let Y and S be as defined in Theorem 11. Then for any $x \in H$, $S^{-1}x = \{\Gamma_j x : j \in \mathcal{N}\}$. So

$$\begin{aligned} \sup_N \|S_N x\| &= \sup_N \left\| \sum_{j=1}^N \Lambda_j^* \Gamma_j x \right\| = \|\{\Gamma_j x\}\|_Y \\ &= \|S^{-1}x\| \leq \|S^{-1}\| \|x\| < \infty. \end{aligned} \quad (16)$$

(2) Since $\|S_N x\| \leq \sup_Q \|S_Q x\| \leq \|S^{-1}\| \|x\|$, $\|S_N\| \leq \|S^{-1}\|$. Thus $C = \sup_N \|S_N\| < \infty$.

(3) It is obvious that $\|\cdot\|$ is a seminorm. It is sufficient to show that $\|\cdot\| \leq \|\cdot\| \leq C\|\cdot\|$. For any $x \in H$, we have

$$\|x\| = \sup_N \|S_N x\| \leq \sup_N \|S_N\| \|x\| = C\|x\|. \quad (17)$$

On the other hand,

$$\begin{aligned} \|x\| &= \left\| \sum_{j=1}^{\infty} \Lambda_j^* \Gamma_j x \right\| = \lim_{N \rightarrow \infty} \left\| \sum_{j=1}^N \Lambda_j^* \Gamma_j x \right\| \\ &\leq \sup_Q \left\| \sum_{j=1}^Q \Lambda_j^* \Gamma_j x \right\| = \sup_Q \|S_Q x\| = \|x\|. \end{aligned} \quad (18)$$

□

Theorem 13. Suppose that $\Lambda_j \in B(H, H_j)$ and Λ_j is surjective. $\{\Lambda_j : j \in \mathcal{N}\}$ is a g -basis for H with respect to $\{H_j : j \in \mathcal{N}\}$ and $\{\Gamma_j : j \in \mathcal{N}\}$ is the g -dual sequence of $\{\Lambda_j : j \in \mathcal{N}\}$. Then Γ_j is linear and bounded; that is, $\Gamma_j \in B(H, H_j)$ for each $j \in \mathcal{N}$.

Proof. Suppose that $x = \sum_{j \in \mathcal{N}} \Lambda_j^* g_j$ and $y = \sum_{j \in \mathcal{N}} \Lambda_j^* h_j$; then $\Gamma_j x = g_j$ and $\Gamma_j y = h_j$ by Definition 7. Since $x + y = \sum_{j \in \mathcal{N}} \Lambda_j^* (g_j + h_j)$, $\Gamma_j(x + y) = g_j + h_j = \Gamma_j x + \Gamma_j y$ and $\Gamma_j(ax) = ag_j = a\Gamma_j x$. It implies that Γ_j is linear, since, for any $x \in H$, we have

$$\begin{aligned} \|\Lambda_j^* \Gamma_j x\| &= \left\| \sum_{i=1}^j \Lambda_i^* \Gamma_i x - \sum_{i=1}^{j-1} \Lambda_i^* \Gamma_i x \right\| \\ &\leq \left\| \sum_{i=1}^j \Lambda_i^* \Gamma_i x \right\| + \left\| \sum_{i=1}^{j-1} \Lambda_i^* \Gamma_i x \right\| \\ &= \|S_j x\| + \|S_{j-1} x\| \leq 2C \cdot \|x\|. \end{aligned} \quad (19)$$

Since Λ_j is surjective, $\|\Gamma_j x\| \leq \|(\Lambda_j^\dagger)^* \| \cdot \| \Lambda_j^* \Gamma_j x\|$. It follows that

$$\|\Gamma_j x\| \leq \|(\Lambda_j^\dagger)^* \| \cdot 2C \cdot \|x\|. \quad (20)$$

So Γ_j is bounded for each $j \in \mathcal{N}$. □

Theorem 14. Suppose that $\Lambda_j \in B(H, H_j)$ and Λ_j is surjective. $\{\Lambda_j : j \in \mathcal{N}\}$ is a g -basis for H with respect to $\{H_j : j \in \mathcal{N}\}$ and $\{\Gamma_j : j \in \mathcal{N}\}$ is the dual sequence of $\{\Lambda_j : j \in \mathcal{N}\}$. Then $\{\Gamma_j : j \in \mathcal{N}\}$ is also a g -basis for H and $\{\Lambda_j : j \in \mathcal{N}\}$ and $\{\Gamma_j : j \in \mathcal{N}\}$ are g -biorthonormal.

Proof. By Theorem 13, we know that $\Gamma_j \in B(H, H_j)$. Since $\{\Lambda_j : j \in \mathcal{N}\}$ is a g -basis for H with respect to $\{H_j : j \in \mathcal{N}\}$ and $\{\Gamma_j : j \in \mathcal{N}\}$ is the dual sequence of $\{\Lambda_j : j \in \mathcal{N}\}$, $x = \sum_{j \in \mathcal{N}} \Lambda_j^* \Gamma_j x$, for all $x \in H$. In particular, for each $i \in \mathcal{N}$, $g_i \in H_i$, we have

$$\begin{aligned} \Lambda_i^* g_i &= \sum_{j \in \mathcal{N}} \Lambda_j^* \Gamma_j \Lambda_i^* g_i \\ &= \Lambda_i^* \Gamma_i \Lambda_i^* g_i + \sum_{j \neq i} \Lambda_j^* \Gamma_j \Lambda_i^* g_i. \end{aligned} \quad (21)$$

Since the representation is unique, $\Gamma_i \Lambda_i^* g_i = g_i$ and $\Gamma_j \Lambda_i^* g_i = 0$ for $j \neq i$. It implies that

$$\begin{aligned} \langle \Lambda_i^* g_i, \Gamma_j^* g_j \rangle &= \langle \Gamma_j \Lambda_i^* g_i, g_j \rangle \\ &= \delta_{i,j} \langle g_i, g_j \rangle. \end{aligned} \quad (22)$$

So, $\{\Lambda_j : j \in \mathcal{N}\}$ and $\{\Gamma_j : j \in \mathcal{N}\}$ are g -biorthonormal. Since $\sum_{j \in \mathcal{N}} \Lambda_j^* \Gamma_j = I$, $\sum_{j \in \mathcal{N}} \Gamma_j^* \Lambda_j = I$. So for every $x \in H$, we have that $x = \sum_{j \in \mathcal{N}} \Gamma_j^* \Lambda_j x$. To prove that $\{\Gamma_j : j \in \mathcal{N}\}$ is a g -basis, it is sufficient to show that the representation is unique. It is sufficient to prove that $\{\Gamma_j : j \in \mathcal{N}\}$ is g -linearly independent. Suppose that $\sum_{j \in \mathcal{N}} \Gamma_j^* g_j = 0$; then, for each $i \in \mathcal{N}$, we have

$$g_i = \Lambda_i \sum_{j \in \mathcal{N}} \Gamma_j^* g_j = 0. \quad (23)$$

It implies that $\{\Gamma_j : j \in \mathcal{N}\}$ is g -linearly independent. □

Theorem 15. Suppose that $\Lambda_j \in B(H, H_j)$ and Λ_j is surjective. $\{\Lambda_j : j \in \mathcal{N}\}$ is a g -basis for H with respect to $\{H_j : j \in \mathcal{N}\}$ and $\{\Gamma_j : j \in \mathcal{N}\}$ is the dual g -basis of $\{\Lambda_j : j \in \mathcal{N}\}$. $S_N x = \sum_{j=1}^N \Lambda_j^* \Gamma_j x$; that is, S_N is the partial sum operator associated with $(\{\Lambda_j\}, \{\Gamma_j\})$. Then under the norm $\|\cdot\|$ defined in Theorem 12(3), $\|S_N\| = 1$, for all $N > 0$.

Proof. Since $\{\Lambda_j : j \in \mathcal{N}\}$ and $\{\Gamma_j : j \in \mathcal{N}\}$ are g -biorthonormal by Theorem 14, if $K \leq L$, then

$$\begin{aligned} S_K S_L x &= \sum_{j=1}^K \Lambda_j^* \Gamma_j (S_L x) = \sum_{j=1}^K \Lambda_j^* \Gamma_j \sum_{i=1}^L \Lambda_i^* \Gamma_i x \\ &= \sum_{j=1}^K \sum_{i=1}^L \Lambda_j^* \Gamma_j \Lambda_i^* \Gamma_i x = \sum_{j=1}^K \Lambda_j^* \Gamma_j x = S_K x; \end{aligned} \quad (24)$$

if $K \geq L$, then

$$\begin{aligned} S_K S_L x &= \sum_{j=1}^K \Lambda_j^* \Gamma_j (S_L x) = \sum_{j=1}^K \Lambda_j^* \Gamma_j \sum_{i=1}^L \Lambda_i^* \Gamma_i x \\ &= \sum_{j=1}^L \sum_{i=1}^L \Lambda_j^* \Gamma_j \Lambda_i^* \Gamma_i x = \sum_{j=1}^L \Lambda_j^* \Gamma_j x = S_L x. \end{aligned} \quad (25)$$

Hence

$$\|S_L x\| = \sup_K \|S_K (S_L x)\| = \sup \{\|S_1 x\|, \dots, \|S_L x\|\}. \quad (26)$$

So, for any $L \in \mathcal{N}$, $\|S_L x\| \geq \|S_L x\|$, which follows that

$$\sup_L \|S_L x\| \geq \sup_L \|S_L x\| = \|x\|. \quad (27)$$

On the other hand, for any $L \in \mathcal{N}$, from (26), we also have that

$$\|S_L x\| = \sup_{K \leq L} \|S_K x\| \leq \sup_{K \in \mathcal{N}} \|S_K x\| = \|x\|, \quad (28)$$

which follows that $\sup_L \|S_L x\| \leq \|x\|$. So $\sup_L \|S_L x\| = \|x\|$. So

$$\|S_L\| = \sup_{\|x\|=1} \|S_L x\| = \sup_{\|x\|=1} \|x\| = 1. \quad (29)$$

□

Theorem 16. Suppose that $\Lambda_j \in B(H, H_j)$ and Λ_j is surjective. Then the following statements are equivalent:

- (1) $\{\Lambda_j : j \in \mathcal{N}\}$ is a g -basis for H with respect to $\{H_j : j \in \mathcal{N}\}$;
- (2) there exists $\{\Gamma_j \in B(H, H_j) : j \in \mathcal{N}\}$ such that $\{\Lambda_j : j \in \mathcal{N}\}$ is g -biorthonormal with $\{\Gamma_j : j \in \mathcal{N}\}$ and

$$\forall x \in H, \quad x = \sum_{j \in \mathcal{N}} \Lambda_j^* \Gamma_j x, \quad (30)$$

- (3) $\{\Lambda_j : j \in \mathcal{N}\}$ is g -complete and there exists $\{\Gamma_j \in B(H, H_j) : j \in \mathcal{N}\}$ such that $\{\Lambda_j : j \in \mathcal{N}\}$ is g -biorthonormal with $\{\Gamma_j : j \in \mathcal{N}\}$ and

$$\forall x \in H, \quad \sup_N \|\Lambda_N x\| < \infty, \quad (31)$$

- (4) $\{\Lambda_j : j \in \mathcal{N}\}$ is g -complete and there exists $\{\Gamma_j \in B(H, H_j) : j \in \mathcal{N}\}$ such that $\{\Lambda_j : j \in \mathcal{N}\}$ is g -biorthonormal with $\{\Gamma_j : j \in \mathcal{N}\}$ and

$$\sup_N \|\Lambda_N\| < \infty. \quad (32)$$

Proof. (1) $\Rightarrow$ (2): It is obvious that the g -dual basis $\{\Gamma_j : j \in \mathcal{N}\}$ is g -biorthonormal with $\{\Lambda_j : j \in \mathcal{N}\}$ by Theorem 13 and, for each $x \in H$, $x = \sum_{j \in \mathcal{N}} \Lambda_j^* \Gamma_j x$.

(2) $\Rightarrow$ (1): It is only needed, to show that the representation for x is unique. It is sufficient to show that $\{\Lambda_j : j \in \mathcal{N}\}$ is g -linearly independent. Suppose that $\sum_{j \in \mathcal{N}} \Lambda_j^* g_j = 0$, where $g_j \in H_j$, for all $j \in \mathcal{N}$. Then $g_i = \Gamma_i \sum_{j \in \mathcal{N}} \Lambda_j^* g_j = 0$, for each $i \in \mathcal{N}$, since $\{\Gamma_j : j \in \mathcal{N}\}$ and $\{\Lambda_j : j \in \mathcal{N}\}$ are g -biorthonormal. So $\{\Lambda_j : j \in \mathcal{N}\}$ is g -linearly independent.

(2) $\Rightarrow$ (3): It is obvious that $\{\Lambda_j : j \in \mathcal{N}\}$ is g -complete. It is only needed to prove that, for all $x \in H$, $\sup_N \|\Lambda_N x\| < \infty$. Since $x = \sum_{j \in \mathcal{N}} \Lambda_j^* \Gamma_j x$, $x = \lim_{N \rightarrow \infty} S_N x$. It implies that $\{S_N x\}_{N=1}^\infty$ is convergent for each $x \in H$. Thus $\{S_N x\}_{N=1}^\infty$ is a bounded sequence for each $x \in H$. Hence $\sup_N \|\Lambda_N x\| < \infty$.

(3) $\Rightarrow$ (4): It is only needed, to show that $\sup_N \|\Lambda_N\| < \infty$. Since $\{S_N\}_{N=1}^\infty$ is a sequence of linear bounded operators on H and, for each $x \in H$, $\sup_N \|\Lambda_N x\| < \infty$. So $\sup_N \|\Lambda_N\| < \infty$ by the Banach-Steinhaus theorem.

(4) $\Rightarrow$ (2): For all $x \in \text{span}\{\Lambda_j^*(H_j) : j \in \mathcal{N}\}$, suppose that $x = \sum_{n=1}^M \Lambda_n^* g_n$, $g_n \in H_n$. Since S_N is linear and $\{\Lambda_j : j \in \mathcal{N}\}$ and $\{\Gamma_j : j \in \mathcal{N}\}$ are g -biorthonormal, for $N \geq M$, we have

$$\begin{aligned} S_N x &= \sum_{n=1}^M S_N (\Lambda_n^* g_n) = \sum_{n=1}^M \sum_{i=1}^N \Lambda_i^* \Gamma_i (\Lambda_n^* g_n) \\ &= \sum_{n=1}^M \Lambda_n^* g_n = x. \end{aligned} \quad (33)$$

It implies that, for any $x \in \text{span}\{\Lambda_j^*(H_j) : j \in \mathcal{N}\}$, $x = \lim_{N \rightarrow \infty} S_N x = \sum_{j \in \mathcal{N}} \Lambda_j^* \Gamma_j x$. Next we prove that, for all $x \in H$, it is also correct. Since $\{\Lambda_j : j \in \mathcal{N}\}$ is g -complete, $\overline{\text{span}\{\Lambda_j^*(H_j) : j \in \mathcal{N}\}} = H$ by Lemma 8. For any $\varepsilon > 0$,

there exists $y \in \text{span}\{\Lambda_j(H_j) : j \in \mathcal{N}\}$ such that $\|x - y\| < \varepsilon(1 + \sup_N \|\Lambda_N\|)^{-1}$. Since $y \in \text{span}\{\Lambda_j(H_j) : j \in \mathcal{N}\}$, there exist N such that $S_N y = y$. Since

$$\begin{aligned} \|x - S_N x\| &\leq \|x - y\| + \|y - S_N y\| + \|S_N y - S_N x\| \\ &\leq \|x - y\| + 0 + \|S_N\| \|x - y\| \\ &\leq (1 + \sup_N \|\Lambda_N\|) \|x - y\| < \varepsilon, \end{aligned} \quad (34)$$

thus $\lim_{N \rightarrow \infty} S_N x = x$. $\square$

Theorem 17. Suppose, for each $j \in \mathcal{N}$, $\Lambda_j \in B(H, H_j)$ and Λ_j is surjective. $\{\Lambda_j : j \in \mathcal{N}\}$ is g -complete with respect to $\{H_j : j \in \mathcal{N}\}$. Then $\{\Lambda_j : j \in \mathcal{N}\}$ is a g -basis with respect to $\{H_j : j \in \mathcal{N}\}$ if and only if there exists a constant C such that, for any $g_j \in H_j$, any $m, n \in \mathcal{N}$, and $m \leq n$, we have

$$\left\| \sum_{j=1}^m \Lambda_j^* g_j \right\| \leq C \cdot \left\| \sum_{j=1}^n \Lambda_j^* g_j \right\|. \quad (35)$$

Proof. $\Rightarrow$: Suppose $\{\Lambda_j : j \in \mathcal{N}\}$ is a g -basis with respect to $\{H_j : j \in \mathcal{N}\}$. Then, for any $x \in H$, there exists a unique sequence $\{g_j : j \in \mathcal{N}\}$ with $g_j \in H_j$ for each $j \in \mathcal{N}$ such that $x = \sum_{j=1}^\infty \Lambda_j^* g_j$. Let $\|x\| = \sup_n \left\| \sum_{j=1}^n \Lambda_j^* g_j \right\|$. Then by Theorem 12, $\|\cdot\|$ is a norm on H and it is equivalent to $\|\cdot\|$. So there exists a constant C such that, for any $x \in H$, $\|x\| \leq C \cdot \|x\|$. Hence, for any $n \in \mathcal{N}$, any $g_j \in H_j$, $j = 1, 2, \dots, n$, we choose $x = \sum_{j=1}^n \Lambda_j^* g_j$; then, for any $m \leq n$, we have

$$\left\| \sum_{j=1}^m \Lambda_j^* g_j \right\| \leq C \cdot \left\| \sum_{j=1}^n \Lambda_j^* g_j \right\|. \quad (36)$$

$\Leftarrow$: Let $\mathcal{A} = \{\sum_{k \in \mathcal{N}} \Lambda_k^* g_k, g_k \in H_k, \text{ and } \sum_{k \in \mathcal{N}} \Lambda_k^* g_k \text{ is convergent}\}$. First, we show that $\mathcal{A} = H$. Since $\{\Lambda_j \in B(H, H_j) : j \in \mathcal{N}\}$ is g -complete, $\mathcal{A}$ is dense in H . It is sufficient to show that $\mathcal{A}$ is closed. Suppose $\{y_k\} \subset \mathcal{A}$ and $\lim_{k \rightarrow \infty} y_k = y$. Denote $y_k = \sum_{j \in \mathcal{N}} \Lambda_j^* g_j^{(k)}$. Then for any $j \in \mathcal{N}$ and any $n \geq m \geq j$, we have, for any $k, l \in \mathcal{N}$,

$$\begin{aligned} &\|\Lambda_j^* g_j^{(k)} - \Lambda_j^* g_j^{(l)}\| \\ &\leq C \cdot \left\| \sum_{s=1}^m \Lambda_s^* (g_s^{(k)} - g_s^{(l)}) \right\| \\ &\leq C^2 \cdot \left\| \sum_{s=1}^n \Lambda_s^* (g_s^{(k)} - g_s^{(l)}) \right\| \\ &\leq C^2 \cdot \left(\left\| \sum_{s=1}^n \Lambda_s^* g_s^{(k)} - y_k \right\| + \|y_k - y\| \right) \\ &\quad + C^2 \cdot \left(\|y - y_l\| + \left\| y_l - \sum_{s=1}^n \Lambda_s^* g_s^{(l)} \right\| \right). \end{aligned} \quad (37)$$

Since $\lim_{k \rightarrow \infty} y_k = y$, so, for any $\varepsilon > 0$, there exists $M > 0$, such that whenever $k \geq M$, we have $\|y - y_k\| \leq \varepsilon/2C^2$. In the above inequality, let $n \rightarrow \infty$; we get

$$\begin{aligned} \|\Lambda_j^* g_j^{(k)} - \Lambda_j^* g_j^{(l)}\| &\leq \varepsilon \quad \text{for any } j \in \mathcal{N}, k, l \geq M, \\ \left\| \sum_{s=1}^m \Lambda_s^* (g_s^{(k)} - g_s^{(l)}) \right\| &\leq \frac{\varepsilon}{C} \end{aligned} \quad (38)$$

for any $m \in \mathcal{N}$ and any $k, l \geq M$.

Since Λ_j is surjective, by inequality (10), we have that $\|g_j^{(k)} - g_j^{(l)}\| \leq \|(\Lambda_j^\dagger)^*\| \cdot \|\Lambda_j^* (g_j^{(k)} - g_j^{(l)})\| \leq \|(\Lambda_j^\dagger)^*\| \varepsilon$ for any $j \in \mathcal{N}$ and any $k, l \geq M$. So $\{g_j^{(k)}\}_{k \in \mathcal{N}}$ is convergent for each $j \in \mathcal{N}$. Suppose $\lim_{k \rightarrow \infty} g_j^{(k)} = g_j$. Then

$$\begin{aligned} \|\Lambda_j^* (g_j^{(k)} - g_j)\| &\leq \varepsilon \quad \text{for any } j \in \mathcal{N}, k \geq M, \\ \left\| \sum_{s=1}^m \Lambda_s^* (g_s^{(k)} - g_s) \right\| &\leq \frac{\varepsilon}{C} \end{aligned} \quad (39)$$

for any $m \in \mathcal{N}$ and any $k \geq M$.

Since

$$\begin{aligned} \left\| y - \sum_{j=1}^m \Lambda_j^* g_j \right\| &\leq \|y - y_k\| + \left\| y_k - \sum_{j=1}^m \Lambda_j^* g_j^{(k)} \right\| \\ &\quad + \left\| \sum_{j=1}^m \Lambda_j^* g_j^{(k)} - \sum_{j=1}^m \Lambda_j^* g_j \right\|, \end{aligned} \quad (40)$$

so $\sum_{j=1}^m \Lambda_j^* g_j$ converges to y , which implies that $y \in \mathcal{A}$. Thus $\mathcal{A}$ is a closed set. Now we will show that $\{\Lambda_j : j \in \mathcal{N}\}$ is g -linearly independent. Suppose that $\sum_{j \in \mathcal{N}} \Lambda_j^* g_j = 0$, where $g_j \in H_j$ for each $j \in \mathcal{N}$. Since, for any $n \in \mathcal{N}$ and any $j \leq n$, we have $\|\Lambda_j^* g_j\| \leq C \cdot \|\sum_{s=1}^n \Lambda_s^* g_s\|$, hence $\|\Lambda_j^* g_j\| = 0$ for any $j \leq n$. But since Λ_j is surjective, $\|g_j\| \leq \|(\Lambda_j^\dagger)^*\| \cdot \|\Lambda_j^* g_j\|$. So, for each $j \leq n$, $g_j = 0$. Since n is arbitrary, $g_j = 0$ for any $j \in \mathcal{N}$. Thus $\{\Lambda_j : j \in \mathcal{N}\}$ is g -linearly independent. So $\{\Lambda_j : j \in \mathcal{N}\}$ is a g -basis. $\square$

4. The Property of g -Minimal of g -Bases

In this section, we studied the property of g -minimal of g -bases and the perturbations of g -bases.

Theorem 18. Suppose $\Lambda_j \in B(H, H_j)$ and Λ_j is surjective. Then

- (1) if $\{\Lambda_j : j \in \mathcal{N}\}$ is a g -basis, then $\{\Lambda_j : j \in \mathcal{N}\}$ is g -minimal and g -complete;
- (2) if $\{\Lambda_j : j \in \mathcal{N}\}$ is g -minimal, then $\{\Lambda_j : j \in \mathcal{N}\}$ is g -linearly independent.

Proof. (1) Since $\{\Lambda_j : j \in \mathcal{N}\}$ is a g -basis, it is obvious that $\{\Lambda_j : j \in \mathcal{N}\}$ is g -complete. We only need to prove that

$\{\Lambda_j : j \in \mathcal{N}\}$ is g -minimal. Suppose that $\{\Gamma_j : j \in \mathcal{N}\}$ is the unique dual g -basis of $\{\Lambda_j : j \in \mathcal{N}\}$. Then $\{\Lambda_j : j \in \mathcal{N}\}$ and $\{\Gamma_j : j \in \mathcal{N}\}$ are g -biorthonormal by Theorem 16; that is, $\langle \Lambda_j^* g_j, \Gamma_i^* g_i \rangle = \delta_{i,j} \langle g_j, g_i \rangle$ for any $i, j \in \mathcal{N}$ and $g_j \in H_j, g_i \in H_i$. For any $m \in \mathcal{N}$ and any $g_m \in H_m$ such that $\Lambda_m^* g_m \neq 0$, let $E_m = \overline{\text{span}}_{i \neq m} \{\Lambda_i^* g_i\}$. Then, for any $x \in E_m$, $\langle x, \Gamma_m^* g_m \rangle = 0$, but $\langle \Lambda_m^* g_m, \Gamma_m^* g_m \rangle = \langle g_m, g_m \rangle \neq 0$, so $\Lambda_m^* g_m \notin E_m$. Hence $\{\Lambda_j : j \in \mathcal{N}\}$ is g -minimal.

(2) Suppose $\{\Lambda_j : j \in \mathcal{N}\}$ is g -minimal and $\sum_{j \in \mathcal{N}} \Lambda_j^* g_j = 0$. Since Λ_j is surjective for any $j \in \mathcal{N}$, then $\|g_j\| \leq \|(\Lambda_j^\dagger)^*\| \cdot \|\Lambda_j^* g_j\|$ for any $j \in \mathcal{N}$ and $g_j \in H_j$. If there exists $j_0 \in \mathcal{N}$ such that $g_{j_0} \neq 0$, then $\|\Lambda_{j_0}^* g_{j_0}\| > 0$. It follows that $\Lambda_{j_0}^* g_{j_0} \neq 0$. Since $\Lambda_{j_0}^* g_{j_0} = -\sum_{j \neq j_0} \Lambda_j^* g_j$, $\Lambda_{j_0}^* g_{j_0} \in \overline{\text{span}}_{j \neq j_0} \{\Lambda_j^* g_j\}$, which contradicts with the fact that $\{\Lambda_j : j \in \mathcal{N}\}$ is g -minimal. $\square$

Theorem 19. Suppose that $\Lambda_j \in B(H, H_j)$ and Λ_j is surjective for each $j \in \mathcal{N}$. Then

- (1) there exists a sequence $\{\Gamma_j \in M(H_j, H) : j \in \mathcal{N}\}$, such that $\{\Lambda_j^* : j \in \mathcal{N}\}$ and $\{\Gamma_j : j \in \mathcal{N}\}$ are biorthonormal if and only if $\{\Lambda_j : j \in \mathcal{N}\}$ is g -minimal,
- (2) there exists a unique sequence $\{\Gamma_j \in M(H_j, H) : j \in \mathcal{N}\}$ such that $\{\Lambda_j^* : j \in \mathcal{N}\}$ and $\{\Gamma_j : j \in \mathcal{N}\}$ are biorthonormal, if and only if $\{\Lambda_j : j \in \mathcal{N}\}$ is g -minimal and g -complete.

Proof. (1) $\Rightarrow$: Suppose that $\{\Gamma_j : j \in \mathcal{N}\}$ is biorthonormal with $\{\Lambda_j^* : j \in \mathcal{N}\}$; that is, $\langle \Lambda_j^* g_j, \Gamma_i^* g_i \rangle = \delta_{i,j} \langle g_j, g_i \rangle$ for any $i, j \in \mathcal{N}$ and $g_j \in H_j, g_i \in H_i$. For any $m \in \mathcal{N}$ and any $g_m \in H_m$ such that $\Lambda_m^* g_m \neq 0$, let $E_m = \overline{\text{span}}_{i \neq m} \{\Lambda_i^* (H_i)\}$. Then, for any $x \in E_m$, $\langle x, \Gamma_m^* g_m \rangle = 0$, but $\langle \Lambda_m^* g_m, \Gamma_m^* g_m \rangle = \langle g_m, g_m \rangle \neq 0$, so $\Lambda_m^* g_m \notin E_m$. Hence $\{\Lambda_j : j \in \mathcal{N}\}$ is g -minimal.

$\Leftarrow$: Suppose that $\{\Lambda_j : j \in \mathcal{N}\}$ is g -minimal. For all $m \in \mathcal{N}$, for all $g_m \in H_m$ such that $g_m \neq 0$, let $E_m = \overline{\text{span}}_{j \neq m} \{\Lambda_j^* (H_j)\}$. Since Λ_j is surjective, $\|(\Lambda_j^\dagger)^*\| \cdot \|\Lambda_m^* g_m\| \geq \|g_m\| > 0$. Since $\{\Lambda_j : j \in \mathcal{N}\}$ is g -minimal, $\Lambda_m^* g_m \notin E_m$. So there exists $x_m \in H$ such that $\langle \Lambda_m^* g_m, x_m \rangle = \|g_m\|^2$ and $\langle y, x_m \rangle = 0$, for all $y \in E_m$. Keep in mind that the notation $M(H_j, H)$ in statement (1) means just a general map with no linearity and bounded requirements, defining a map $\Gamma_m : H_m \rightarrow H$ by $\Gamma_m g_m = x_m$ if $g_m \neq 0$, $\Gamma_m g_m = 0$ if $g_m = 0$. Then $\{\Gamma_j : j \in \mathcal{N}\}$ is biorthonormal with $\{\Lambda_j^* : j \in \mathcal{N}\}$, since $\langle \Lambda_j^* g_j, \Gamma_i^* g_i \rangle = \delta_{i,j} \langle g_j, g_i \rangle$ for every $i, j \in \mathcal{N}$ and $g_j \in H_j, g_i \in H_i$.

(2) $\Rightarrow$: By (1) we know that $\{\Lambda_j : j \in \mathcal{N}\}$ is g -minimal. So it is only needed to show that $\{\Lambda_j : j \in \mathcal{N}\}$ is g -complete. Suppose that $x \in H$ and $\langle x, \Lambda_j^* g_j \rangle = 0$ for any $j \in \mathcal{N}$ and any $g_j \in H_j$. Since $\langle \Lambda_j^* g_j, \Gamma_i^* g_i \rangle = \delta_{i,j} \langle g_j, g_i \rangle$, so $\langle \Lambda_j^* g_j, x + \Gamma_i^* g_i \rangle = \delta_{i,j} \langle g_j, g_i \rangle$, which implies that $\{\delta_x + \Gamma_j : j \in \mathcal{N}\}$ and $\{\Lambda_j^* : j \in \mathcal{N}\}$ are biorthonormal, where $\delta_x \in B(H_j, H)$ for each $j \in \mathcal{N}$ defined by $\delta_x(f) = x$ for any $f \in H_j$. But it is assumed that there exists a unique sequence $\{\Gamma_j \in M(H_j, H) : j \in \mathcal{N}\}$ such that $\{\Lambda_j^* : j \in \mathcal{N}\}$ and $\{\Gamma_j : j \in \mathcal{N}\}$ are biorthonormal,

so $\delta_x = 0$; hence $x = \delta_x(f) = 0$, which implies that $\overline{\text{span}}\{\Lambda_j^* g_j\} = H$. So $\{\Lambda_j : j \in \mathcal{N}\}$ is g -complete.

$\Leftarrow$: By (1), we know that if $\{\Lambda_j : j \in \mathcal{N}\}$ is g -minimal, then there exists a sequence $\{\Gamma_j \in M(H_j, H) : j \in \mathcal{N}\}$ such that $\{\Lambda_j^* : j \in \mathcal{N}\}$ and $\{\Gamma_j : j \in \mathcal{N}\}$ are biorthonormal. Now we only need to show that such sequence is unique. If there is another sequence $\{T_j \in M(H_j, H) : j \in \mathcal{N}\}$ such that $\{\Lambda_j^* : j \in \mathcal{N}\}$ and $\{T_j : j \in \mathcal{N}\}$ are biorthonormal; then $\langle \Lambda_j^* g_j, (\Gamma_i - T_i)g_i \rangle = 0$ for any $i, j \in \mathcal{N}$ and $g_i \in H_i, g_j \in H_j$. Since $\{\Lambda_j : j \in \mathcal{N}\}$ is g -complete, $\overline{\text{span}}\{\Lambda_j^*(H_j) : j \in \mathcal{N}\} = H$ by Lemma 8. Thus $(\Gamma_i - T_i)g_i = 0$ for every $i \in \mathcal{N}$ and $g_i \in H_i$. It follows that $T_i = \Gamma_i$ for every $i \in \mathcal{N}$. So the sequence which is biorthonormal with $\{\Lambda_j^* : j \in \mathcal{N}\}$ is unique. $\square$

Theorem 20. Suppose that $\Lambda_j \in B(H, H_j)$ and Λ_j is surjective for each $j \in \mathcal{N}$. Then the following statements are equivalent:

- (1) there exists a sequence $\{\Gamma_j \in B(H, H_j) : j \in \mathcal{N}\}$ which is g -biorthonormal with $\{\Lambda_j : j \in \mathcal{N}\}$,
- (2) for all $M > 0$, there exists a constant $C_M \geq 1$ such that, for any $N \geq M$ and $g_j \in H_j$ ($j = 1, \dots, N$), we have

$$\left\| \sum_{j=1}^M \Lambda_j^* g_j \right\| \leq C_M \left\| \sum_{j=1}^N \Lambda_j^* g_j \right\|. \quad (41)$$

Proof. (1) $\Rightarrow$ (2): For any $N > 0$, any $x \in H$, let $S_N x = \sum_{j=1}^N \Lambda_j^* \Gamma_j x$. Then for any $N \geq M$ and any $g_j \in H_j$ ($j = 1, \dots, N$), we have

$$\begin{aligned} S_M \left(\sum_{j=1}^N \Lambda_j^* g_j \right) &= \sum_{j=1}^N S_M (\Lambda_j^* g_j) \\ &= \sum_{j=1}^N \sum_{i=1}^M \Lambda_i^* \Gamma_i (\Lambda_j^* g_j) = \sum_{j=1}^M \Lambda_j^* g_j. \end{aligned} \quad (42)$$

Hence

$$\left\| \sum_{j=1}^M \Lambda_j^* g_j \right\| = \left\| S_M \left(\sum_{j=1}^N \Lambda_j^* g_j \right) \right\| \leq \|S_M\| \left\| \sum_{j=1}^N \Lambda_j^* g_j \right\|. \quad (43)$$

Then $\|S_M\| \geq 1$. So $C_M = \|S_M\|$ satisfies the requirement.

(2) $\Rightarrow$ (1): Let $E = \text{span}\{\Lambda_j^*(H_j) : j \in \mathcal{N}\}$ and $x \in E$. Suppose that $x = \sum_{j=1}^N \Lambda_j^* g_j$ and $M \leq N$. Since

$$\begin{aligned} \|g_M\| &\leq \|(\Lambda_M^\dagger)^*\| \cdot \|\Lambda_M^* g_M\| \\ &\leq \|(\Lambda_M^\dagger)^*\| \left(\left\| \sum_{j=1}^M \Lambda_j^* g_j \right\| + \left\| \sum_{j=1}^{M-1} \Lambda_j^* g_j \right\| \right) \\ &\leq \|(\Lambda_M^\dagger)^*\| \left(C_M \left\| \sum_{j=1}^N \Lambda_j^* g_j \right\| + C_{M-1} \left\| \sum_{j=1}^N \Lambda_j^* g_j \right\| \right) \\ &= \|(\Lambda_M^\dagger)^*\| \cdot (C_M + C_{M-1}) \|x\|, \end{aligned} \quad (44)$$

so, if $x = 0$, then $g_j = 0$ for $j = 1, \dots, N$, which follows that the vectors $g_j, j = 1, \dots, N$ are uniquely defined by $x = \sum_{j=1}^N \Lambda_j^* g_j$. Now, for each $m \in \mathcal{N}$, we define operator $\Gamma_m : E \rightarrow H_m$ by

$$\Gamma_m \left(\sum_{j=1}^N \Lambda_j^* g_j \right) = \begin{cases} g_m, & m \leq N, \\ 0, & m > N. \end{cases} \quad (45)$$

Then Γ_m is linear and

$$\begin{aligned} \left\| \Gamma_m \left(\sum_{j=1}^N \Lambda_j^* g_j \right) \right\| \\ \leq \|g_m\| \leq \|(\Lambda_m^\dagger)^*\| \cdot (C_m + C_{m-1}) \left\| \sum_{j=1}^N \Lambda_j^* g_j \right\|. \end{aligned} \quad (46)$$

So Γ_m is bounded on E ; hence it can be linearly extended to $\bar{E}$. Let $\Gamma_m = 0$ on $\bar{E}^\perp$; then $\Gamma_m \in B(H, H_m)$ for each $m \in \mathcal{N}$ and

$$\langle \Lambda_j^* g_j, \Gamma_i^* g_i \rangle = \langle \Gamma_i \Lambda_j^* g_j, g_i \rangle = \delta_{i,j} \langle g_j, g_i \rangle; \quad (47)$$

that is, $\{\Gamma_j : j \in \mathcal{N}\}$ is g -biorthonormal with $\{\Lambda_j : j \in \mathcal{N}\}$. $\square$

5. Perturbation of g -Bases

In this section, we give one result about the stability of g -bases under certain perturbation.

Theorem 21. Suppose that $\Lambda_j \in B(H, H_j)$ and Λ_j is surjective for each $j \in \mathcal{N}$. $\{\Lambda_j : j \in \mathcal{N}\}$ is a g -basis for H with $\{\Gamma_j : j \in \mathcal{N}\}$ as its g -dual basis and its basis constant is C . If $\{L_j \in B(H, H_j) : j \in \mathcal{N}\}$ is a sequence such that $R = \sum_{j \in \mathcal{N}} \|\Lambda_j^* - L_j^*\| \cdot \|\Gamma_j\| < 1$, then $\{L_j : j \in \mathcal{N}\}$ is a g -basis for H with basis constant $C_1 \leq ((1 + R)/(1 - R))C$.

Proof. Suppose $x = \sum_{j=m}^n \Lambda_j^* g_j, g_j \in H_j$ ($j = m, \dots, n$). Then $g_j = \Gamma_j x$ ($j = m, \dots, n$), since $\{\Lambda_j : j \in \mathcal{N}\}$ and $\{\Gamma_j : j \in \mathcal{N}\}$ are g -biorthonormal by Theorem 16. Since

$$\begin{aligned} \left\| \sum_{j=m}^n (\Lambda_j^* - L_j^*) g_j \right\| \\ \leq \sum_{j=m}^n \|\Lambda_j^* - L_j^*\| \cdot \|g_j\| \\ = \sum_{j=m}^n \|\Lambda_j^* - L_j^*\| \cdot \|\Gamma_j x\| \\ \leq \sum_{j=m}^n \|\Lambda_j^* - L_j^*\| \cdot \|\Gamma_j\| \cdot \|x\| \leq R \cdot \|x\| \\ = R \cdot \left\| \sum_{j=m}^n \Lambda_j^* g_j \right\|, \end{aligned} \quad (48)$$

which implies that if $\sum_{j \in \mathcal{N}} \Lambda_j^* g_j$ is convergent, then $\sum_{j \in \mathcal{N}} (\Lambda_j^* - L_j^*) g_j$ is convergent, the following operator is well defined:

$$T : H \longrightarrow H, T \left(\sum_{j \in \mathcal{N}} \Lambda_j^* g_j \right) = \sum_{j \in \mathcal{N}} (\Lambda_j^* - L_j^*) g_j. \quad (49)$$

It is obvious that T is linear and $\|T\| \leq R \leq 1$. It implies that $I - T$ is invertible. Since, for any $j \in \mathcal{N}$ and $g_j \in H_j$, we have

$$\begin{aligned} (I - T) \Lambda_j^* g_j &= \Lambda_j^* g_j - T(\Lambda_j^* g_j) \\ &= \Lambda_j^* g_j - (\Lambda_j^* - L_j^*) g_j = L_j^* g_j, \end{aligned} \quad (50)$$

so $L_j = \Lambda_j(I - T)^*$ for any $j \in \mathcal{N}$. Since $(I - T)^*$ is invertible, it is easy to see that $\{L_j : j \in \mathcal{N}\}$ is a g -basis for H . Now we start to show the basis constants inequality. For any $N \geq M$, we have

$$\begin{aligned} \left\| \sum_{j=1}^M L_j^* g_j \right\| &\leq \left\| \sum_{j=1}^M \Lambda_j^* g_j \right\| + \left\| \sum_{j=1}^M (\Lambda_j^* - L_j^*) g_j \right\| \\ &\leq \left\| \sum_{j=1}^M \Lambda_j^* g_j \right\| + R \cdot \left\| \sum_{j=1}^M \Lambda_j^* g_j \right\| \\ &= (1 + R) \cdot \left\| \sum_{j=1}^M \Lambda_j^* g_j \right\|. \end{aligned} \quad (51)$$

On the other hand, we have

$$\begin{aligned} \left\| \sum_{j=1}^N \Lambda_j^* g_j \right\| &\leq \left\| \sum_{j=1}^N L_j^* g_j \right\| + \left\| \sum_{j=1}^N (\Lambda_j^* - L_j^*) g_j \right\| \\ &\leq \left\| \sum_{j=1}^N L_j^* g_j \right\| + R \cdot \left\| \sum_{j=1}^M \Lambda_j^* g_j \right\|. \end{aligned} \quad (52)$$

It follows that

$$\left\| \sum_{j=1}^N L_j^* g_j \right\| \geq (1 - R) \cdot \left\| \sum_{j=1}^N \Lambda_j^* g_j \right\|. \quad (53)$$

Since $\{\Lambda_j : j \in \mathcal{N}\}$ is a g -basis for H , by Theorem 17, we have that

$$\left\| \sum_{j=1}^M \Lambda_j^* g_j \right\| \leq C \left\| \sum_{j=1}^N \Lambda_j^* g_j \right\|. \quad (54)$$

So, by inequalities (51), (53) and (54), we have

$$\begin{aligned} \left\| \sum_{j=1}^M L_j^* g_j \right\| &\leq (1 + R) \cdot \left\| \sum_{j=1}^M \Lambda_j^* g_j \right\| \leq (1 + R) C \cdot \left\| \sum_{j=1}^N \Lambda_j^* g_j \right\| \\ &\leq \frac{1 + R}{1 - R} C \cdot \left\| \sum_{j=1}^N L_j^* g_j \right\|. \end{aligned} \quad (55)$$

□

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