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Research Article

Exponential Stability of a Linear Distributed Parameter Bioprocess with Input Delay in Boundary Control

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We consider a linear distributed parameter bioprocess with boundary control input possessing a time delay. Using a simple boundary feedback law, we show that the closed-loop system generates a uniformly bounded C_0-semigroup of linear operators under a certain condition with respect to the feedback gain. After analyzing the spectrum configuration of closed-loop system and verifying the spectrum determined growth assumption, we show that the closed-loop system is exponentially stable. Thus, we demonstrate that the linear distributed parameter bioprocess preserves the exponential stability for arbitrary time delays.

1. Introduction

In a practical control system, there is often a time delay between the controller to be implemented and the information via the observation of the system. These hereditary effects are sometime unavoidable because they might turn a well-behaved system into a wild one. A simple example can be found in Gumowski and Mira [1], where they demonstrated that the occurrence of delays could destroy the stability and cause periodic oscillations in a system governed by differential equation. Datko [2, 3] illustrated that an arbitrary small time delay in the control could destabilize a boundary feedback hyperbolic control system as well. On the other side, the inclusion of an appropriate time delay effect can sometime improve the performance of the system (e.g., see [3–7]). When the time delay appears, redesigning a stabilizing controller becomes thereby sometimes necessary because the stabilization by the PI output feedback becomes defective or the stabilization is not robust to time delay. The stabilization with time delay in observation or control represents difficult mathematical challenges in the control of distributed parameter systems. However, this does not mean that there is no stabilizing controller in the presence of time delay. You can refer to [8–12] for some successful examples.

Motivated by these works, we will introduce time delays to a linear distributed parameter bioprocess and investigate the effect of time delays on exponential stability of the system. The linear distributed parameter bioprocess treated here was firstly discussed by Bourrel and Dochain in [13]. They showed that the system with zero boundary input is exponentially stable. Following [13], Sano considered the linear distributed parameter bioprocess from the feedback control point of view in [14]. Namely, the control input and the measured output were imposed on the boundaries, and a simply proportional feedback controller was designed. By using Huang's result in [15], He showed that the closed-loop system is exponentially stable under a certain condition with respect to the feedback gain and further that the exponential decay rate of the system with zero input was derived by letting the feedback gain tend to zero. However, if time delays in the boundary input arise in this linear distributed parameter bioprocess, we want to pose a question. Is the stabilization robust to time delays for the proportional feedback controller? The present paper is devoted to answering this question.

The content of this paper is organized as follows. In Section 2 we will introduce the linear distributed parameter bioprocess mentioned previously and formulate our problem in a suitable Hilbert space. We show that the closed-loop

system generates a uniformly bounded C_0-semigroup of linear operators and obtain the wellposedness of the system as well. In Section 3, we carry out a spectral analysis and obtain the spectrum configuration of the closed-loop system. From verifying the spectrum determined growth assumption, we show that the closed-loop system is exponentially stable. In the last section, a concise conclusion is given.

2. System Description and Wellposedness of the System

We will consider the following type of linear distributed parameter bioprocess model in which time delays occur in boundary control input:

$$\begin{gathered}\frac{\partial z_1}{\partial t}(t,x) = -\nu\frac{\partial z_1}{\partial x}(t,x) - a_1 z_1(t,x) - a_2 z_2(t,x), \\ (t,x) \in (0,\infty)\times(0,1), \\ \frac{\partial z_2}{\partial t}(t,x) = a_3 z_1(t,x) - a_4 z_2(t,x), \\ (t,x) \in (0,\infty)\times(0,1), \\ z_1(t,0) = u(t-\tau), \quad t \in (0,\infty), \\ z_1(0,x) = z_{10}(x), \quad z_2(0,x) = z_{20}(x), \quad x \in (0,1), \\ y(t) = z_1(t,1), \quad t \in (0,\infty),\end{gathered} \tag{1}$$

where $z_1(t,x), z_2(t,x) \in \mathbb{R}$ are the deviations of substrate and biomass concentrations from steady-state values at the time t and at the point $x \in (0,1)$, respectively. And $u(t) \in \mathbb{R}$ is the control input, $y(t) \in \mathbb{R}$ is the measured output, $\nu > 0$ is the fluid superficial velocity, a_1, a_2, a_3, and a_4 are positive constants, and $\tau \geq 0$ is the length of time delay.

As usual, we adopt the simple feedback control law $u(t) = -ky(t)$ with $k > 0$ which results in the following closed-loop system:

$$\begin{gathered}\frac{\partial z_1}{\partial t}(t,x) = -\nu\frac{\partial z_1}{\partial x}(t,x) - a_1 z_1(t,x) - a_2 z_2(t,x), \\ (t,x) \in (0,\infty)\times(0,1), \\ \frac{\partial z_2}{\partial t}(t,x) = a_3 z_1(t,x) - a_4 z_2(t,x), \\ (t,x) \in (0,\infty)\times(0,1), \\ z_1(t,0) = -kz_1(t-\tau,1), \quad t \in (0,\infty), \\ z_1(0,x) = z_{10}(x), \quad z_2(0,x) = z_{20}(x), \quad x \in (0,1).\end{gathered} \tag{2}$$

Setting $z_3(t,x) = z_1(t - x\tau, 1)$, (2) is equivalent to

$$\begin{gathered}\frac{\partial z_1}{\partial t}(t,x) = -\nu\frac{\partial z_1}{\partial x}(t,x) - a_1 z_1(t,x) - a_2 z_2(t,x), \\ (t,x) \in (0,\infty)\times(0,1), \\ \frac{\partial z_2}{\partial t}(t,x) = a_3 z_1(t,x) - a_4 z_2(t,x), \\ (t,x) \in (0,\infty)\times(0,1), \\ \frac{\partial z_3}{\partial t}(t,x) = -\tau^{-1}\frac{\partial z_3}{\partial x}(t,x), \quad (t,x) \in (0,\infty)\times(0,1), \\ z_3(t,0) = z_1(t,1), \quad z_1(t,0) = -kz_3(t,1), \quad t \in (0,\infty), \\ z_1(0,x) = z_{10}(x), \quad z_2(0,x) = z_{20}(x), \\ z_3(0,x) = z_{30}(-\tau x), \quad x \in (0,1).\end{gathered} \tag{3}$$

We take the state Hilbert space $\mathscr{H}$,

$$\mathscr{H} = L^2[0,1]\times L^2[0,1]\times L^2[0,1] = \left(L^2[0,1]\right)^3, \tag{4}$$

equipped with inner product

$$\begin{aligned}\langle f,g\rangle_{\mathscr{H}} = &\int_0^1 f_1(x)\overline{g_1(x)}dx + \int_0^1 f_2(x)\overline{g_2(x)}dx \\ &+ \int_0^1 f_3(x)\overline{g_3(x)}dx, \quad f,g \in \mathscr{H}.\end{aligned} \tag{5}$$

Define the operator $A : D(A) \subset \mathscr{H} \to \mathscr{H}$ as

$$\begin{gathered}Af = \begin{pmatrix} -\nu\dfrac{d}{dx} - a_1 & -a_2 & 0 \\ a_3 & -a_4 & 0 \\ 0 & 0 & -\tau^{-1}\dfrac{d}{dx}\end{pmatrix}\begin{pmatrix} f_1 \\ f_2 \\ f_3\end{pmatrix}, \\ f = (f_1, f_2, f_3)^{\perp} \in D(A), \\ D(A) = \left\{f \in H^1(0,1)\times L^2[0,1]\right. \\ \left.\times H^1(0,1) \mid f_1(0) = -kf_3(1), f_1(1) = f_3(0)\right\}.\end{gathered} \tag{6}$$

Then the system (3) can be written as

$$\frac{d}{dt}\begin{pmatrix} z_1(t) \\ z_2(t) \\ z_3(t)\end{pmatrix} = A\begin{pmatrix} z_1(t) \\ z_2(t) \\ z_3(t)\end{pmatrix}, \begin{pmatrix} z_1(0) \\ z_2(0) \\ z_3(0)\end{pmatrix} = \begin{pmatrix} z_{10} \\ z_{20} \\ z_{30}\end{pmatrix}, \tag{7}$$

where z_{10}, z_{20}, $z_{30} \in L^2[0,1]$. Therefore, if the operator A generates a C_0-semigroup $T(t)$ on $\mathscr{H}$, then a unique solution of (7) is expressed as

$$\begin{pmatrix} z_1(t) \\ z_2(t) \\ z_3(t)\end{pmatrix} = T(t)\begin{pmatrix} z_{10} \\ z_{20} \\ z_{30}\end{pmatrix}, \quad \forall t \geq 0, \tag{8}$$

which means that the unique solution to (2) or (3) exists.

Let us define the operator $M \in L(\mathscr{H})$ as

$$M = \begin{pmatrix} \sqrt{a_2} & 0 & 0 \\ 0 & \sqrt{a_3} & 0 \\ 0 & 0 & 1\end{pmatrix} \tag{9}$$

and consider the properties of a semigroup generated by the operator $M^{-1}AM$. The operator $M^{-1}AM$ is expressed as

$$\begin{aligned} M^{-1}AMf &= \begin{pmatrix} -\nu\dfrac{d}{dx}-a_1 & -\sqrt{a_2a_3} & 0 \\ \sqrt{a_2a_3} & -a_4 & 0 \\ 0 & 0 & -\tau^{-1}\dfrac{d}{dx} \end{pmatrix}\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}, \\ f &= (f_1, f_2, f_3)^{\perp} \in D\left(M^{-1}AM\right), \\ D\left(M^{-1}AM\right) &= \left\{f \in H^1(0,1)\times L^2[0,1]\times H^1(0,1) \right. \\ &\qquad \left. \mid f_1(0) = -kf_3(1), f_1(1) = f_3(0)\right\}. \end{aligned} \tag{10}$$

Firstly, we have the following result.

Theorem 1. *Suppose that the feedback gain k is chosen such that $0 < k < 1$. Then, the operator A defined by (6) generates a uniformly bounded C_0-semigroup $T(t)$ on $\mathscr{H}$.*

Proof. In order to prove that A generates a uniformly bounded C_0-semigroup, we introduce a new equivalent inner product in $\mathscr{H}$:

$$\begin{aligned} \langle f, g\rangle_{\mathscr{H}_1} &= \frac{1}{\nu}\int_0^1 f_1(x)\overline{g_1(x)}dx + \frac{1}{\nu}\int_0^1 f_2(x)\overline{g_2(x)}dx \\ &\quad + \tau\int_0^1 f_3(x)\overline{g_3(x)}dx, \quad f, g \in \mathscr{H}. \end{aligned} \tag{11}$$

From the domain of the operator $M^{-1}AM$ it follows that the identities

$$\begin{aligned} &\operatorname{Re}\left\langle M^{-1}AMf, f\right\rangle_{\mathscr{H}_1} \\ &\quad = -\frac{a_1}{\nu}\|f_1\|_{L^2(0,1)} - \frac{a_4}{\nu}\|f_2\|_{L^2(0,1)} \\ &\qquad - \left(|f_1(1)|^2 - |f_1(0)|^2\right) - \left(|f_3(1)|^2 - |f_3(0)|^2\right) \\ &\quad = -\frac{a_1}{\nu}\|f_1\|_{L^2(0,1)} - \frac{a_4}{\nu}\|f_2\|_{L^2(0,1)} - \left(1-k^2\right)|f_3(1)|^2 \end{aligned} \tag{12}$$

hold for all $f \in D(M^{-1}AM)$. If the feedback gain k satisfies $0 < k < 1$, then it is easy to see

$$\operatorname{Re}\left\langle M^{-1}AMf, f\right\rangle_{\mathscr{H}_1} \le 0, \quad \forall f \in D\left(M^{-1}AM\right). \tag{13}$$

Next, for all $f \in D(M^{-1}AM)$ and $g \in \mathscr{H}$, we have

$$\begin{aligned} &\left\langle M^{-1}AMf, g\right\rangle_{\mathscr{H}_1} \\ &\quad = \frac{1}{\nu}\int_0^1 \left(-\nu f_1'(x) - a_1 f_1(x)\right)\overline{g_1(x)}dx \\ &\qquad - \frac{a_4}{\nu}\int_0^1 f_2(x)\overline{g_2(x)}dx - \frac{\sqrt{a_2a_3}}{\nu}\int_0^1 f_2(x)\overline{g_1(x)}dx \\ &\qquad + \frac{\sqrt{a_2a_3}}{\nu}\int_0^1 f_1(x)\overline{g_2(x)}dx - \int_0^1 f_3'(x)\overline{g_3(x)}dx \\ &\quad = \frac{1}{\nu}\int_0^1 f_1(x)\left(\nu\overline{g_1'(x)} - a_1\overline{g_1(x)} + \sqrt{a_2a_3}\overline{g_2(x)}\right)dx \\ &\qquad - \frac{1}{\nu}\int_0^1 f_2(x)\left(a_4\overline{g_2(x)} + \sqrt{a_2a_3}\overline{g_1(x)}\right)dx \\ &\qquad + \frac{1}{\tau}\int_0^1 f_3(x)\overline{g_3'(x)}dx \\ &\qquad - \left[f_1(x)\overline{g_1'(x)}\right]_0^1 - \frac{1}{\tau}\left[f_3(x)\overline{g_3(x)}\right]_0^1. \end{aligned} \tag{14}$$

From the definition of the adjoint operator and the conditions in the domain of the $M^{-1}AM$, we know that

$$\begin{aligned} \left(M^{-1}AM\right)^* f &= \begin{pmatrix} \nu\dfrac{d}{dx}-a_1 & \sqrt{a_2a_3} & 0 \\ -\sqrt{a_2a_3} & -a_4 & 0 \\ 0 & 0 & \tau^{-1}\dfrac{d}{dx} \end{pmatrix}\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}, \\ f &= (f_1, f_2, f_3)^{\perp} \in D\left(\left(M^{-1}AM\right)^*\right), \\ D\left(\left(M^{-1}AM\right)^*\right) &= \left\{f \in H^1(0,1)\times L^2[0,1]\times H^1(0,1) \right. \\ &\qquad \left. \mid f_3(1) = -kf_1(0), f_3(0) = f_1(1)\right\} \end{aligned} \tag{15}$$

because it is easily verified that $M^{-1}AM$ is a closed and densely defined linear operator. Thus, by similar arguments as previously, we obtain that, for all $f \in D((M^{-1}AM)^*)$,

$$\begin{aligned} &\operatorname{Re}\left\langle \left(M^{-1}AM\right)^* f, f\right\rangle_{\mathscr{H}_1} \\ &\quad = -\frac{a_1}{\nu}\|f_1\|_{L^2(0,1)} - \frac{a_4}{\nu}\|f_2\|_{L^2(0,1)} \\ &\qquad + \left(|f_1(1)|^2 - |f_1(0)|^2\right) + \left(|f_3(1)|^2 - |f_3(0)|^2\right) \\ &\quad = -\frac{a_1}{\nu}\|f_1\|_{L^2(0,1)} - \frac{a_4}{\nu}\|f_2\|_{L^2(0,1)} - \left(1-k^2\right)|f_1(0)|^2 \le 0. \end{aligned} \tag{16}$$

It follows from $\operatorname{Re}\langle M^{-1}AMf, f\rangle_{\mathscr{H}_1} \le 0$ and $\operatorname{Re}\langle (M^{-1}AM)^* f, f\rangle_{\mathscr{H}_1} \le 0$ that the operators $M^{-1}AM$

and $(M^{-1}AM)^*$ are dissipative. According to Proposition 3.1.11 of [16], the closed operator $M^{-1}AM$ is m-dissipative. Therefore, Lumer-Philips theorem implies that $M^{-1}AM$ generates contraction semigroups $S(t)$ on state space $\mathscr{H}$. For all $t \geq 0$, if we define $T(t)$ by $T(t) = MS(t)M^{-1}$, then the semigroups $T(t)$ and $S(t)$ are similar. This means that the C_0-semigroups $T(t)$ are uniformly bounded (i.e., a C_0-semigroup with the operator norm bound $\| T(t)\|_{\mathscr{L}(\mathscr{H}_1)} \leq M$, for some $M > 0$ and $\forall t \geq 0$) and their generator is A. Thus, the proof of the theorem is complete since the new inner product is equivalent to the original one. □

3. Exponential Stability of the System (7)

In order to show the exponential stability of the system (7), we will verify that the operator A satisfies the conditions of Theorem 1.1 of [14], which is a summarized edition of Huang's result on the spectrum determined growth assumption in [15]. To this end, we should analyze the spectrum configuration of the operator A and show that the norm of the resolvent is uniformly bounded in any given right half-plane. All these results are collected in the following two lemmas.

Lemma 2. *Suppose that the assumption of Theorem 1 is satisfied. Then, the following inequality holds:*

$$\sup \{\mathrm{Re}(\lambda) : \lambda \in \sigma(A)\} \leq \mu(k,\tau), \tag{17}$$

where $\mu(k,\tau)$ is defined by

$$\begin{aligned}
&\mu(k,\tau) \\
&:= \begin{cases} \max\left\{-a_4, -\dfrac{a_1 - \nu\log k}{\nu\tau+1}, \beta_1(k,\tau), \beta_2(k,\tau)\right\} \\ \quad \text{if } \dfrac{(a_1+a_4-\nu\log k)^2}{\nu\tau+1} > 4\left[a_2a_3 + a_4(a_1-\nu\log k)\right], \\ \max\left\{-a_4, -\dfrac{a_1 - \nu\log k}{\nu\tau+1}, -\dfrac{(a_1+a_4-\nu\log k)^2}{2(\nu\tau+1)}\right\} \\ \quad \text{if } \dfrac{(a_1+a_4-\nu\log k)^2}{\nu\tau+1} = 4\left[a_2a_3 + a_4(a_1-\nu\log k)\right], \end{cases} \\
&\mu(k,\tau) := \max\left\{-a_4, \frac{-a_1+\nu\log k}{\nu\tau+1}\right\} \\
&\qquad \text{if } \frac{(a_1+a_4-\nu\log k)^2}{\nu\tau+1} \\
&\qquad\qquad < 4\left[a_2a_3 + a_4(a_1-\nu\log k)\right],
\end{aligned} \tag{18}$$

with $\beta_1(k,\tau)$ and $\beta_2(k,\tau)$ being

$$\begin{aligned}
\beta_1(k,\tau) &= \frac{-(a_1+a_4-\nu\log k) + \sqrt{(a_1+a_4-\nu\log k)^2 - 4(\nu\tau+1)\left[a_2a_3 + a_4(a_1-\nu\log k)\right]}}{2(\nu\tau+1)}, \\
\beta_2(k,\tau) &= \frac{-(a_1+a_4-\nu\log k) - \sqrt{(a_1+a_4-\nu\log k)^2 - 4(\nu\tau+1)\left[a_2a_3 + a_4(a_1-\nu\log k)\right]}}{2(\nu\tau+1)}.
\end{aligned} \tag{19}$$

Proof. First, let us calculate the eigenvalues of the operator A. It is easy to see that, for $\lambda \in \mathbb{C}$ and $f = (f_1(x), f_2(x), f_3(x)) \in D(A)$, $Af = \lambda f$ is equivalent to

$$-\nu f_1'(x) - a_1 f_1(x) - a_2 f_2(x) = \lambda f_1(x), \tag{20}$$

$$a_3 f_1(x)(x) - a_4 f_2(x) = \lambda f_2(x), \tag{21}$$

$$-\tau^{-1} f_3'(x) = \lambda f_3(x), \tag{22}$$

$$f_1(0) = -k f_3(1), \qquad f_3(0) = f_1(1). \tag{23}$$

By using similar argument of the appendix of [14], it is easy to know that $\lambda = -a_4$ belongs to the continuous spectrum $\sigma_C(A)$ of A. When $\lambda \neq -a_4$, solving (21) and (22), we have

$$f_2(x) = \frac{a_3}{\lambda + a_4} f_1(x), \tag{24}$$

$$f_3(x) = f_3(0)\, e^{-\tau\lambda x}. \tag{25}$$

Set

$$\alpha(\lambda) = -\frac{1}{\nu}\left(\lambda + a_1 + \frac{a_2a_3}{\lambda + a_4}\right), \qquad \beta(\lambda) = \tau\lambda - \alpha(\lambda). \tag{26}$$

Substituting (24) to (20), we have

$$f_1(x) = f_1(0)\, e^{\alpha(\lambda)x}. \tag{27}$$

It follows from (23), (25), and (27) that

$$e^{\beta(\lambda)} = -k. \tag{28}$$

In order to solve (28) with respect to λ, let us set $\lambda = x+iy$, $x, y \in \mathbb{R}$, and

$$\begin{aligned}
u(x,y) &= (\nu\tau+1)x + a_1 + \frac{a_2a_3(x+a_4)}{(x+a_4)^2 + y^2}, \\
v(x,y) &= y\left[1 + \nu\tau - \frac{a_2a_3}{(x+a_4)^2 + y^2}\right].
\end{aligned} \tag{29}$$

Then, (28) becomes

$$e^{\beta(\lambda)} = e^{(1/\nu)[u(x,y)+i\nu(x,y)]} = -k, \tag{30}$$

which is equivalent to

$$\begin{gathered} e^{(1/\nu)u(x,y)} \cos \frac{1}{\nu} \nu(x,y) = -k, \\ \sin \frac{1}{\nu} \nu(x,y) = 0. \end{gathered} \tag{31}$$

Thus, it follows from the previous equations that

$$u(x,y) = \nu \log k, \quad \nu(x,y) = (2n+1)\nu\pi, \quad n \in \mathbb{Z}, \tag{32}$$

which are equivalent to

$$(\nu\tau + 1)x + a_1 + \frac{a_2 a_3 (x + a_4)}{(x + a_4)^2 + y^2} = \nu \log k, \tag{33}$$

$$y\left[1 + \nu\tau - \frac{a_2 a_3}{(x + a_4)^2 + y^2}\right] = (2n+1)\nu\pi. \tag{34}$$

Combining (33) with (34), we get

$$y = \frac{(2n+1)\nu\pi(x + a_4)}{2(\nu\tau + 1)x + a_1 + a_4(\nu\tau + 1) - \nu \log k}, \quad n \in \mathbb{Z}. \tag{35}$$

On the other hand, solving (33) with respect to y, we have

$$y = \pm\sqrt{\frac{a_2 a_3 (x + a_4)}{\nu \log k - (\nu\tau + 1)x - a_1} - (x + a_4)^2}. \tag{36}$$

As a result, introducing two sets

$$\begin{aligned} S_1 &= \left\{x + iy : y = \frac{(2n+1)\nu\pi(x + a_4)}{2(\nu\tau + 1)x + a_1 + a_4(\nu\tau + 1) - \nu \log k}, \right. \\ &\qquad \left. x, y \in \mathbb{R}, x \neq a_4, n \in \mathbb{Z}\right\}, \\ S_2 &= \left\{x + iy : y \right. \\ &\qquad = \pm\sqrt{\frac{a_2 a_3 (x + a_4)}{\nu \log k - (\nu\tau + 1)x - a_1} - (x + a_4)^2}, \\ &\qquad \left. x, y \in \mathbb{R}, x \neq a_4\right\}, \end{aligned} \tag{37}$$

we see that the point spectrum $\sigma_p(A)$ of A is given by $\sigma_p(A) = S_1 \cap S_2$. But we remark that the resolvent set $\rho(A)$ of A is $(S_1^c \cup S_2^c) \setminus \{-a_4\}$ (see the Appendix). This means that

$$\sigma(A) = (S_1 \cap S_2) \cup \{-a_4\}. \tag{38}$$

When $\lambda = x + iy \in \sigma_p(A)$, from the definition of the set S_2, x must satisfy the inequality

$$\frac{a_2 a_3 (x + a_4)}{\nu \log k - (\nu\tau + 1)x - a_1} - (x + a_4)^2 \geq 0, \tag{39}$$

which is equivalent to

$$\begin{aligned} &(x + a_4)\left(x - \frac{\nu \log k - a_1}{\nu\tau + 1}\right) \\ &\quad \times \left(x^2 + \frac{a_1 - \nu \log k + a_4}{\nu\tau + 1}x \right. \\ &\quad \left. + \frac{a_2 a_3 + a_4(a_1 - \nu \log k)}{\nu\tau + 1}\right) \leq 0. \end{aligned} \tag{40}$$

From the inequality and the definition of the $\mu(k,\tau)$, it is obvious that

$$\sup\{\operatorname{Re}(\lambda) : \lambda \in \sigma(A)\} \leq \mu(k,\tau). \tag{41}$$

□

Lemma 3. *Suppose that the assumption of Theorem 1 is satisfied. Then, for any $\varepsilon > 0$, the following holds:*

$$\sup\left\{\left\|(\lambda I - A)^{-1}\right\|_{\mathscr{L}(\mathscr{H})} : \lambda \in \mathbb{C} \text{ and } \operatorname{Re}\lambda \geq \mu(k,\tau) + \varepsilon\right\} < \infty, \tag{42}$$

in which $\mu(k,\tau)$ is the number defined in Lemma 2.

Proof. In Theorem 1, it is shown that the operator A generates a uniformly bounded C_0-semigroup $T(t)$ on $\mathscr{H}$ when the feedback gain k is chosen such that $k^2 < 1/(\nu\tau) < 1$. Then it follows from Theorem 5.3 and Remark 5.4 of [17] that, for any $\varepsilon > 0$, there exists some constant M such that

$$\left\|(\lambda I - A)^{-1}\right\|_{\mathscr{L}(\mathscr{H})} \leq \frac{M}{\operatorname{Re}(\lambda)} \leq \frac{M}{\varepsilon} \tag{43}$$

holds for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}(\lambda) \geq \varepsilon$.

Now, let the subset E_1 of the complex domain $\mathbb{C}$ be given by

$$E_1 = \{\lambda \in \mathbb{C} : \mu(k,\tau) + \varepsilon \leq \operatorname{Re}(\lambda) \leq \varepsilon\}. \tag{44}$$

In order to apply Theorem 1.1 of [14], it must be shown that

$$\sup\left\{\left\|(\lambda I - A)^{-1}\right\|_{\mathscr{L}(\mathscr{H})} : \lambda \in E_1\right\} < \infty. \tag{45}$$

First, for each $\lambda \in E_1$ and each $g = (g_1, g_2, g_3)^T \in \mathscr{H}$, we consider the resolvent equation $(\lambda I - A)f = g$, which is equivalent to

$$\nu f_1'(x) + (\lambda + a_1) f_1(x) + a_2 f_2(x) = g_1(x), \tag{46}$$

$$-a_3 f_1(x)(x) + (\lambda + a_4) f_2(x) = g_2(x), \tag{47}$$

$$f_3'(x) + \tau\lambda f_3(x) = \tau g_3(x). \tag{48}$$

Solving (47) and (48), we have

$$f_2(x) = \frac{a_3}{\lambda + a_4} f_1(x) + \frac{1}{\lambda + a_4} g_2(x), \tag{49}$$

$$f_3(x) = f_3(0) e^{-\tau\lambda x} + \int_0^x \tau e^{-\tau\lambda(x-s)} g_3(s)\, ds. \tag{50}$$

Substituting (49) with (46) and solving it, we have

$$\begin{aligned} f_1(x) = {} & f_1(0) e^{\alpha(\lambda)x} + \int_0^x \tau e^{\alpha(\lambda)(x-s)} \\ & \times \left[\frac{1}{\nu} g_1(s) - \frac{a_2}{\nu(\lambda + a_4)} g_2(s) \right] ds. \end{aligned} \tag{51}$$

Since $f = (f_1, f_2, f_3)^T$ belongs to the domain $D(A)$ of A, $f_1(x)$ and $f_3(x)$ should satisfy the relations

$$f_1(0) = -k f_3(1), \qquad f_1(1) = f_3(0). \tag{52}$$

Putting $x = 1$ in (50) and (51), respectively, we obtain

$$\begin{aligned} f_3(1) = {} & \frac{e^{-\tau\lambda}}{1 + k e^{-\beta(\lambda)}} \\ & \times \left[\int_0^1 e^{\alpha(\lambda)(1-s)} \left(\frac{1}{\nu} g_1(s) - \frac{a_2}{\nu(\lambda + a_4)} g_2(s) \right) ds \right. \\ & \left. + \int_0^1 \tau e^{\tau\lambda s} g_3(s)\, ds \right], \\ f_3(0) = {} & e^{\tau\lambda} f_3(1) - \int_0^1 \tau e^{\tau\lambda s} g_3(s)\, ds. \end{aligned} \tag{53}$$

If substitute them into (50) and (51), then we have

$$\begin{aligned} f_1(x) = {} & \frac{-k e^{-\tau\lambda} e^{\alpha(\lambda)x}}{1 + k e^{-\beta(\lambda)}} \\ & \times \left[\int_0^1 e^{\alpha(\lambda)(1-s)} \left(\frac{1}{\nu} g_1(s) - \frac{a_2}{\nu(\lambda + a_4)} g_2(s) \right) ds \right. \\ & \left. + \int_0^1 \tau e^{\tau\lambda s} g_3(s)\, ds \right] \\ & + \int_0^x \tau e^{\alpha(\lambda)(x-s)} \left[\frac{1}{\nu} g_1(s) - \frac{a_2}{\nu(\lambda + a_4)} g_2(s) \right] ds, \\ f_3(x) = {} & \frac{e^{\tau\lambda x}}{1 + k e^{-\beta(\lambda)}} \\ & \times \left[\int_0^1 e^{\alpha(\lambda)(1-s)} \left(\frac{1}{\nu} g_1(s) - \frac{a_2}{\nu(\lambda + a_4)} g_2(s) \right) ds \right. \\ & \left. + \int_0^1 \tau e^{-\tau\lambda(1-s)} g_3(s)\, ds \right] \\ & - \int_x^1 \tau e^{-\tau\lambda(x-s)} g_3(s)\, ds. \end{aligned} \tag{54}$$

Also, $f_2(x)$ can be obtained from the previous equations and (49).

Next, we will estimate a bound of $\mathscr{H}$ norm of $f = (f_1, f_2, f_3)^T$. It follows from

$$\begin{aligned} \varepsilon & \le \mu(k, \tau) + \varepsilon + a_4 \\ & = \left| \mu(k, \tau) + \varepsilon + a_4 \right| \le \operatorname{Re} \lambda + a_4 = \left| \lambda + a_4 \right| \end{aligned} \tag{55}$$

that

$$\begin{aligned} & \left| e^{-(1/\nu)(a_2 a_3/(\lambda + a_4))x} \right| \\ & \quad = e^{-(1/\nu) a_2 a_3 (\operatorname{Re} \lambda + a_4) x / ((\operatorname{Re} \lambda + a_4)^2 + (\operatorname{Im} \lambda)^2)} \\ & \quad \le e^{-(1/\nu) a_2 a_3 \varepsilon x / ((\operatorname{Re} \lambda + a_4)^2 + (\operatorname{Im} \lambda)^2)} \le 1. \end{aligned} \tag{56}$$

Noting that

$$\begin{aligned} \nu \log k + (1 + \nu\tau) \varepsilon & \le a_1 + (1 + \nu\tau) \left[\mu(k, \tau) + \varepsilon \right] \\ & \le a_1 + (1 + \nu\tau) \operatorname{Re} \lambda, \\ \nu \log k + \varepsilon & \le \nu \log k + \varepsilon + \nu\tau (\varepsilon - \operatorname{Re} \lambda) \le a_1 + \operatorname{Re} \lambda, \end{aligned} \tag{57}$$

we have

$$\begin{aligned} & \left| e^{-\beta(\lambda)x} \right| \\ & \quad = e^{-(1/\nu)[(1+\nu\tau) \operatorname{Re} \lambda + a_1]} \left| e^{-(1/\nu)(a_2 a_3/(\lambda + a_4))x} \right| \\ & \quad \le e^{-(1/\nu)[\nu \log k + (1+\nu\tau)\varepsilon] x} \\ & \quad = k^{-1} e^{-(1/\nu)(1+\nu\tau)\varepsilon x} < k^{-1}, \end{aligned} \tag{58}$$

$$\begin{aligned} & \left| e^{\alpha(\lambda)x} \right| \\ & \quad = e^{-(1/\nu)[\operatorname{Re} \lambda + a_1]} \left| e^{-(1/\nu)(a_2 a_3/(\lambda + a_4))x} \right| \\ & \quad \le e^{-(1/\nu)[\nu \log k + \varepsilon] x} = k^{-1} e^{-(1/\nu)\varepsilon x} < k^{-1}, \end{aligned} \tag{59}$$

for all $x \in [0, 1]$. Putting $x = 1$ in (58), we have

$$k \left| e^{-\beta(\lambda)} \right| \le e^{-(1/\nu)(1+\nu\tau)\varepsilon} < 1. \tag{60}$$

Moreover, the continuous function $h(x, y) := e^{-\tau x y}$ defined on the compact set $[0, 1] \times [\mu(k, \tau) + \varepsilon, \varepsilon]$ has absolute maximum and absolute minimum, which are denoted by L and l, respectively. Thus, we have

$$\begin{aligned} \left| f_1(x) \right| \le {} & \frac{k \left| e^{-\tau\lambda} \right| \left| e^{\alpha(\lambda)x} \right|}{\left| 1 + k e^{-\beta(\lambda)} \right|} \\ & \times \left[\int_0^1 \left| e^{\alpha(\lambda)(1-s)} \right| \left(\frac{1}{\nu} \left| g_1(s) \right| + \frac{a_2}{\nu \left| \lambda + a_4 \right|} \left| g_2(s) \right| \right) ds \right. \\ & \left. + \int_0^1 \tau \left| e^{\tau\lambda s} \right| \left| g_3(s) \right| ds \right] \\ & + \int_0^x \tau \left| e^{\alpha(\lambda)(x-s)} \right| \end{aligned}$$

$$\times \left[\frac{1}{\nu} \left| g_1(s) \right| + \frac{a_2}{\nu \left| \lambda + a_4 \right|} \left| g_2(s) \right| \right] ds$$

$$\leq \frac{e^{-\tau \operatorname{Re} \lambda}}{1 - \left| k e^{-\beta(\lambda)} \right|}$$

$$\times \left[\int_0^1 \frac{1}{k\nu} \left| g_1(s) \right| + \frac{a_2}{k\nu\varepsilon} \left| g_2(s) \right| ds \right.$$

$$\left. + \int_0^1 \tau e^{\tau s \operatorname{Re} \lambda} \left| g_3(s) \right| ds \right]$$

$$+ \int_0^1 \frac{\tau}{k\nu} \left| g_1(s) \right| + \frac{\tau a_2}{k\nu\varepsilon} \left| g_2(s) \right| ds$$

$$\leq \frac{L}{1 - e^{-(1/\nu)(1+\nu\tau)\varepsilon}}$$

$$\times \left[\int_0^1 \frac{1}{k\nu} \left| g_1(s) \right| + \frac{a_2}{k\nu\varepsilon} \left| g_2(s) \right| ds \right.$$

$$\left. + \int_0^1 \frac{\tau}{l} \left| g_3(s) \right| ds \right]$$

$$+ \int_0^1 \frac{\tau}{k\nu} \left| g_1(s) \right| + \frac{\tau a_2}{k\nu\varepsilon} \left| g_2(s) \right| ds$$

$$\leq \left(\frac{L}{k\nu \left[1 - e^{-(1/\nu)(1+\nu\tau)\varepsilon} \right]} + \frac{\tau}{k\nu} \right) \sqrt{\int_0^1 \left| g_1(s) \right|^2 dx}$$

$$+ \frac{a_2 L}{k\nu\varepsilon \left[1 - e^{-(1/\nu)(1+\nu\tau)\varepsilon} \right]} \sqrt{\int_0^1 \left| g_2(s) \right|^2 dx}$$

$$+ \left(\frac{\tau L}{l \left[1 - e^{-(1/\nu)(1+\nu\tau)\varepsilon} \right]} + \frac{\tau a_2}{k\nu\varepsilon} \right) \sqrt{\int_0^1 \left| g_3(s) \right|^2 dx}. \tag{61}$$

The Cauchy-Schwarz inequality is applied in the last step. This means that

$$\left| f_1(x) \right| \leq M_1 \|g\|_{\mathscr{H}} \quad \text{or} \quad \|f_1(x)\|_{L^2[0,1]} \leq M_1 \|g\|_{\mathscr{H}}, \tag{62}$$

in which

$$M_1 = \max \left\{ \frac{L}{k\nu \left[1 - e^{-(1/\nu)(1+\nu\tau)\varepsilon} \right]} + \frac{\tau}{k\nu}, \right.$$

$$\frac{a_2 L}{k\nu\varepsilon \left[1 - e^{-(1/\nu)(1+\nu\tau)\varepsilon} \right]}, \tag{63}$$

$$\left. \frac{\tau L}{l \left[1 - e^{-(1/\nu)(1+\nu\tau)\varepsilon} \right]} + \frac{\tau a_2}{k\nu\varepsilon} \right\}.$$

Similarly, we have

$$\left| f_3(x) \right| \leq \frac{\left| e^{\tau\lambda x} \right|}{\left| 1 + k e^{-\beta(\lambda)} \right|}$$

$$\times \left[\int_0^1 \left| e^{\alpha(\lambda)(1-s)} \right| \right.$$

$$\times \left(\frac{1}{\nu} \left| g_1(s) \right| + \frac{a_2}{\nu \left| \lambda + a_4 \right|} \left| g_2(s) \right| \right) ds$$

$$\left. + \int_0^1 \tau \left| e^{-\tau\lambda(1-s)} \right| \left| g_3(s) \right| ds \right]$$

$$+ \int_x^1 \tau \left| e^{-\tau\lambda(x-s)} \right| \left| g_3(s) \right| ds$$

$$\leq \frac{e^{\tau x \operatorname{Re} \lambda}}{1 - e^{-(1/\nu)(1+\nu\tau)\varepsilon}}$$

$$\times \left[\int_0^1 \left(\frac{1}{\nu} \left| g_1(s) \right| + \frac{a_2}{\nu\varepsilon} \left| g_2(s) \right| \right) ds \right.$$

$$\left. + \int_0^1 \tau e^{-\tau \operatorname{Re} \lambda (1-s)} \left| g_3(s) \right| ds \right]$$

$$+ \int_x^1 \tau e^{-\tau \operatorname{Re} \lambda (x-s)} \left| g_3(s) \right| ds$$

$$\leq \frac{1}{l \left[1 - e^{-(1/\nu)(1+\nu\tau)\varepsilon} \right]}$$

$$\times \left[\int_0^1 \left(\frac{1}{\nu} \left| g_1(s) \right| + \frac{a_2}{\nu\varepsilon} \left| g_2(s) \right| \right) ds \right.$$

$$\left. + \int_0^1 \frac{\tau L}{l} \left| g_3(s) \right| ds \right]$$

$$+ \int_0^1 \frac{\tau L}{l} \left| g_3(s) \right| ds$$

$$\leq \frac{1}{l\nu \left[1 - e^{-(1/\nu)(1+\nu\tau)\varepsilon} \right]} \sqrt{\int_0^1 \left| g_1(s) \right|^2 dx}$$

$$+ \frac{a_2}{l\nu\varepsilon \left[1 - e^{-(1/\nu)(1+\nu\tau)\varepsilon} \right]} \sqrt{\int_0^1 \left| g_2(s) \right|^2 dx}$$

$$+ \left(\frac{\tau L}{l^2 \left[1 - e^{-(1/\nu)(1+\nu\tau)\varepsilon} \right]} + \frac{\tau L}{l} \right) \sqrt{\int_0^1 \left| g_3(s) \right|^2 dx}. \tag{64}$$

This means that

$$\left| f_3(x) \right| \leq M_3 \|g\|_{\mathscr{H}} \quad \text{or} \quad \|f_3(x)\|_{L^2[0,1]} \leq M_3 \|g\|_{\mathscr{H}}, \tag{65}$$

in which

$$M_3 = \max\left\{\frac{1}{l\nu\left[1-e^{-(1/\nu)(1+\nu\tau)\varepsilon}\right]}, \frac{a_2}{l\nu\varepsilon\left[1-e^{-(1/\nu)(1+\nu\tau)\varepsilon}\right]}, \frac{\tau L}{l^2\left[1-e^{-(1/\nu)(1+\nu\tau)\varepsilon}\right]}+\frac{\tau L}{l}\right\}. \tag{66}$$

It follows from (49) and (62) that

$$\left|f_2(x)\right| \le \frac{a_3}{\varepsilon}\left|f_1(x)\right| + \frac{1}{\varepsilon}\left|g_2(x)\right| \le \frac{a_3 M_1}{\varepsilon}\|g\|_{\mathscr{H}} + \frac{1}{\varepsilon}\left|g_2(x)\right|. \tag{67}$$

This implies that

$$\|f_2(x)\|_{L^2[0,1]} \le M_2\|g\|_{\mathscr{H}} \tag{68}$$

in which

$$M_2 = \frac{a_3 M_1}{\varepsilon} + \frac{1}{\varepsilon}. \tag{69}$$

Therefore, by using (62), (65), and (68), we can estimate a bound of $\mathscr{H}$ norm of $f = (f_1, f_2, f_3)^T$ as follows:

$$\begin{aligned}\|f\|_{\mathscr{H}}^2 &= \|f_1(x)\|_{L^2[0,1]}^2 + \|f_2(x)\|_{L^2[0,1]}^2 + \|f_3(x)\|_{L^2[0,1]}^2 \\ &\le \left(M_1^2 + M_2^2 + M_3^2\right)\|g\|_{\mathscr{H}},\end{aligned} \tag{70}$$

which is equivalent to

$$\left\|(\lambda I - A)^{-1}\right\|_{L(\mathscr{H})} \le \sqrt{M_1^2 + M_2^2 + M_3^2}. \tag{71}$$

Since the inequality holds for all $g \in \mathscr{H}$ and for all $\lambda \in E_1$, we have

$$\sup_{\lambda\in E_1}\left\|(\lambda I - A)^{-1}\right\|_{L(\mathscr{H})} \le \sqrt{M_1^2 + M_2^2 + M_3^2} < \infty. \tag{72}$$

This shows that (45) holds. In this way, we finally obtain

$$\sup\left\{\left\|(\lambda I - A)^{-1}\right\|_{\mathscr{L}(\mathscr{H})} : \lambda \in \mathbb{C} \text{ and } \operatorname{Re}\lambda \ge \mu(k,\tau) + \varepsilon\right\} < \infty. \tag{73}$$

□

Theorem 4. *Suppose that the assumption of Theorem 1 is satisfied. Then, for any $\varepsilon > 0$, there exists a constant $M_{\varepsilon,\tau}$ such that*

$$\|T(t)\|_{\mathscr{L}(\mathscr{H})} \le M_{\varepsilon,\tau} e^{[\mu(k,\tau)+\varepsilon]t} \tag{74}$$

in which $\mu(k,\tau)$ is defined in Lemma 2.

Proof. According to Theorem 1.1 of [14], Theorem 4 is direct consequence of Lemmas 2 and 3. □

4. Conclusion

In the present paper, we have considered a linear distributed parameter bioprocess with boundary control input possessing a time delay. Using a simple boundary feedback law, we have shown that the closed-loop system generates a uniformly bounded C_0-semigroup of linear operators if the feedback gain k satisfies $0 < k < 1$. After analyzing the spectrum configuration of the closed-loop system and verifying the spectrum determined growth assumption, we have demonstrated that the closed-loop system becomes exponentially stable. Our main result implies that the linear distributed parameter bioprocess preserves the exponential stability for arbitrary time delay. This means that the answer to the question posed in Section 1 is positive.

Appendix

Let S_1 and S_2 be the sets defined in the proof of Lemma 2. To show that the resolvent set $\rho(A)$ of A is $(S_1^c \cup S_2^c) \setminus \{-a_4\}$, we have to prove that the operator $\lambda I - A$ is bijective for each $\lambda \in (S_1^c \cup S_2^c) \setminus \{-a_4\}$. Thus, for each $\lambda \in (S_1^c \cup S_2^c) \setminus \{-a_4\}$ and each $g = (g_1, g_2, g_3)^T \in \mathscr{H}$, we consider the resolvent equation $(\lambda I - A)f = g$, which is equivalent to

$$\begin{aligned}\nu f_1'(x) + (\lambda + a_1) f_1(x) + a_2 f_2(x) &= g_1(x), \\ -a_3 f_1(x)(x) + (\lambda + a_4) f_2(x) &= g_2(x), \\ f_3'(x) + \tau\lambda f_3(x) &= \tau g_3(x).\end{aligned} \tag{A.1}$$

It follows from the proof of Lemma 3 that

$$\begin{aligned}f_1(x) &= f_1(0)e^{\alpha(\lambda)x} + \int_0^x \tau e^{\alpha(\lambda)(x-s)}\left[\frac{1}{\nu}g_1(s) - \frac{a_2}{\nu(\lambda+a_4)}g_2(s)\right]ds, \\ f_2(x) &= \frac{a_3}{\lambda+a_4}f_1(x) + \frac{1}{\lambda+a_4}g_2(x), \\ f_3(x) &= f_3(0)e^{-\tau\lambda x} + \int_0^x \tau e^{-\tau\lambda(x-s)}g_3(s)\,ds, \\ \left(1+ke^{-\beta(\lambda)}\right)f_3(1) &= e^{-\tau\lambda}\left[\int_0^1 e^{\alpha(\lambda)(1-s)}\left(\frac{1}{\nu}g_1(s) - \frac{a_2}{\nu(\lambda+a_4)}g_2(s)\right)ds + \int_0^1 \tau e^{\tau\lambda s}g_3(s)\,ds\right].\end{aligned} \tag{A.2}$$

It is easy to see that $\lambda I - A$ is bijective if and only if $1 + ke^{-\beta(\lambda)} \neq 0$. From the proof of Lemma 3, we know that $e^{\beta(\lambda)} = -k$ for $\lambda \in S_1 \cap S_2$. This implies that $\rho(A) = (S_1^c \cup S_2^c) \setminus \{-a_4\}$.

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