Research Article

Stability of a Generalized Trigonometric-Quadratic Functional Equation

Janyarak Tongsonporn¹ and Vichian Laohakosol²

¹ School of Science, Walailak University, Nakhon Si Thammarat 80160, Thailand

² Department of Mathematics, Faculty of Science, Kasetsart University, Bangkok 10900, Thailand

Correspondence should be addressed to Vichian Laohakosol; fscivil@ku.ac.th

Received 28 May 2013; Accepted 25 September 2013

Academic Editor: Jaeyoung Chung

Copyright © 2013 J. Tongsonporn and V. Laohakosol. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

A stability result for the generalized trigonometric-quadratic functional equation $\mathcal{F}(x+y) - \mathcal{G}(x-y) = 2\mathcal{H}(x)\mathcal{K}(y) + \mathcal{L}(x) + \mathcal{M}(y)$ over the domain of a normed space and the range of a Banach space is derived. It states that under appropriate conditions on the function $\mathcal{H}(x)\mathcal{K}(y)$, the four functions $\mathcal{F}(x)$, $\mathcal{G}(x)$, $\mathcal{L}(x)$, and $\mathcal{M}(x)$ are approximately equal to combinations of certain quadratic and additive functions.

1. Introduction

A generalized trigonometric functional equation is a functional equation of the form

$$F(x+y) - G(x-y) = 2H(x)K(y), \quad (1)$$

which is so named because the two best known trigonometric functions, sine and cosine, are solutions of special cases of this equation. In [1], the authors investigated the stability of (1), where F, G, H , and K are nonzero functions from an abelian group $(G, +)$ to the complex field $\mathbb{C}$. Later in the same year, a generalized trigonometric functional equation with either the function H or K being bounded was investigated to complement the above earlier work where such function boundedness was not treated. A generalized quadratic functional equation is a functional equation of the form

$$f_1(x+y) + f_2(x-y) = f_3(x) + f_4(y), \quad (2)$$

which is so named because the quadratic function x^2 is a solution of one of its particular cases, namely,

$$f(x+y) + f(x-y) = 2f(x) + 2f(y), \quad (3)$$

and any solution of (3) is referred to as a *quadratic* function. The stability of the quadratic functional equation was first

proved by Skof [2] in 1983 for functions from a normed space to a Banach space. In 1984, Cholewa [3] demonstrated that Skof's theorem is also valid if a normed space is replaced by an abelian group. Later, Czerwik [4] extended Cholewa's result by changing a control function and considering functions from a normed space to a Banach space. In another paper, [5], Czerwik investigated the stability of the "partially pexiderized" quadratic functional equation of the form $f(x+y) + f(x-y) = g(x) + g(y)$. In 2000, Jung [6] considered (2), where f_1, f_2, f_3 , and f_4 are functions from a normed space E_1 to a Banach space E_2 . In the paper [7], the authors considered the stability of a combined generalized trigonometric-quadratic functional equation of the form

$$\mathcal{F}(x+y) + \mathcal{G}(x-y) = 2\mathcal{H}(x)\mathcal{K}(y) + \mathcal{L}(x) + \mathcal{M}(y) \quad (4)$$

over the domain of an abelian group $(G, +)$ and the range $\mathbb{C}$, except the control function ϕ whose range is taken to be the nonnegative real numbers $[0, \infty)$. There, the functions $\mathcal{H}$ and $\mathcal{K}$ are explicitly determined under certain restrictions on the remaining functions $\mathcal{F}$, $\mathcal{G}$, $\mathcal{L}$, and $\mathcal{M}$. It is natural to ask whether the restrictions on the functions $\mathcal{F}$, $\mathcal{G}$, $\mathcal{L}$, and $\mathcal{M}$ can be removed and/or altered to obtain some stability result of (4). We answer this question affirmatively using Jung's techniques elaborated in [6, 8].

2. Notation

We collect in this section the notation and terminology that will be kept standard throughout. A normed space E_1 is the domain and a Banach space $(E_2, \|\cdot\|)$ is the range of all functions involved. Let $\mathcal{H}, \mathcal{K} : E_1 \rightarrow E_2$. Define

$$\begin{aligned} HK(x, y) &:= \mathcal{H}(x)\mathcal{K}(y) - \mathcal{H}(0)\mathcal{K}(0), \\ HK_e(x, y) &:= \frac{HK(x, y) + HK(-x, -y)}{2}, \\ HK_o(x, y) &:= \frac{HK(x, y) - HK(-x, -y)}{2}. \end{aligned} \quad (5)$$

Let $\varphi : E_1 \times E_1 \rightarrow [0, \infty)$ be a given function satisfying the following three conditions (i) $\varphi(y, x) = \varphi(x, y)$, (ii) $\varphi(x, -y) = \varphi(x, y)$, and (iii) there is an integer $k \geq 2$ such that

$$\begin{aligned} \sum_{i=0}^{\infty} \frac{1}{k^i} \varphi(k^i x, k^i y) &< \infty, \\ \text{or } \sum_{i=0}^{\infty} k^i \varphi\left(\frac{x}{k^i}, \frac{y}{k^i}\right) &< \infty \quad (x, y \in E_1). \end{aligned} \quad (6)$$

For $x, y \in E_1$, define

$$\begin{aligned} \varphi_e(x, y) &= 2\varphi(x, y) + 4\varphi(x, 0) \\ &\quad + 3\varphi(x + y, 0) + 3\varphi(x - y, 0) \\ &\quad + 6\varphi\left(\frac{x + y}{2}, \frac{x + y}{2}\right) \\ &\quad + 2\varphi\left(\frac{x - y}{2}, \frac{x - y}{2}\right) + 20\varphi(0, 0), \\ \varphi_o(x, y) &= 2\varphi(x, y) + 2\varphi(x, x) \\ &\quad + 2\varphi(y, y) + 2\varphi(2x, 0) + 2\varphi(2y, 0) \\ &\quad + 2\varphi(x + y, 0) + 2\varphi(x - y, 0) + 2\varphi(0, 0), \\ \psi_k(x, y) &= \sum_{i=0}^{\infty} \frac{1}{k^{2i}} \varphi(k^i x, k^i y), \\ \tilde{\psi}_k(x, y) &= \sum_{i=0}^{\infty} k^{2(i+1)} \varphi\left(\frac{x}{k^{i+1}}, \frac{y}{k^{i+1}}\right), \\ \Phi_k(x, y) &= \sum_{i=0}^{\infty} \frac{1}{k^{2i}} \sum_{j=1}^{k-1} b_j \varphi_e(k^i(k-j)x, k^i y), \\ \tilde{\Phi}_k(x, y) &= \sum_{i=0}^{\infty} k^{2(i+1)} \sum_{j=1}^{k-1} b_j \varphi_e\left(\frac{(k-j)x}{k^{i+1}}, \frac{y}{k^{i+1}}\right), \\ \Phi'_k(x, y) &= \sum_{i=0}^{\infty} \frac{1}{k^i} \sum_{j=1}^{k-1} \varphi_o\left(k^i \frac{jx + y}{2}, k^i \frac{jx - y}{2}\right), \end{aligned}$$

$$\begin{aligned} \tilde{\Phi}'_k(x, y) &= \sum_{i=0}^{\infty} k^{i+1} \sum_{j=1}^{k-1} \varphi_o\left(\frac{jx + y}{2k^{i+1}}, \frac{jx - y}{2k^{i+1}}\right), \\ \Phi''_k(x, y) &= \sum_{i=0}^{\infty} \frac{1}{k^i} \sum_{j=1}^{k-1} \varphi_o\left(k^i \left(jx + \frac{y}{2}\right), k^i \frac{y}{2}\right), \\ \tilde{\Phi}''_k(x, y) &= \sum_{i=0}^{\infty} k^{i+1} \sum_{j=1}^{k-1} \varphi_o\left(\frac{2jx + y}{2k^{i+1}}, \frac{y}{2k^{i+1}}\right), \\ \psi_{HK,k}(x, y) &= \sum_{i=0}^{\infty} \frac{1}{k^{2i}} \|2HK_e(k^i x, k^i y) \\ &\quad + HK_e\left(\frac{k^i(x+y)}{2}, -\frac{k^i(x+y)}{2}\right) \\ &\quad - HK_e\left(\frac{k^i(x+y)}{2}, \frac{k^i(x+y)}{2}\right) \\ &\quad + HK_e\left(\frac{k^i(x-y)}{2}, \frac{k^i(x-y)}{2}\right) \\ &\quad - HK_e\left(\frac{k^i(x-y)}{2}, -\frac{k^i(x-y)}{2}\right) \\ &\quad + 2HK_e(0, k^i x) - 2HK_e(k^i x, 0) \\ &\quad - HK_e(0, k^i(x+y)) \\ &\quad - HK_e(0, k^i(x-y))\|, \\ \tilde{\psi}_{HK,k}(x, y) &= \sum_{i=0}^{\infty} k^{2(i+1)} \|2HK_e\left(\frac{x}{k^{i+1}}, \frac{y}{k^{i+1}}\right) \\ &\quad + HK_e\left(\frac{x+y}{2k^{i+1}}, -\frac{x+y}{2k^{i+1}}\right) \\ &\quad - HK_e\left(\frac{x+y}{2k^{i+1}}, \frac{x+y}{2k^{i+1}}\right) \\ &\quad + HK_e\left(\frac{x-y}{2k^{i+1}}, \frac{x-y}{2k^{i+1}}\right) \\ &\quad - HK_e\left(\frac{x-y}{2k^{i+1}}, -\frac{x-y}{2k^{i+1}}\right) \\ &\quad + 2HK_e\left(0, \frac{x}{k^{i+1}}\right) - 2HK_e\left(\frac{x}{k^{i+1}}, 0\right) \\ &\quad - HK_e\left(0, \frac{x+y}{k^{i+1}}\right) - HK_e\left(0, \frac{x-y}{k^{i+1}}\right)\|, \\ \Phi_{HK,k}(x, y) &= \frac{2}{k^2} \sum_{i=0}^{\infty} \frac{1}{k^{2i}} \sum_{j=1}^{k-1} \|2HK_e((k-j)k^i x, k^i y) \end{aligned}$$

$$\begin{aligned}
& + HK_e \left(\left(\frac{(k-j)x+y}{2} \right) k^i, \right. \\
& \quad \left. - \left(\frac{(k-j)x+y}{2} \right) k^i \right) \\
& - HK_e \left(\left(\frac{(k-j)x+y}{2} \right) k^i, \right. \\
& \quad \left. \left(\frac{(k-j)x+y}{2} \right) k^i \right) \\
& + HK_e \left(\left(\frac{(k-j)x-y}{2} \right) k^i, \right. \\
& \quad \left. \left(\frac{(k-j)x-y}{2} \right) k^i \right) \\
& - HK_e \left(\left(\frac{(k-j)x-y}{2} \right) k^i, \right. \\
& \quad \left. - \left(\frac{(k-j)x-y}{2} \right) k^i \right) \\
& + 2HK_e(0, (k-j)k^i x) \\
& - 2HK_e((k-j)k^i x, 0) \\
& - HK_e(0, ((k-j)x+y)k^i) \\
& - HK_e(0, ((k-j)x-y)k^i) \parallel,
\end{aligned}$$

$$\widetilde{\Phi}_{HK,k}(x, y)$$

$$\begin{aligned}
& = \frac{2}{k^2} \sum_{i=0}^{\infty} \frac{1}{k^{2i}} \sum_{j=1}^{k-1} \parallel 2HK_e(jk^i x, k^i y) \\
& \quad + HK_e \left(\left(\frac{jx+y}{2} \right) k^i, - \left(\frac{jx+y}{2} \right) k^i \right) \\
& \quad - HK_e \left(\left(\frac{jx+y}{2} \right) k^i, \left(\frac{jx+y}{2} \right) k^i \right) \\
& \quad + HK_e \left(\left(\frac{jx-y}{2} \right) k^i, \left(\frac{jx-y}{2} \right) k^i \right) \\
& \quad - HK_e \left(\left(\frac{jx-y}{2} \right) k^i, - \left(\frac{jx-y}{2} \right) k^i \right) \\
& \quad + 2HK_e(0, jk^i x) - 2HK_e(jk^i x, 0) \\
& \quad - HK_e(0, (jx+y)k^i) \\
& \quad - HK_e(0, (jx-y)k^i) \parallel,
\end{aligned}$$

$$\Phi'_{HK,k}(x, y)$$

$$= \frac{2}{k} \sum_{i=0}^{\infty} \frac{1}{k^i} \sum_{j=1}^{k-1} \parallel HK_o \left(\left(\frac{jx+y}{2} \right) k^i, \left(\frac{jx-y}{2} \right) k^i \right)$$

$$\begin{aligned}
& - HK_o \left(- \left(\frac{jx+y}{2} \right) k^i, \left(\frac{jx-y}{2} \right) k^i \right) \\
& - HK_o \left(\left(\frac{jx+y}{2} \right) k^i, \left(\frac{jx+y}{2} \right) k^i \right) \\
& + HK_o \left(- \left(\frac{jx+y}{2} \right) k^i, \left(\frac{jx+y}{2} \right) k^i \right) \\
& + HK_o((jx+y)k^i, 0) \\
& - HK_o(jk^i x, 0) - HK_o(k^i x, 0) \parallel,
\end{aligned}$$

$$\widetilde{\Phi}'_{HK,k}(x, y)$$

$$\begin{aligned}
& = \frac{2}{k} \sum_{i=0}^{\infty} k^{i+1} \sum_{j=1}^{k-1} \parallel HK_o \left(\frac{jx+y}{2k^{i+1}}, \frac{jx-y}{2k^{i+1}} \right) \\
& \quad - HK_o \left(- \frac{jx+y}{2k^{i+1}}, \frac{jx-y}{2k^{i+1}} \right) \\
& \quad - HK_o \left(\frac{jx+y}{2k^{i+1}}, \frac{jx+y}{2k^{i+1}} \right) \\
& \quad + HK_o \left(- \frac{jx+y}{2k^{i+1}}, \frac{jx+y}{2k^{i+1}} \right) \\
& \quad + HK_o \left(\frac{jx+y}{k^{i+1}}, 0 \right) \\
& \quad - HK_o \left(\frac{jx}{k^{i+1}}, 0 \right) - HK_o \left(\frac{x}{k^{i+1}}, 0 \right) \parallel.
\end{aligned} \tag{7}$$

Define a real sequence (b_i) by

$$b_1 = 1, \quad b_2 = 2, \quad b_i = 2b_{i-1} + b_{i-2} \quad (i \geq 3), \tag{8}$$

and let

$$\begin{aligned}
\Psi_k(x, y) &= b_1 \psi_k((k-1)x, y) \\
&\quad + b_2 \psi_k((k-2)x, y) + \cdots + b_{k-1} \psi_k(x, y), \\
\widetilde{\Psi}_k(x, y) &= b_1 \widetilde{\psi}_k((k-1)x, y) \\
&\quad + b_2 \widetilde{\psi}_k((k-2)x, y) + \cdots + b_{k-1} \widetilde{\psi}_k(x, y), \\
\Psi_{HK,k}(x, x) &= b_1 \psi_{HK,k}((k-1)x, y) \\
&\quad + b_2 \psi_{HK,k}((k-2)x, y) + \cdots \\
&\quad + b_{k-1} \psi_{HK,k}(x, y), \\
\widetilde{\Psi}_{HK,k}(x, x) &= b_1 \widetilde{\psi}_{HK,k}((k-1)x, y) \\
&\quad + b_2 \widetilde{\psi}_{HK,k}((k-2)x, y) + \cdots \\
&\quad + b_{k-1} \widetilde{\psi}_{HK,k}(x, y).
\end{aligned} \tag{9}$$

Clearly, the functions φ_e, φ_o satisfy the conditions (i), (ii), and (iii).

3. Auxiliary Results

We start with an important lemma.

Lemma 1. *If $f: E_1 \rightarrow E_2$, $f(0) = 0$, satisfies*

$$\begin{aligned} & \|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \\ & \leq \varphi_e(x, y) \\ & + 2\|2HK_e(x, y) \\ & + HK_e\left(\frac{x+y}{2}, -\frac{x+y}{2}\right) \\ & - HK_e\left(\frac{x+y}{2}, \frac{x+y}{2}\right) \\ & + HK_e\left(\frac{x-y}{2}, \frac{x-y}{2}\right) \\ & - HK_e\left(\frac{x-y}{2}, -\frac{x-y}{2}\right) \\ & + 2HK_e(0, x) - 2HK_e(x, 0) \\ & - HK_e(0, x+y) - HK_e(0, x-y)\|, \end{aligned} \quad (10)$$

then one has, for integer $k \geq 2$, $x \in E_1$ and $n \in \mathbb{N}$,

$$\begin{aligned} & \|f(k^n x) - k^{2n} f(x)\| \\ & \leq \sum_{i=0}^{n-1} k^{2(n-1-i)} \sum_{j=1}^{k-1} b_j \varphi((k-j)k^i x, k^i x) \\ & + 2 \sum_{i=0}^{n-1} k^{2(n-1-i)} \\ & \times \sum_{j=1}^{k-1} b_j \|2HK_e((k-j)k^i x, k^i x) \\ & + HK_e\left(\frac{(k-j+1)k^i x}{2}, -\frac{(k-j+1)k^i x}{2}\right) \\ & - HK_e\left(\frac{(k-j+1)k^i x}{2}, \frac{(k-j+1)k^i x}{2}\right) \\ & + HK_e\left(\frac{(k-j-1)k^i x}{2}, \frac{(k-j-1)k^i x}{2}\right) \\ & - HK_e\left(\frac{(k-j-1)k^i x}{2}, -\frac{(k-j-1)k^i x}{2}\right) \\ & + 2HK_e(0, (k-j)k^i x) \\ & - 2HK_e((k-j)k^i x, 0) \\ & - HK_e(0, (k-j+1)k^i x) \\ & - HK_e(0, (k-j-1)k^i x)\|. \end{aligned} \quad (11)$$

Proof. We prove the lemma by induction on n . To verify the case $n = 1$, we use induction on $k \geq 2$. Putting $y = x$ in (10), we obtain

$$\begin{aligned} & \|f(2x) - 2^2 f(x)\| \\ & \leq \varphi_e(x, x) \\ & + 2\|HK_e(x, x) + HK_e(x, -x) \\ & + 2HK_e(0, x) - 2HK_e(x, 0) - HK_e(0, 2x)\|; \end{aligned} \quad (12)$$

that is, the assertion is true for $k = 2$. Assume that the statement is true for all $m \in \{2, \dots, k\}$. Replacing x and y in (10) by kx and x , we get

$$\begin{aligned} & \|f((k+1)x) + f((k-1)x) - 2f(kx) - 2f(x)\| \\ & \leq \varphi_e(kx, x) \\ & + 2\|2HK_e(kx, x) + HK_e\left(\frac{(k+1)x}{2}, -\frac{(k+1)x}{2}\right) \\ & - HK_e\left(\frac{(k+1)x}{2}, \frac{(k+1)x}{2}\right) \\ & + HK_e\left(\frac{(k-1)x}{2}, \frac{(k-1)x}{2}\right) \\ & - HK_e\left(\frac{(k-1)x}{2}, -\frac{(k-1)x}{2}\right) \\ & + 2HK_e(0, kx) - 2HK_e(kx, 0) \\ & - HK_e(0, (k+1)x) - HK_e(0, (k-1)x)\|, \end{aligned} \quad (13)$$

and so

$$\begin{aligned} & \|f((k+1)x) - (k+1)^2 f(x)\| \\ & \leq \|f((k+1)x) + f((k-1)x) - 2f(kx) - 2f(x)\| \\ & + 2\|f(kx) - k^2 f(x)\| \\ & + \|f((k-1)x) - (k-1)^2 f(x)\| \\ & \leq \sum_{j=1}^k b_j \varphi_e((k+1-j)x, x) \\ & + 2 \sum_{j=1}^k b_j \|2HK_e((k+1-j)x, x) \\ & + HK_e\left(\frac{(k+2-j)x}{2}, -\frac{(k+2-j)x}{2}\right) \\ & - HK_e\left(\frac{(k+2-j)x}{2}, \frac{(k+2-j)x}{2}\right) \\ & + HK_e\left(\frac{(k+2-j)x}{2}, \frac{(k+2-j)x}{2}\right) \\ & - HK_e\left(\frac{(k+2-j)x}{2}, -\frac{(k+2-j)x}{2}\right)\|. \end{aligned}$$

$$\begin{aligned}
& + HK_e \left(\frac{(k-j)x}{2}, \frac{(k-j)x}{2} \right) \\
& - HK_e \left(\frac{(k-j)x}{2}, -\frac{(k-j)x}{2} \right) \\
& + 2HK_e(0, (k+1-j)x) \\
& - 2HK_e((k+1-j)x, 0) \\
& - HK_e(0, (k+2-j)x) \\
& - HK_e(0, (k-j)x) \Big\|,
\end{aligned} \tag{14}$$

and we are done for $n = 1$. Assuming that the lemma is true for some $n \geq 1$, we have

$$\begin{aligned}
& \|f(k^{n+1}x) - k^{2(n+1)}f(x)\| \\
& \leq \|f(k^{n+1}x) - k^2f(k^n x)\| \\
& \quad + k^2\|f(k^n x) - k^{2n}f(x)\| \\
& \leq \sum_{i=0}^n k^{2(n-i)} \sum_{j=1}^{k-1} b_j \varphi((k-j)k^i x, k^i x) + 2 \sum_{i=0}^n k^{2(n-i)} \\
& \quad \times \sum_{j=1}^{k-1} b_j \|2HK_e((k-j)k^i x, k^i x) \\
& \quad + HK_e\left(\frac{(k-j+1)k^i x}{2}, -\frac{(k-j+1)k^i x}{2}\right) \\
& \quad - HK_e\left(\frac{(k-j+1)k^i x}{2}, \frac{(k-j+1)k^i x}{2}\right) \\
& \quad + HK_e\left(\frac{(k-j-1)k^i x}{2}, \frac{(k-j-1)k^i x}{2}\right) \\
& \quad - HK_e\left(\frac{(k-j-1)k^i x}{2}, -\frac{(k-j-1)k^i x}{2}\right) \\
& \quad + 2HK_e(0, (k-j)k^i x) \\
& \quad - 2HK_e((k-j)k^i x, 0) \\
& \quad - HK_e(0, (k-j+1)k^i x) \\
& \quad - HK_e(0, (k-j-1)k^i x)\|,
\end{aligned} \tag{15}$$

verifying the lemma in general. $\square$

We come now to our first main proposition.

Proposition 2. Assume that $f : E_1 \rightarrow E_2$ satisfies

$$\begin{aligned}
& \|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \\
& \leq \varphi_e(x, y)
\end{aligned}$$

$$\begin{aligned}
& + 2\|2HK_e(x, y) + HK_e\left(\frac{x+y}{2}, -\frac{x+y}{2}\right) \\
& \quad - HK_e\left(\frac{x+y}{2}, \frac{x+y}{2}\right) \\
& \quad + HK_e\left(\frac{x-y}{2}, \frac{x-y}{2}\right) \\
& \quad - HK_e\left(\frac{x-y}{2}, -\frac{x-y}{2}\right) \\
& \quad + 2HK_e(0, x) - 2HK_e(x, 0) \\
& \quad - HK_e(0, x+y) - HK_e(0, x-y)\|.
\end{aligned} \tag{16}$$

When $\tilde{\psi}_k(x, y) < \infty$ for all $x, y \in E_1$, one further assumes that $f(0) = 0$. Then there exists a unique quadratic function $Q : E_1 \rightarrow E_2$ satisfying

$$\|Q(x) - f(x)\| \leq \frac{1}{k^2} \Psi_k(x, x) + \frac{2}{k^2} \Psi_{HK,k}(x, x) \tag{17}$$

provided that $\psi_k(x, y) < \infty$ and $\psi_{HK,k}(x, y) < \infty$ for all $x, y \in E_1$, or

$$\|Q(x) - f(x)\| \leq \frac{1}{k^2} \tilde{\Psi}_k(x, x) + \frac{2}{k^2} \tilde{\Psi}_{HK,k}(x, x) \tag{18}$$

provided that $\tilde{\psi}_k(x, y) < \infty$ and $\tilde{\psi}_{HK,k}(x, x) < \infty$ for all $x, y \in E_1$. Moreover, if f is measurable or $f(tx)$ is continuous in $t \in \mathbb{R}$ for every fixed $x \in E_1$, then the function Q satisfies

$$Q(tx) = t^2 Q(x) \quad (x \in E_1, t \in \mathbb{R}). \tag{19}$$

Proof. From Lemma 1, we have

$$\begin{aligned}
& \left\| \frac{1}{k^{2n}} f(k^n x) - f(x) \right\| \\
& \leq \frac{1}{k^2} \sum_{i=0}^{n-1} \frac{1}{k^{2i}} \sum_{j=1}^{k-1} b_j \varphi((k-j)k^i x, k^i x) + \frac{2}{k^2} \sum_{i=0}^{n-1} \frac{1}{k^{2i}} \\
& \quad \times \sum_{j=1}^{k-1} b_j \|2HK_e((k-j)k^i x, k^i x) \\
& \quad + HK_e\left(\frac{(k-j+1)k^i x}{2}, -\frac{(k-j+1)k^i x}{2}\right) \\
& \quad - HK_e\left(\frac{(k-j+1)k^i x}{2}, \frac{(k-j+1)k^i x}{2}\right) \\
& \quad + HK_e\left(\frac{(k-j-1)k^i x}{2}, \frac{(k-j-1)k^i x}{2}\right) \\
& \quad - HK_e\left(\frac{(k-j-1)k^i x}{2}, -\frac{(k-j-1)k^i x}{2}\right) \\
& \quad + 2HK_e(0, (k-j)k^i x)
\end{aligned}$$

$$\begin{aligned}
& -2HK_e((k-j)k^i x, 0) \\
& -HK_e(0, (k-j+1)k^i x) \\
& -HK_e(0, (k-j-1)k^i x) \Big\|.
\end{aligned} \tag{20}$$

For $m, n \in \mathbb{N}$ with $n > m$, from (20), we obtain

$$\begin{aligned}
& \left\| \frac{1}{k^{2n}} f(k^n x) - \frac{1}{k^{2m}} f(k^m x) \right\| \\
& = \frac{1}{k^{2m}} \left\| \frac{1}{k^{2(n-m)}} f(k^{n-m} k^m x) - f(k^m x) \right\| \\
& \leq \frac{1}{k^2} \sum_{j=1}^{k-1} b_j \sum_{i=m}^{n-1} \frac{1}{k^{2i}} \varphi((k-j)k^i x, k^i x) + \frac{2}{k^2} \sum_{j=1}^{k-1} b_j \\
& \quad \times \sum_{i=m}^{n-1} \frac{1}{k^{2i}} \left\| 2HK_e((k-j)k^i x, k^i x) \right. \\
& \quad + HK_e\left(\frac{(k-j+1)k^i x}{2}, -\frac{(k-j+1)k^i x}{2}\right) \\
& \quad - HK_e\left(\frac{(k-j+1)k^i x}{2}, \frac{(k-j+1)k^i x}{2}\right) \\
& \quad + HK_e\left(\frac{(k-j-1)k^i x}{2}, \frac{(k-j-1)k^i x}{2}\right) \\
& \quad - HK_e\left(\frac{(k-j-1)k^i x}{2}, -\frac{(k-j-1)k^i x}{2}\right) \\
& \quad + 2HK_e(0, (k-j)k^i x) \\
& \quad - 2HK_e((k-j)k^i x, 0) \\
& \quad - HK_e(0, (k-j+1)k^i x) \\
& \quad \left. - HK_e(0, (k-j-1)k^i x) \right\|.
\end{aligned} \tag{21}$$

In (21), since the last term can be made as small as we wish by choosing m sufficiently large, we deduce that $\{(1/k^{2n})f(k^n x)\}$ is a Cauchy sequence in E_2 . Since E_2 is a Banach space, the limit

$$Q(x) = \lim_{n \rightarrow \infty} \frac{1}{k^{2n}} f(k^n x) \quad (x \in E_1) \tag{22}$$

is well defined. Using Q and (20), the validity of the inequality (17) is easily verified. Next, replacing x, y in (16) by $k^n x, k^n y$, respectively, dividing the resulting inequality by k^{2n} , and using $\psi_k(x, y) < \infty, \psi_{HK,k}(x, y) < \infty$, we get

$$\begin{aligned}
& \left\| \frac{1}{k^{2n}} f(k^n(x+y)) + \frac{1}{k^{2n}} f(k^n(x-y)) \right. \\
& \quad \left. - \frac{2}{k^{2n}} f(k^n x) - \frac{2}{k^{2n}} f(k^n y) \right\|
\end{aligned}$$

$$\begin{aligned}
& \leq \frac{1}{k^{2n}} \varphi(k^n x, k^n y) + \frac{2}{k^{2n}} \\
& \quad \times \left\| 2HK_e(k^n x, k^n y) \right. \\
& \quad + HK_e\left(\frac{k^n(x+y)}{2}, -\frac{k^n(x+y)}{2}\right) \\
& \quad - HK_e\left(\frac{k^n(x+y)}{2}, \frac{k^n(x+y)}{2}\right) \\
& \quad + HK_e\left(\frac{k^n(x-y)}{2}, \frac{k^n(x-y)}{2}\right) \\
& \quad - HK_e\left(\frac{k^n(x-y)}{2}, -\frac{k^n(x-y)}{2}\right) \\
& \quad + 2HK_e(0, k^n x) - 2HK_e(k^n x, 0) \\
& \quad - HK_e(0, k^n(x+y)) \\
& \quad \left. - HK_e(0, k^n(x-y)) \right\| \rightarrow 0 \quad (n \rightarrow \infty),
\end{aligned} \tag{23}$$

which implies that Q is a quadratic function. If $Q_1 : E_1 \rightarrow E_2$ is another quadratic function satisfying the inequality (17), since Q and Q_1 are quadratic functions, we have

$$Q(k^n x) = k^{2n} Q(x), \tag{24}$$

$$Q_1(k^n x) = k^{2n} Q_1(x) \quad (n \in \mathbb{N}).$$

It thus follows from (24), (17), $\psi_k(x, y) < \infty$, and $\psi_{HK,k}(x, y) < \infty$ that

$$\begin{aligned}
& \|Q(x) - Q_1(x)\| \\
& = \frac{1}{k^{2n}} \|Q(k^n x) - Q_1(k^n x)\| \\
& \leq \frac{1}{k^{2n}} \frac{2}{k^2} \Psi_k(k^n x, k^n x) \\
& \quad + \frac{1}{k^{2n}} \frac{2}{k^2} \tilde{\Psi}_k(k^n x, k^n x) \rightarrow 0 \quad (n \rightarrow \infty);
\end{aligned} \tag{25}$$

that is, $Q(x) = Q_1(x)$.

The other provision $\tilde{\psi}_k(x, y) < \infty$ and $\tilde{\psi}_{HK,k}(x, x) < \infty$ is treated analogously by replacing x by x/k^n in (11) to obtain an upper bound for $\|f(x) - k^{2n} f(x/k^n)\|$, which is then used to show that $\{k^{2n} f(x/k^n)\}$ is a Cauchy sequence in E_2 so that the limit

$$Q(x) = \lim_{n \rightarrow \infty} k^{2n} f\left(\frac{x}{k^n}\right) \quad (x \in E_1) \tag{26}$$

is well defined and satisfies (18). Finally, the proof that (19) holds when f is measurable or $f(tx)$ is continuous is the same as that in [4]. $\square$

We now proceed to our second main proposition.

Proposition 3. If $f : E_1 \rightarrow E_2$ satisfies

$$\begin{aligned} & \|f(x+y) - f(x) - f(y)\| \\ & \leq \varphi_0(x, y) + 2 \left\| HK_o \left(\frac{x+y}{2}, \frac{x-y}{2} \right) \right. \\ & \quad - HK_o \left(-\frac{x+y}{2}, \frac{x-y}{2} \right) \\ & \quad - HK_o \left(\frac{x+y}{2}, \frac{x+y}{2} \right) \\ & \quad + HK_o \left(-\frac{x+y}{2}, \frac{x+y}{2} \right) \\ & \quad + HK_o(x+y, 0) - HK_o(x, 0) \\ & \quad \left. - HK_o(y, 0) \right\|, \end{aligned} \quad (27)$$

then there exists a unique additive function $A : E_1 \rightarrow E_2$ such that either

$$\|A(x) - f(x)\| \leq \frac{1}{k} \Phi'_k(x, x) + \Phi'_{HK,k}(x, x) \quad (28)$$

provided that for $x, y \in E_1$, one has $\sum_{i=0}^{\infty} (1/k^i) \varphi(k^i x, k^i y) < \infty$ and

$$\begin{aligned} & \sum_{i=0}^{\infty} \left\| HK_o(k^i(x+y), k^i(x-y)) \right. \\ & \quad - HK_o(-k^i(x+y), k^i(x-y)) \\ & \quad - HK_o(k^i(x+y), k^i(x+y)) \\ & \quad + HK_o(-k^i(x+y), k^i(x+y)) \\ & \quad + HK_o(k^i(x+y), 0) - HK_o(2k^i x, 0) \\ & \quad \left. - HK_o(2k^i y, 0) \right\| < \infty \end{aligned} \quad (29)$$

or

$$\|A(x) - f(x)\| \leq \frac{1}{k} \bar{\Phi}'_k(x, x) + \bar{\Phi}'_{HK,k}(x, x) \quad (30)$$

provided that for $x, y \in E_1$, one has $\sum_{i=0}^{\infty} k^i \varphi(x/k^i, y/k^i) < \infty$ and

$$\begin{aligned} & \sum_{i=0}^{\infty} k^{i+1} \left\| HK_o \left(\frac{x+y}{k^{i+1}}, \frac{x-y}{k^{i+1}} \right) \right. \\ & \quad - HK_o \left(-\frac{x+y}{k^{i+1}}, \frac{x-y}{k^{i+1}} \right) \\ & \quad - HK_o \left(\frac{x+y}{k^{i+1}}, \frac{x+y}{k^{i+1}} \right) \\ & \quad + HK_o \left(-\frac{x+y}{k^{i+1}}, \frac{x+y}{k^{i+1}} \right) \\ & \quad \left. + HK_o \left(\frac{2(x+y)}{k^{i+1}}, 0 \right) \right\| \end{aligned}$$

$$\begin{aligned} & - HK_o \left(\frac{2x}{k^{i+1}}, 0 \right) \\ & - HK_o \left(\frac{2y}{k^{i+1}}, 0 \right) \right\| < \infty. \end{aligned} \quad (31)$$

Moreover, if $f(tx)$ is continuous in $t \in \mathbb{R}$ for every fixed $x \in E_1$, then A is linear.

Proof. First, we show that

$$\begin{aligned} & \|f(kx) - kf(x)\| \\ & \leq \sum_{j=1}^{k-1} \varphi_0 \left(\left(\frac{j+1}{2} \right) x, \left(\frac{j-1}{2} \right) x \right) \\ & \quad + 2 \sum_{j=1}^{k-1} \left\| HK_o \left(\frac{(j+1)x}{2}, \frac{(j-1)x}{2} \right) \right. \\ & \quad - HK_o \left(-\frac{(j+1)x}{2}, \frac{(j-1)x}{2} \right) \\ & \quad - HK_o \left(\frac{(j+1)x}{2}, \frac{(j+1)x}{2} \right) \\ & \quad + HK_o \left(-\frac{(j+1)x}{2}, \frac{(j+1)x}{2} \right) \\ & \quad + HK_o((j+1)x, 0) - HK_o(jx, 0) \\ & \quad \left. - HK_o(x, 0) \right\|. \end{aligned} \quad (32)$$

Putting $y = x$ in (27), we obtain

$$\begin{aligned} & \|f(2x) - 2f(x)\| \leq \varphi_0(x, 0) \\ & \quad + 2 \left\| -HK_o(x, x) + HK_o(-x, x) \right. \\ & \quad \left. + HK_o(2x, 0) \right\|. \end{aligned} \quad (33)$$

This shows that (32) holds for $k = 2$. Assume that (32) is valid for $k = m$; that is,

$$\begin{aligned} & \|f(mx) - mf(x)\| \\ & \leq \sum_{j=1}^{m-1} \varphi_0 \left(\left(\frac{j+1}{2} \right) x, \left(\frac{j-1}{2} \right) x \right) \\ & \quad + 2 \sum_{j=1}^{m-1} \left\| HK_o \left(\frac{(j+1)x}{2}, \frac{(j-1)x}{2} \right) \right. \\ & \quad - HK_o \left(-\frac{(j+1)x}{2}, \frac{(j-1)x}{2} \right) \\ & \quad - HK_o \left(\frac{(j+1)x}{2}, \frac{(j+1)x}{2} \right) \\ & \quad + HK_o \left(-\frac{(j+1)x}{2}, \frac{(j+1)x}{2} \right) \\ & \quad \left. + HK_o \left(\frac{(j+1)x}{2}, \frac{(j+1)x}{2} \right) \right\| \end{aligned}$$

$$\begin{aligned}
& + HK_o((j+1)x) - HK_o(jx, 0) \\
& - HK_o(x, 0) \Big\| .
\end{aligned}
\tag{34}$$

Replacing y by mx in (27) and using (34), we have

$$\begin{aligned}
& \|f((m+1)x) - (m+1)f(x)\| \\
& \leq \|f((m+1)x) - f(mx) - f(x)\| \\
& \quad + \|f(mx) - mf(x)\| \\
& \leq \sum_{j=1}^m \varphi_o\left(\left(\frac{j+1}{2}\right)x, \left(\frac{j-1}{2}\right)x\right) \\
& \quad + 2 \sum_{j=1}^m \left\| HK_o\left(\frac{(j+1)x}{2}, \frac{(j-1)x}{2}\right) \right. \\
& \quad \left. - HK_o\left(-\frac{(j+1)x}{2}, \frac{(j-1)x}{2}\right) \right. \\
& \quad \left. - HK_o\left(\frac{(j+1)x}{2}, \frac{(j+1)x}{2}\right) \right. \\
& \quad \left. + HK_o\left(-\frac{(j+1)x}{2}, \frac{(j+1)x}{2}\right) \right. \\
& \quad \left. + HK_o((j+1)x) - HK_o(jx, 0) \right. \\
& \quad \left. - HK_o(x, 0) \right\| ,
\end{aligned}
\tag{35}$$

verifying the case $m+1$, and so the assertion (32) holds for all $n > 1$ and all $x \in E_1$.

Next, we use induction on n to show that

$$\begin{aligned}
& \|f(k^n x) - k^n f(x)\| \\
& \leq \sum_{i=1}^n k^{n-i} \sum_{j=1}^{k-1} \varphi_o\left(\left(\frac{j+1}{2}\right)k^{i-1}x, \left(\frac{j-1}{2}\right)k^{i-1}x\right) \\
& \quad + 2 \sum_{i=1}^n k^{n-i} \\
& \quad \times \sum_{j=1}^{k-1} \left\| HK_o\left(\left(\frac{j+1}{2}\right)k^{i-1}x, \left(\frac{j-1}{2}\right)k^{i-1}x\right) \right. \\
& \quad \left. - HK_o\left(-\left(\frac{j+1}{2}\right)k^{i-1}x, \left(\frac{j-1}{2}\right)k^{i-1}x\right) \right. \\
& \quad \left. - HK_o\left(\left(\frac{j+1}{2}\right)k^{i-1}x, \left(\frac{j+1}{2}\right)k^{i-1}x\right) \right. \\
& \quad \left. + HK_o\left(-\left(\frac{j+1}{2}\right)k^{i-1}x, \left(\frac{j+1}{2}\right)k^{i-1}x\right) \right.
\end{aligned}$$

$$\begin{aligned}
& + HK_o((j+1)k^{i-1}x, 0) \\
& - HK_o(jk^{i-1}x, 0) - HK_o(k^{i-1}x, 0) \Big\| .
\end{aligned}
\tag{36}$$

The case $n = 1$ follows from (32). Assuming that this holds for $n = m$, putting $n = 1$, and then substituting $k^m x$ for x in (36), we obtain

$$\begin{aligned}
& \|f(k^{m+1}x) - k^{m+1}f(x)\| \\
& \leq \|f(k \cdot k^m x) - kf(k^m x)\| \\
& \quad + k \|f(k^m x) - k^m f(x)\| \\
& \leq \sum_{i=0}^m k^{m-i} \sum_{j=1}^{k-1} \varphi_o\left(\left(\frac{j+1}{2}\right)k^i x, \left(\frac{j-1}{2}\right)k^i x\right) \\
& \quad + 2 \sum_{i=0}^m k^{m-i} \\
& \quad \times \sum_{j=1}^{k-1} \left\| HK_o\left(\left(\frac{j+1}{2}\right)k^i x, \left(\frac{j-1}{2}\right)k^i x\right) \right. \\
& \quad \left. - HK_o\left(-\left(\frac{j+1}{2}\right)k^i x, \left(\frac{j-1}{2}\right)k^i x\right) \right. \\
& \quad \left. - HK_o\left(\left(\frac{j+1}{2}\right)k^i x, \left(\frac{j+1}{2}\right)k^i x\right) \right. \\
& \quad \left. + HK_o\left(-\left(\frac{j+1}{2}\right)k^i x, \left(\frac{j+1}{2}\right)k^i x\right) \right. \\
& \quad \left. + HK_o((j+1)k^i x, 0) \right. \\
& \quad \left. - HK_o(jk^i x, 0) - HK_o(k^i x, 0) \right\| ,
\end{aligned}
\tag{37}$$

verifying (36) generally. Rewriting (36), we get

$$\begin{aligned}
& \left\| \frac{1}{k^n} f(k^n x) - f(x) \right\| \\
& \leq \sum_{i=1}^n \frac{1}{k^i} \sum_{j=1}^{k-1} \varphi_o\left(\left(\frac{j+1}{2}\right)k^{i-1}x, \left(\frac{j-1}{2}\right)k^{i-1}x\right) \\
& \quad + 2 \sum_{i=1}^n \frac{1}{k^i} \\
& \quad \times \sum_{j=1}^{k-1} \left\| HK_o\left(\left(\frac{j+1}{2}\right)k^{i-1}x, \left(\frac{j-1}{2}\right)k^{i-1}x\right) \right. \\
& \quad \left. - HK_o\left(-\left(\frac{j+1}{2}\right)k^{i-1}x, \left(\frac{j-1}{2}\right)k^{i-1}x\right) \right. \\
& \quad \left. - HK_o\left(\left(\frac{j+1}{2}\right)k^{i-1}x, \left(\frac{j+1}{2}\right)k^{i-1}x\right) \right.
\end{aligned}$$

$$\begin{aligned}
& + HK_o \left(- \left(\frac{j+1}{2} \right) k^{i-1} x, \left(\frac{j+1}{2} \right) k^{i-1} x \right) \\
& + HK_o \left((j+1) k^{i-1} x, 0 \right) \\
& - HK_o \left(j k^{i-1} x, 0 \right) - HK_o \left(k^{i-1} x, 0 \right) \Big\|.
\end{aligned} \tag{38}$$

Using (38), $\sum_{i=0}^{\infty} k^i \varphi(x/k^i, y/k^i) < \infty$, and the inequality

$$\begin{aligned}
& \sum_{i=0}^{\infty} \left\| HK_o \left(k^i (x+y), k^i (x-y) \right) \right. \\
& \quad - HK_o \left(-k^i (x+y), k^i (x-y) \right) \\
& \quad - HK_o \left(k^i (x+y), k^i (x+y) \right) \\
& \quad + HK_o \left(-k^i (x+y), k^i (x+y) \right) \\
& \quad + HK_o \left(k^i (x+y), 0 \right) - HK_o \left(2k^i x, 0 \right) \\
& \quad \left. - HK_o \left(2k^i y, 0 \right) \right\| < \infty
\end{aligned} \tag{39}$$

and proceeding as in the proof of Proposition 2, we deduce that $\{k^{-n} f(k^n x)\}$ is a Cauchy sequence. Thus, the limit

$$A(x) = \lim_{n \rightarrow \infty} \frac{1}{k^n} f(k^n x) \quad (x \in E_1) \tag{40}$$

is well defined. To show that A is additive, we substitute $k^n x$ and $k^n y$ for x and y in (27), respectively, to get

$$\begin{aligned}
& \|f(k^n x + k^n y) - f(k^n x) - f(k^n y)\| \\
& \leq \varphi_o(k^n x, k^n y) \\
& + 2 \left\| HK_o \left(\frac{k^n (x+y)}{2}, \frac{k^n (x-y)}{2} \right) \right. \\
& \quad - HK_o \left(-\frac{k^n (x+y)}{2}, \frac{k^n (x-y)}{2} \right) \\
& \quad - HK_o \left(\frac{k^n (x+y)}{2}, \frac{k^n (x+y)}{2} \right) \\
& \quad + HK_o \left(-\frac{k^n (x+y)}{2}, \frac{k^n (x+y)}{2} \right) \\
& \quad + HK_o(k^n x + k^n y, 0) - HK_o(k^n x, 0) \\
& \quad \left. - HK_o(k^n y, 0) \right\|.
\end{aligned} \tag{41}$$

From $\sum_{i=0}^{\infty} (1/k^i) \varphi(k^i x, k^i y) < \infty$ and

$$\begin{aligned}
& \sum_{i=0}^{\infty} \left\| HK_o \left(k^i (x+y), k^i (x-y) \right) \right. \\
& \quad - HK_o \left(-k^i (x+y), k^i (x-y) \right) \\
& \quad - HK_o \left(k^i (x+y), k^i (x+y) \right) \\
& \quad + HK_o \left(-k^i (x+y), k^i (x+y) \right) \\
& \quad + HK_o \left(k^i (x+y), 0 \right) - HK_o \left(2k^i x, 0 \right) \\
& \quad \left. - HK_o \left(2k^i y, 0 \right) \right\| < \infty,
\end{aligned} \tag{42}$$

it follows that

$$\begin{aligned}
& \frac{1}{k^n} \left\{ \varphi_o(k^n x, k^n y) \right. \\
& \quad + 2 \left\| HK_o \left(\frac{k^n (x+y)}{2}, \frac{k^n (x-y)}{2} \right) \right. \\
& \quad \quad - HK_o \left(-\frac{k^n (x+y)}{2}, \frac{k^n (x-y)}{2} \right) \\
& \quad \quad - HK_o \left(\frac{k^n (x+y)}{2}, \frac{k^n (x+y)}{2} \right) \\
& \quad \quad + HK_o \left(-\frac{k^n (x+y)}{2}, \frac{k^n (x+y)}{2} \right) \\
& \quad \quad + HK_o(k^n x + k^n y, 0) - HK_o(k^n x, 0) \\
& \quad \quad \left. \left. - HK_o(k^n y, 0) \right\| \right\} \rightarrow 0 \quad (n \rightarrow \infty).
\end{aligned} \tag{43}$$

Using this last inequality, dividing both sides of (41) by k^n and letting $n \rightarrow \infty$, we obtain $A(x+y) = A(x) + A(y)$.

According to (38), we have

$$\begin{aligned}
& \|f(x) - A(x)\| \\
& \leq \sum_{i=1}^{\infty} \frac{1}{k^i} \sum_{j=1}^{k-1} \varphi_o \left(\left(\frac{j+1}{2} \right) k^{i-1} x, \left(\frac{j-1}{2} \right) k^{i-1} x \right) \\
& \quad + 2 \sum_{i=1}^{\infty} \frac{1}{k^i} \\
& \quad \times \sum_{j=1}^{k-1} \left\| HK_o \left(\left(\frac{j+1}{2} \right) k^{i-1} x, \left(\frac{j-1}{2} \right) k^{i-1} x \right) \right. \\
& \quad \quad - HK_o \left(-\left(\frac{j+1}{2} \right) k^{i-1} x, \left(\frac{j-1}{2} \right) k^{i-1} x \right) \\
& \quad \quad \left. - HK_o \left(\left(\frac{j+1}{2} \right) k^{i-1} x, \left(\frac{j+1}{2} \right) k^{i-1} x \right) \right.
\end{aligned}$$

$$\begin{aligned}
& + HK_o \left(- \left(\frac{j+1}{2} \right) k^{i-1} x, \left(\frac{j+1}{2} \right) k^{i-1} x \right) \\
& + HK_o \left((j+1) k^{i-1} x, 0 \right) - HK_o \left(j k^{i-1} x, 0 \right) \\
& - HK_o \left(k^{i-1} x, 0 \right) \Big\|.
\end{aligned} \tag{44}$$

To show that A is unique, let $A_1 : E_1 \rightarrow E_2$ be another additive mapping satisfying (44). Thus,

$$\begin{aligned}
& \|A(x) - A_1(x)\| \\
& \leq \frac{1}{k^n} \|A(k^n x) - f(k^n x)\| \\
& \quad + \frac{1}{k^n} \|f(k^n x) - A_1(k^n x)\| \\
& \leq 2 \sum_{i=n+1}^{\infty} \frac{1}{k^i} \sum_{j=1}^{k-1} \varphi_o \left(\left(\frac{j+1}{2} \right) k^{i-1} x, \left(\frac{j-1}{2} \right) k^{i-1} x \right) \\
& \quad + 4 \sum_{i=n+1}^{\infty} \frac{1}{k^i} \\
& \quad \times \sum_{j=1}^{k-1} \left\| HK_o \left(\left(\frac{j+1}{2} \right) k^{i-1} x, \left(\frac{j-1}{2} \right) k^{i-1} x \right) \right. \\
& \quad - HK_o \left(- \left(\frac{j+1}{2} \right) k^{i-1} x, \left(\frac{j-1}{2} \right) k^{i-1} x \right) \\
& \quad - HK_o \left(\left(\frac{j+1}{2} \right) k^{i-1} x, \left(\frac{j+1}{2} \right) k^{i-1} x \right) \\
& \quad + HK_o \left(- \left(\frac{j+1}{2} \right) k^{i-1} x, \left(\frac{j+1}{2} \right) k^{i-1} x \right) \\
& \quad + HK_o \left((j+1) k^{i-1} x, 0 \right) - HK_o \left(j k^{i-1} x, 0 \right) \\
& \quad \left. - HK_o \left(k^{i-1} x, 0 \right) \right\|.
\end{aligned} \tag{45}$$

Using $\sum_{i=0}^{\infty} (1/k^i) \varphi(k^i x, k^i y) < \infty$ and

$$\begin{aligned}
& \sum_{i=0}^{\infty} \left\| HK_o \left(k^i (x+y), k^i (x-y) \right) \right. \\
& \quad - HK_o \left(-k^i (x+y), k^i (x-y) \right) \\
& \quad - HK_o \left(k^i (x+y), k^i (x+y) \right) \\
& \quad + HK_o \left(-k^i (x+y), k^i (x+y) \right) \\
& \quad + HK_o \left(k^i (x+y), 0 \right) - HK_o \left(2k^i x, 0 \right) \\
& \quad \left. - HK_o \left(2k^i y, 0 \right) \right\| < \infty,
\end{aligned} \tag{46}$$

we see that the last term becomes arbitrarily small for sufficiently large n , implying $A = A_1$. Finally, the same proof as in

[9] shows that A is linear if E_1 is a Banach space and $f(tx)$ is continuous in t for every fixed $x \in E_1$.

The other provision $\sum_{i=0}^{\infty} k^i \varphi(x/k^i, y/k^i) < \infty$ and

$$\begin{aligned}
& \sum_{i=0}^{\infty} k^{i+1} \left\| HK_o \left(\frac{x+y}{k^{i+1}}, \frac{x-y}{k^{i+1}} \right) \right. \\
& \quad - HK_o \left(-\frac{x+y}{k^{i+1}}, \frac{x-y}{k^{i+1}} \right) \\
& \quad - HK_o \left(\frac{x+y}{k^{i+1}}, \frac{x+y}{k^{i+1}} \right) \\
& \quad + HK_o \left(-\frac{x+y}{k^{i+1}}, \frac{x+y}{k^{i+1}} \right) \\
& \quad + HK_o \left(\frac{2(x+y)}{k^{i+1}}, 0 \right) \\
& \quad - HK_o \left(\frac{2x}{k^{i+1}}, 0 \right) \\
& \quad \left. - HK_o \left(\frac{2y}{k^{i+1}}, 0 \right) \right\| < \infty
\end{aligned} \tag{47}$$

is treated analogously by replacing x by x/k^n in (36) to obtain

$$\begin{aligned}
& \left\| f(x) - k^n f \left(\frac{x}{k^n} \right) \right\| \\
& \leq \frac{1}{k} \sum_{i=0}^{n-1} k^{i+1} \sum_{j=1}^{k-1} \varphi_o \left(\frac{(j+1)x}{2k^{i+1}}, \left(\frac{j-1}{2} \right) \frac{x}{k^{i+1}} \right) \\
& \quad + \frac{2}{k} \sum_{i=0}^{n-1} k^{i+1} \\
& \quad \times \sum_{j=1}^{k-1} \left\| HK_o \left(\frac{(j+1)x}{2k^{i+1}}, \frac{(j-1)x}{2k^{i+1}} \right) \right. \\
& \quad - HK_o \left(-\frac{(j+1)x}{2k^{i+1}}, \frac{(j-1)x}{2k^{i+1}} \right) \\
& \quad - HK_o \left(\frac{(j+1)x}{2k^{i+1}}, \frac{(j+1)x}{2k^{i+1}} \right) \\
& \quad + HK_o \left(-\frac{(j+1)x}{2k^{i+1}}, \frac{(j+1)x}{2k^{i+1}} \right) \\
& \quad + HK_o \left((j+1) \frac{x}{k^{i+1}}, 0 \right) \\
& \quad \left. - HK_o \left(j \frac{x}{k^{i+1}}, 0 \right) - HK_o \left(\frac{x}{k^{i+1}}, 0 \right) \right\|,
\end{aligned} \tag{48}$$

which is then used to show that $\{k^n f(x/k^n)\}$ is a Cauchy sequence in E_2 for all $x \in E_1$, so that the limit

$$A(x) = \lim_{n \rightarrow \infty} k^n f \left(\frac{x}{k^n} \right) \quad (x \in E_1) \tag{49}$$

is a well-defined additive function. Using

$$\begin{aligned}
 & \|A(x) - f(x)\| \\
 & \leq \frac{1}{k} \sum_{i=0}^{\infty} k^{i+1} \sum_{j=1}^{k-1} \varphi_o \left(\frac{(j+1)x}{2k^{i+1}}, \frac{(j-1)x}{2k^{i+1}} \right) \\
 & \quad + \frac{2}{k} \sum_{i=0}^{\infty} k^{i+1} \\
 & \quad \times \sum_{j=1}^{k-1} \left\| HK_o \left(\frac{(j+1)x}{2k^{i+1}}, \frac{(j-1)x}{2k^{i+1}} \right) \right. \\
 & \quad \left. - HK_o \left(-\frac{(j+1)x}{2k^{i+1}}, \frac{(j-1)x}{2k^{i+1}} \right) \right. \\
 & \quad \left. - HK_o \left(\frac{(j+1)x}{2k^{i+1}}, \frac{(j+1)x}{2k^{i+1}} \right) \right. \\
 & \quad \left. + HK_o \left(-\frac{(j+1)x}{2k^{i+1}}, \frac{(j+1)x}{2k^{i+1}} \right) \right. \\
 & \quad \left. + HK_o \left(\frac{(j+1)x}{k^{i+1}}, 0 \right) - HK_o \left(\frac{jx}{k^{i+1}}, 0 \right) \right. \\
 & \quad \left. - HK_o \left(\frac{x}{k^{i+1}}, 0 \right) \right\|, \quad (50)
 \end{aligned}$$

we can show that A is unique. Finally, that A is linear if E_1 is a Banach space and $f(tx)$ is continuous in t for every fixed $x \in E_1$ is proved by the same technique as that in [9]. $\square$

4. Main Result

Theorem 4. Adopting the notation of Section 2, if the functions $\mathcal{F}, \mathcal{G}, \mathcal{H}, \mathcal{K}, \mathcal{L}, \mathcal{M} : E_1 \rightarrow E_2$ satisfy

$$\begin{aligned}
 & \|\mathcal{F}(x+y) + \mathcal{G}(x-y) - 2\mathcal{H}(x)\mathcal{K}(y) \\
 & - \mathcal{L}(x) - \mathcal{M}(y)\| \leq \varphi(x, y) \quad (x, y \in E_1), \quad (51)
 \end{aligned}$$

then there exist a quadratic function $Q : E_1 \rightarrow E_2$ and two additive functions $A_1, A_2 : E_1 \rightarrow E_2$ such that, for all $x \in E_1$, one has

$$\begin{aligned}
 & \|\mathcal{F}(x) - Q(x) - A_1(x) - A_2(x) - \mathcal{F}(0)\| \\
 & \leq \frac{1}{2k^2} \widehat{\Phi}_k(x, x) + \frac{1}{2k} \widehat{\Phi}'_k(x, x) + \frac{1}{2k} \widehat{\Phi}''_k(x, x) \\
 & \quad + \frac{1}{2k^2} \widehat{\Phi}_{HK,k}(x, x) + \frac{1}{2k} \widehat{\Phi}'_{HK,k}(x, x) \\
 & \quad + \frac{1}{2k} \widehat{\Phi}''_{HK,k}(x, x) + \|HK_o(x, 0)\|
 \end{aligned}$$

$$\begin{aligned}
 & + \|HK_o(0, x)\| + \|HK_e(x, 0)\| \\
 & + 3\varphi\left(\frac{x}{2}, \frac{x}{2}\right) + \frac{5}{2}\varphi(x, 0) + \frac{11}{2}\varphi(0, 0), \\
 & \|\mathcal{G}(x) - Q(x) - A_1(x) - A_2(x) - \mathcal{G}(0)\| \\
 & \leq \frac{1}{2k^2} \widehat{\Phi}_k(x, x) + \frac{1}{2k} \widehat{\Phi}'_k(x, x) + \frac{1}{2k} \widehat{\Phi}''_k(x, x) \\
 & \quad + \frac{1}{2k^2} \widehat{\Phi}_{HK,k}(x, x) + \frac{1}{2k} \widehat{\Phi}'_{HK,k}(x, x) \\
 & \quad + \frac{1}{2k} \widehat{\Phi}''_{HK,k}(x, x) + \|HK_o(x, 0)\| \\
 & \quad + \|HK_o(0, x)\| + \|HK_e(x, 0)\| + \varphi\left(\frac{x}{2}, \frac{x}{2}\right) \\
 & \quad + \frac{5}{2}\varphi(x, 0) + \frac{7}{2}\varphi(0, 0), \\
 & \|\mathcal{L}(x) - 2Q(x) - 2A_1(x) - \mathcal{L}(0)\| \\
 & \leq \frac{1}{k^2} \widehat{\Phi}_k(x, x) + \frac{1}{k} \widehat{\Phi}'_k(x, x) + 2\varphi(x, 0) + 2\varphi(0, 0) \\
 & \quad + \frac{1}{k^2} \widehat{\Phi}_{HK,k}(x, x) + \frac{1}{k} \widehat{\Phi}'_{HK,k}(x, x), \\
 & \|\mathcal{M}(x) - 2Q(x) - 2A_2(x) - \mathcal{M}(0)\| \\
 & \leq \frac{1}{k^2} \widehat{\Phi}_k(x, x) + \frac{1}{k} \widehat{\Phi}''_k(x, x) \\
 & \quad + \frac{1}{k^2} \widehat{\Phi}_{HK,k}(x, x) + \frac{1}{k} \widehat{\Phi}''_{HK,k}(x, x), \quad (52)
 \end{aligned}$$

where

$$\begin{aligned}
 & \widehat{\Phi}_k(x, y) \\
 & = \begin{cases} \Phi_k(x, y) & \text{if the first condition in} \\ & \text{definition (iii) is true,} \\ \widetilde{\Phi}_k(x, y) & \text{if the second condition in} \\ & \text{definition (iii) is true,} \end{cases} \\
 & \widehat{\Phi}'_k(x, y) \\
 & = \begin{cases} \Phi'_k(x, y) & \text{if the first condition in} \\ & \text{definition (iii) is true,} \\ \widetilde{\Phi}'_k(x, y) & \text{if the second condition in} \\ & \text{definition (iii) is true,} \end{cases} \\
 & \widehat{\Phi}''_k(x, y) \\
 & = \begin{cases} \Phi''_k(x, y) & \text{if the first condition in} \\ & \text{definition (iii) is true,} \\ \widetilde{\Phi}''_k(x, y) & \text{if the second condition in} \\ & \text{definition (iii) is true,} \end{cases}
 \end{aligned}$$

$$\begin{aligned}
& \widehat{\Phi}_{HK,k}(x, y) \\
&= \begin{cases} \Phi_{HK,k}(x, y) & \text{if the first condition in} \\ & \text{definition (iii) is true,} \\ \widetilde{\Phi}_{HK,k}(x, y) & \text{if the second condition in} \\ & \text{definition (iii) is true,} \end{cases} \\
& \widehat{\Phi}'_{HK,k}(x, y) \\
&= \begin{cases} \Phi'_{HK,k}(x, y) & \text{if the first condition in} \\ & \text{definition (iii) is true,} \\ \widetilde{\Phi}'_{HK,k}(x, y) & \text{if the second condition in} \\ & \text{definition (iii) is true,} \end{cases} \\
& \widehat{\Phi}''_{HK,k}(x, y) \\
&= \begin{cases} \Phi''_{HK,k}(x, y) & \text{if the first condition in} \\ & \text{definition (iii) is true,} \\ \widetilde{\Phi}''_{HK,k}(x, y) & \text{if the second condition in} \\ & \text{definition (iii) is true.} \end{cases}
\end{aligned} \tag{53}$$

Moreover, if $\mathcal{L}(tx)$ and $\mathcal{M}(tx)$ are continuous in $t \in \mathbb{R}$ for each $x \in E_1$, then the two additive functions A_1, A_2 are linear, while the quadratic function Q satisfies

$$Q(tx) = t^2 Q(x). \tag{54}$$

Proof. Putting $x = y = 0$ in (51), we obtain

$$\|\mathcal{F}(0) + \mathcal{G}(0) - 2\mathcal{H}(0)\mathcal{K}(0) - \mathcal{L}(0) - \mathcal{M}(0)\| \leq \varphi(0, 0). \tag{55}$$

For $x, y \in E_1$, setting

$$\begin{aligned}
F(x) &:= \mathcal{F}(x) - \mathcal{F}(0), & G(x) &:= \mathcal{G}(x) - \mathcal{G}(0), \\
L(x) &:= \mathcal{L}(x) - \mathcal{L}(0), & M(x) &:= \mathcal{M}(x) - \mathcal{M}(0), \\
HK(x, y) &:= \mathcal{H}(x)\mathcal{K}(y) - \mathcal{H}(0)\mathcal{K}(0),
\end{aligned} \tag{56}$$

we get $F(0) = G(0) = HK(0, 0) = L(0) = M(0) = 0$. Subtracting (51) and (55), we have

$$\begin{aligned}
& \|F(x+y) + G(x-y) - 2HK(x, y) - L(x) - M(y)\| \\
& \leq \varphi(x, y) + \varphi(0, 0).
\end{aligned} \tag{57}$$

Substituting $-x$ for x and $-y$ for y in (57), we get

$$\begin{aligned}
& \|F(-(x+y)) + G(-(x-y)) \\
& - 2HK(-x, -y) - L(-x) - M(-y)\| \\
& \leq \varphi(x, y) + \varphi(0, 0).
\end{aligned} \tag{58}$$

Combining (57) and (58), we obtain

$$\begin{aligned}
& \|\{F(x+y) + F(-(x+y))\} \\
& + \{G(x-y) + G(-(x-y))\} \\
& - 2\{HK(x, y) + HK(-x, -y)\} \\
& - \{L(x) + L(-x)\} - \{M(y) + M(-y)\}\| \\
& \leq 2\varphi(x, y) + 2\varphi(0, 0).
\end{aligned} \tag{59}$$

Let

$$\begin{aligned}
F_e(x) &:= \frac{F(x) + F(-x)}{2}, & G_e(x) &:= \frac{G(x) + G(-x)}{2}, \\
L_e(x) &:= \frac{L(x) + L(-x)}{2}, & M_e(x) &:= \frac{M(x) + M(-x)}{2}, \\
HK_e(x, y) &:= \frac{HK(x, y) + HK(-x, -y)}{2} \quad (x, y \in E_1).
\end{aligned} \tag{60}$$

Note that F_e, G_e, L_e, M_e are the even parts of the functions F, G, L, M , respectively, and that

$$\begin{aligned}
HK_e(-x, -y) &= HK_e(x, y), \\
HK_e(x, -y) &= HK_e(-x, y), \\
HK_e(0, -x) &= HK_e(0, x), \\
HK_e(-x, 0) &= HK_e(x, 0).
\end{aligned} \tag{61}$$

From (59), we obtain

$$\begin{aligned}
& \|F_e(x+y) + G_e(x-y) - 2HK_e(x, y) - L_e(x) - M_e(y)\| \\
& \leq \varphi(x, y) + \varphi(0, 0).
\end{aligned} \tag{62}$$

Substituting $y = 0$ in (62), we get

$$\|F_e(x) + G_e(x) - 2HK_e(x, 0) - L_e(x)\| \leq \varphi(x, 0) + \varphi(0, 0). \tag{63}$$

Putting $x = 0$ in (62) and replacing y by x , we get

$$\begin{aligned}
& \|F_e(x) + G_e(x) - 2HK_e(0, x) - M_e(x)\| \\
& \leq \varphi(x, 0) + \varphi(0, 0).
\end{aligned} \tag{64}$$

Putting $y = x$ in (62), we have

$$\begin{aligned}
& \|F_e(2x) - 2HK_e(x, x) - L_e(x) - M_e(x)\| \\
& \leq \varphi(x, x) + \varphi(0, 0).
\end{aligned} \tag{65}$$

Substituting $y = -x$ in (62), we get

$$\begin{aligned}
& \|G_e(2x) - 2HK_e(x, -x) - L_e(x) - M_e(x)\| \\
& \leq \varphi(x, x) + \varphi(0, 0).
\end{aligned} \tag{66}$$

From (64) and (63), we have

$$\begin{aligned} & \|L_e(x) - M_e(x) - 2[HK_e(0, x) - HK_e(x, 0)]\| \\ & \leq 2\varphi(x, 0) + 2\varphi(0, 0). \end{aligned} \quad (67)$$

From (65) and (66) and replacing x by $x/2$, we obtain

$$\begin{aligned} & \left\| F_e(x) - G_e(x) - 2 \left\{ HK_e\left(\frac{x}{2}, \frac{x}{2}\right) - HK_e\left(\frac{x}{2}, \frac{-x}{2}\right) \right\} \right\| \\ & \leq 2\varphi\left(\frac{x}{2}, \frac{x}{2}\right) + 2\varphi(0, 0). \end{aligned} \quad (68)$$

From (62), (67), and (68), we get

$$\begin{aligned} & \left\| G_e(x+y) + G_e(x-y) - M_e(x) - M_e(y) \right. \\ & \quad - 2 \left\{ HK_e(x, y) + HK_e\left(\frac{x+y}{2}, -\frac{x+y}{2}\right) \right. \\ & \quad \quad \left. - HK_e\left(\frac{x+y}{2}, \frac{x+y}{2}\right) \right. \\ & \quad \quad \left. + HK_e(0, x) - HK_e(x, 0) \right\} \left\| \right. \\ & \leq \varphi(x, y) + 2\varphi(x, 0) \\ & \quad + 2\varphi\left(\frac{x+y}{2}, \frac{x+y}{2}\right) + 5\varphi(0, 0). \end{aligned} \quad (69)$$

Substituting $y = 0$ in (69), we get

$$\begin{aligned} & \left\| 2G_e(x) - M_e(x) \right. \\ & \quad \left. - 2 \left\{ HK_e\left(\frac{x}{2}, -\frac{x}{2}\right) - HK_e\left(\frac{x}{2}, \frac{x}{2}\right) + HK_e(0, x) \right\} \right\| \\ & \leq 3\varphi(x, 0) + 2\varphi\left(\frac{x}{2}, \frac{x}{2}\right) + 5\varphi(0, 0). \end{aligned} \quad (70)$$

From (69) and (70), we obtain

$$\begin{aligned} & \left\| M_e(x+y) + M_e(x-y) - 2M_e(x) - 2M_e(y) \right. \\ & \quad - 2 \left\{ 2HK_e(x, y) + HK_e\left(\frac{x+y}{2}, -\frac{x+y}{2}\right) \right. \\ & \quad \quad \left. - HK_e\left(\frac{x+y}{2}, \frac{x+y}{2}\right) + HK_e\left(\frac{x-y}{2}, \frac{x-y}{2}\right) \right. \\ & \quad \quad \left. - HK_e\left(\frac{x-y}{2}, -\frac{x-y}{2}\right) + 2HK_e(0, x) \right. \\ & \quad \quad \left. - 2HK_e(x, 0) - HK_e(0, x+y) \right. \\ & \quad \quad \left. - HK_e(0, x-y) \right\} \left\| \right. \end{aligned}$$

$$\begin{aligned} & \leq 2\varphi(x, y) + 6\varphi\left(\frac{x+y}{2}, \frac{x+y}{2}\right) \\ & \quad + 2\varphi\left(\frac{x-y}{2}, \frac{x-y}{2}\right) + 3\varphi(x+y, 0) \\ & \quad + 3\varphi(x-y, 0) + 4\varphi(x, 0) \\ & \quad + 20\varphi(0, 0) \\ & = \varphi_e(x, y), \end{aligned} \quad (71)$$

and so

$$\begin{aligned} & \left\| M_e(x+y) + M_e(x-y) - 2M_e(x) - 2M_e(y) \right\| \\ & \leq \varphi_e(x, y) \\ & \quad + 2 \left\| \left\{ 2HK_e(x, y) + HK_e\left(\frac{x+y}{2}, -\frac{x+y}{2}\right) \right. \right. \\ & \quad \quad \left. - HK_e\left(\frac{x+y}{2}, \frac{x+y}{2}\right) + HK_e\left(\frac{x-y}{2}, \frac{x-y}{2}\right) \right. \\ & \quad \quad \left. - HK_e\left(\frac{x-y}{2}, -\frac{x-y}{2}\right) + 2HK_e(0, x) \right. \\ & \quad \quad \left. - 2HK_e(x, 0) - HK_e(0, x+y) \right. \\ & \quad \quad \left. \left. - HK_e(0, x-y) \right\} \right\|. \end{aligned} \quad (72)$$

By Proposition 2, there is a unique quadratic function $Q : E_1 \rightarrow E_2$ such that

$$\|M_e(x) - 2Q(x)\| \leq \frac{1}{k^2} \widehat{\Phi}_k(x, x) + \frac{1}{k^2} \widehat{\Phi}_{HK,k}(x, x). \quad (73)$$

If $\mathcal{M}(tx)$ is continuous in $t \in \mathbb{R}$ for each $x \in E_1$, then the quadratic function Q satisfies $Q(tx) = t^2Q(x)$. From (68), (70), and (73), we have

$$\begin{aligned} & \|F_e(x) - Q(x)\| \\ & \leq \|F_e(x) - G_e(x)\| \\ & \quad + \left\| G_e(x) - \frac{1}{2}M_e(x) - HK_e(x, 0) \right\| \\ & \quad + \left\| \frac{1}{2}M_e(x) - Q(x) \right\| \\ & \leq \|HK_e(x, 0)\| + 3\varphi\left(\frac{x}{2}, \frac{x}{2}\right) + \frac{3}{2}\varphi(x, 0) + \frac{9}{2}\varphi(0, 0) \\ & \quad + \frac{1}{2k^2} \widehat{\Phi}_k(x, x) + \frac{1}{2k^2} \widehat{\Phi}_{HK,k}(x, x). \end{aligned} \quad (74)$$

From (70) and (73), we get

$$\begin{aligned}
 & \|G_e(x) - Q(x)\| \\
 & \leq \left\| G_e(x) - \frac{1}{2}M_e(x) - HK_e(x, 0) \right\| \\
 & \quad + \left\| \frac{1}{2}M_e(x) - Q(x) \right\| \\
 & \leq \|HK_e(x, 0)\| + \varphi\left(\frac{x}{2}, \frac{x}{2}\right) + \frac{3}{2}\varphi(x, 0) \\
 & \quad + \frac{5}{2}\varphi(0, 0) + \frac{1}{2k^2}\widehat{\Phi}_k(x, x) + \frac{1}{2k^2}\widehat{\Phi}_{HK,k}(x, x).
 \end{aligned} \tag{75}$$

From (67) and (73), we obtain

$$\begin{aligned}
 & \|L_e(x) - 2Q(x)\| \\
 & \leq \|L_e(x) - M_e(x)\| + \|M_e(x) - 2Q(x)\| \\
 & \leq 2\varphi(x, 0) + 2\varphi(0, 0) \\
 & \quad + \frac{1}{2k^2}\widehat{\Phi}_k(x, x) + \frac{1}{2k^2}\widehat{\Phi}_{HK,k}(x, x).
 \end{aligned} \tag{76}$$

Next, we consider the odd part of functions. Subtracting (57) and (58), we obtain

$$\begin{aligned}
 & \left\| \{F(x+y) - F(-(x+y))\} \right. \\
 & \quad + \{G(x-y) - G(-(x-y))\} \\
 & \quad - 2\{HK(x, y) - HK(-x, -y)\} \\
 & \quad \left. - \{L(x) - L(-x)\} - \{M(y) - M(-y)\} \right\| \\
 & \leq 2\varphi(x, y) + 2\varphi(0, 0).
 \end{aligned} \tag{77}$$

For $x, y \in E_1$, let

$$\begin{aligned}
 F_o(x) &:= \frac{F(x) - F(-x)}{2}, & G_o(x) &:= \frac{G(x) - G(-x)}{2}, \\
 L_o(x) &:= \frac{L(x) - L(-x)}{2}, & M_o(x) &:= \frac{M(x) - M(-x)}{2}, \\
 HK_o(x, y) &:= \frac{HK(x, y) - HK(-x, -y)}{2}.
 \end{aligned} \tag{78}$$

Clearly, F_o, G_o, L_o, M_o are the odd parts of F, G, L, M , respectively. Note that

$$\begin{aligned}
 HK_o(-x, -y) &= -HK_o(x, y), \\
 HK_o(x, -y) &= -HK_o(-x, y), \\
 HK_o(0, -x) &= -HK_o(0, x), \\
 HK_o(-x, 0) &= -HK_o(x, 0).
 \end{aligned} \tag{79}$$

From (77), we get

$$\begin{aligned}
 & \|F_o(x+y) + G_o(x-y) - 2HK_o(x, y) - L_o(x) - M_o(y)\| \\
 & \leq \varphi(x, y) + \varphi(0, 0).
 \end{aligned} \tag{80}$$

Substituting $y = 0$ in (80), we get

$$\begin{aligned}
 & \|F_o(x) + G_o(x) - 2HK_o(x, 0) - L_o(x)\| \\
 & \leq \varphi(x, 0) + \varphi(0, 0).
 \end{aligned} \tag{81}$$

Putting $x = 0$ in (80) and replacing y by x , we get

$$\begin{aligned}
 & \|F_o(x) - G_o(x) - 2HK_o(0, x) - M_o(x)\| \\
 & \leq \varphi(x, 0) + \varphi(0, 0).
 \end{aligned} \tag{82}$$

Letting $y = x$ in (80), we have

$$\begin{aligned}
 & \|F_o(2x) - 2HK_o(x, x) - L_o(x) - M_o(x)\| \\
 & \leq \varphi(x, x) + \varphi(0, 0).
 \end{aligned} \tag{83}$$

Substituting $y = -x$ in (80), we get

$$\begin{aligned}
 & \|G_o(2x) - 2HK_o(x, -x) - L_o(x) + M_o(x)\| \\
 & \leq \varphi(x, x) + \varphi(0, 0).
 \end{aligned} \tag{84}$$

From (81) and (82), we obtain

$$\begin{aligned}
 & \|2F_o(x) - L_o(x) - M_o(x) - 2[HK_o(x, 0) + HK_o(0, x)]\| \\
 & \leq 2\varphi(x, 0) + 2\varphi(0, 0),
 \end{aligned} \tag{85}$$

$$\begin{aligned}
 & \|2G_o(x) - L_o(x) + M_o(x) - 2[HK_o(x, 0) - HK_o(0, x)]\| \\
 & \leq 2\varphi(x, 0) + 2\varphi(0, 0).
 \end{aligned} \tag{86}$$

From (83) and (85), we have

$$\begin{aligned}
 & \|L_o(2x) - 2L_o(x) + M_o(2x) - 2M_o(x) \\
 & \quad - 2[2HK_o(x, x) - HK_o(2x, 0) - HK_o(0, 2x)]\| \\
 & \leq 2\varphi(x, x) + 2\varphi(2x, 0) + 4\varphi(0, 0).
 \end{aligned} \tag{87}$$

From (84) and (86), we get

$$\begin{aligned}
 & \|L_o(2x) - 2L_o(x) - M_o(2x) + 2M_o(x) \\
 & \quad + 2[2HK_o(-x, x) + HK_o(2x, 0) - HK_o(0, 2x)]\| \\
 & \leq 2\varphi(x, x) + 2\varphi(2x, 0) + 4\varphi(0, 0).
 \end{aligned} \tag{88}$$

From (87) and (88), we have

$$\begin{aligned}
 & \|L_o(2x) - 2L_o(x) \\
 & \quad - 2[HK_o(x, x) - HK_o(-x, x) - HK_o(2x, 0)]\| \\
 & \leq 2\varphi(x, x) + 2\varphi(2x, 0) + 4\varphi(0, 0),
 \end{aligned} \tag{89}$$

$$\begin{aligned}
 & \|M_o(2x) - 2M_o(x) \\
 & \quad - 2[HK_o(x, x) + HK_o(-x, x) - HK_o(0, 2x)]\| \\
 & \leq 2\varphi(x, x) + 2\varphi(2x, 0) + 4\varphi(0, 0).
 \end{aligned}$$

In view of (80), (85), (86), and (89), we have

$$\begin{aligned}
& \|L_o(x+y) + M_o(x+y) + L_o(x-y) \\
& - M_o(x-y) - L_o(2x) - M_o(2y) \\
& - 2[2HK_o(x, y) - HK_o(x, x) + HK_o(-x, x) \\
& - HK_o(y, y) - HK_o(-y, y) + HK_o(2x, 0) \\
& + HK_o(0, 2y) - HK_o(x+y, 0) \\
& - HK_o(0, x+y) - HK_o(x-y, 0) \\
& + HK_o(0, x-y)]\| \\
& \leq 2\varphi(x, y) + 2\varphi(x, x) + 2\varphi(y, y) \\
& + 2\varphi(2x, 0) + 2\varphi(2y, 0) + 2\varphi(x+y, 0) \\
& + 2\varphi(x-y, 0) + 14\varphi(0, 0) \\
& = \varphi_0(x, y).
\end{aligned} \tag{90}$$

Substituting y by $-y$ in (90), we obtain

$$\begin{aligned}
& \|L_o(x-y) + M_o(x-y) + L_o(x+y) \\
& - M_o(x+y) - L_o(2x) + M_o(2y) \\
& - 2[-2HK_o(-x, y) - HK_o(x, x) + HK_o(-x, x) \\
& + HK_o(y, y) + HK_o(-y, y) + HK_o(2x, 0) \\
& - HK_o(0, 2y) - HK_o(x+y, 0) \\
& + HK_o(0, x+y) - HK_o(x-y, 0) \\
& - HK_o(0, x-y)]\| \\
& \leq \varphi_0(x, y).
\end{aligned} \tag{91}$$

From (90) and (91), we have

$$\begin{aligned}
& \|L_o(x+y) + L_o(x-y) - L_o(2x) \\
& - 2[HK_o(x, y) - HK_o(-x, y) - HK_o(x, x) \\
& + HK_o(-x, x) + HK_o(2x, 0) \\
& - HK_o(x+y, 0) - HK_o(x-y, 0)]\| \\
& \leq \varphi_0(x, y), \\
& \|M_o(x+y) - M_o(x-y) - M_o(2y) \\
& - 2[HK_o(x, y) + HK_o(-x, y)
\end{aligned} \tag{92}$$

$$\begin{aligned}
& - HK_o(y, y) - HK_o(-y, y) \\
& + HK_o(0, 2y) - HK_o(0, x+y) \\
& + HK_o(0, x-y)]\| \\
& \leq \varphi_0(x, y).
\end{aligned} \tag{93}$$

Putting $u = x + y$ and $v = x - y$ in (92), we obtain

$$\begin{aligned}
& \|L_o(u) + L_o(v) - L_o(u+v) \\
& - 2\left\{HK_o\left(\frac{u+v}{2}, \frac{u-v}{2}\right) - HK_o\left(-\frac{u+v}{2}, \frac{u-v}{2}\right) \right. \\
& - HK_o\left(\frac{u+v}{2}, \frac{u+v}{2}\right) + HK_o\left(-\frac{u+v}{2}, \frac{u+v}{2}\right) \\
& \left. + HK_o(u+v, 0) - HK_o(u, 0) - HK_o(v, 0)\right\}\| \\
& \leq \varphi_0\left(\frac{u+v}{2}, \frac{u-v}{2}\right).
\end{aligned} \tag{94}$$

By Proposition 3, there exists a unique additive function $A_1 : E_1 \rightarrow E_2$ such that

$$\begin{aligned}
& \|L_o(x) - 2A_1(x)\| \\
& \leq \frac{1}{k}\widehat{\Phi}'_k(x, x) + \frac{1}{k}\widehat{\Phi}'_{HK,k}(x, x) \quad (x \in E_1).
\end{aligned} \tag{95}$$

Moreover, if $\mathcal{L}(tx)$ is continuous in $t \in \mathbb{R}$ for every fixed $x \in E_1$, we have that A_1 is a linear function. Putting $u = x - y$ and $v = 2y$ in (93), we get

$$\begin{aligned}
& \|M_o(u+v) - M_o(u) - M_o(v) \\
& - 2\left[HK_o\left(u + \frac{v}{2}, \frac{v}{2}\right) + HK_o\left(-\left(u + \frac{v}{2}\right), \frac{v}{2}\right) \right. \\
& - HK_o\left(\frac{v}{2}, \frac{v}{2}\right) - HK_o\left(-\frac{v}{2}, \frac{v}{2}\right) + HK_o(0, v) \\
& \left. - HK_o(0, u+v) + HK_o(0, u)\right]\| \\
& \leq \varphi_0\left(u + \frac{v}{2}, \frac{v}{2}\right) \quad (u, v \in E_1).
\end{aligned} \tag{96}$$

Again by Proposition 3, there exists a unique additive function $A_2 : E_1 \rightarrow E_2$ such that

$$\begin{aligned}
& \|M_o(x) - 2A_2(x)\| \\
& \leq \frac{1}{k}\widehat{\Phi}''_k(x, x) + \frac{1}{k}\widehat{\Phi}''_{HK,k}(x, x) \quad (x \in E_1).
\end{aligned} \tag{97}$$

Furthermore, if $\mathcal{M}(tx)$ is continuous in $t \in \mathbb{R}$ for all $x \in E_1$, then A_2 is also linear. From (85), (95), and (97), it follows that

$$\begin{aligned}
 & \|F_o(x) - A_1(x) - A_2(x)\| \\
 & \leq \frac{1}{2k} \widehat{\Phi}'_k(x, x) + \frac{1}{2k} \widehat{\Phi}''_k(x, x) \\
 & \quad + \frac{1}{2k} \widehat{\Phi}'_{HK,k}(x, x) + \frac{1}{2k} \widehat{\Phi}''_{HK,k}(x, x) \\
 & \quad + \varphi(x, 0) + \varphi(0, 0) + \|HK_o(x, 0)\| \\
 & \quad + \|HK_o(0, x)\|, \\
 & \|G_o(x) - A_1(x) - A_2(x)\| \\
 & \leq \frac{1}{2k} \widehat{\Phi}'_k(x, x) + \frac{1}{2k} \widehat{\Phi}''_k(x, x) \\
 & \quad + \frac{1}{2k} \widehat{\Phi}'_{HK,k}(x, x) + \frac{1}{2k} \widehat{\Phi}''_{HK,k}(x, x) \\
 & \quad + \varphi(x, 0) + \varphi(0, 0) + \|HK_o(x, 0)\| \\
 & \quad + \|HK_o(0, x)\|.
 \end{aligned} \tag{98}$$

The theorem now follows from the inequalities (73), (74), (75), (76), (95), (97), and (98). $\square$

References

- [1] J. Tongsomporn, V. Laohakosol, C. Hengkrawit, and P. Udomkavanich, "Stability of a generalized trigonometric functional equation," *Journal of Computational and Applied Mathematics*, vol. 234, no. 5, pp. 1448–1457, 2010.
- [2] F. Skof, "Local properties and approximation of operators," *Rendiconti del Seminario Matematico e Fisico di Milano*, vol. 53, pp. 113–129, 1983.
- [3] P. W. Cholewa, "Remarks on the stability of functional equations," *Aequationes Mathematicae*, vol. 27, no. 1-2, pp. 76–86, 1984.
- [4] S. Czerwik, "On the stability of the quadratic mapping in normed spaces," *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, vol. 62, pp. 59–64, 1992.
- [5] S. Czerwik, "The stability of the quadratic functional equation," in *Stability of Mappings of Hyers-Ulam Type*, Th. M. Rassias and J. Tabor, Eds., pp. 81–91, Hadronic Press, Palm Harbor, Fla, USA, 1994.
- [6] S. M. Jung, "Stability of the quadratic equation of Pexider type," *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, vol. 70, pp. 175–190, 2000.
- [7] C. Hengkrawit, V. Laohakosol, and J. Tongsomporn, "A stability result for a generalized trigonometric-quadratic functional equation with one unbounded function," *WSEAS Transactions on Mathematics*, vol. 10, no. 8, pp. 279–290, 2011.
- [8] S. M. Jung, "On the Hyers-Ulam-Rassias stability of approximately additive mappings," *Journal of Mathematical Analysis and Applications*, vol. 204, no. 1, pp. 221–226, 1996.
- [9] T. M. Rassias, "On the stability of the linear mapping in Banach spaces," *Proceedings of the American Mathematical Society*, vol. 72, no. 2, pp. 297–300, 1978.