Research Article

Topological Properties of the Complex Vector Lattice of Bounded Measurable Functions

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Let Σ be a σ -algebra of subsets of a nonempty set Ω . Let $B(\Sigma)$ be the complex vector lattice of bounded Σ -measurable complex-valued functions on Ω and let $ca(\Sigma)$ be the Banach space of all bounded countably additive complex-valued measures on Ω . We study locally solid topologies on $B(\Sigma)$. In particular, it is shown that the Mackey topology $\tau(B(\Sigma), ca(\Sigma))$ is the finest locally convex-solid σ -Lebesgue topology on $B(\Sigma)$.

1. Introduction and Terminology

For terminology concerning complex vector lattices we refer the reader to [1, Chapter 12, Section 91], [2, Chapter 16], [3]. We denote by $\sigma(L, K)$ and $\tau(L, K)$ the weak topology and the Mackey topology with respect to a dual pair $\langle L, K \rangle$. For a topological vector space (L, η) by $(L, \eta)'$ or L'_η , we denote its topological dual.

We assume that Σ is a σ -algebra of subsets of a nonempty set Ω . Let $B(\Sigma)$ denote the Banach space of all bounded Σ -measurable functions $f : \Omega \rightarrow \mathbb{C}$, provided with the uniform norm $\|\cdot\|$. Then, $B(\Sigma)$ with the positive cone $B(\Sigma)^+ = \{f \in B(\Sigma) : f(\omega) \geq 0 \text{ for all } \omega \in \Omega\}$ and a valuation $B(\Sigma) \ni f \mapsto |f| \in B(\Sigma)^+$ is a *complex vector lattice* (in the sense of [2, Section 16-2.2]). By $B_{\mathbb{R}}(\Sigma)$ we denote the Dedekind σ -complete Banach lattice of all bounded Σ -measurable functions $u : \Omega \rightarrow \mathbb{R}$, with the natural ordering $u \leq v$ if $u(\omega) \leq v(\omega)$ for all $\omega \in \Omega$ (see [4, Chapter 13.1]). Then, $B(\Sigma) = B_{\mathbb{R}}(\Sigma) + iB_{\mathbb{R}}(\Sigma)$, that is, $B(\Sigma)$ is a complexification of $B_{\mathbb{R}}(\Sigma)$ and $B(\Sigma)$ has the dominated decomposition property, that is, if $|f| \leq u_1 + u_2$, where $f \in B(\Sigma)$, $u_1, u_2 \in B(\Sigma)^+$, then there exist $f_1, f_2 \in B(\Sigma)$ such that $f = f_1 + f_2$ and $|f_1| \leq u_1$, $|f_2| \leq u_2$ (see [1, Theorem 91.3], [3]). This means that $B(\Sigma)$ is a *breakable* complex vector lattice (see [2, Section 16-3]).

Denote by $ba(\Sigma)$ the Banach space of all bounded finitely additive measures $\mu : \Sigma \rightarrow \mathbb{C}$ with the norm $\|\mu\| = |\mu|(\Omega)$, where $|\mu|(A)$ denotes the variation of μ on $A \in \Sigma$. It is well

known that the Banach dual $B(\Sigma)^*$ of $B(\Sigma)$ can be identified with $ba(\Sigma)$ through the integration mapping $ba(\Sigma) \ni \mu \mapsto \Phi_\mu \in B(\Sigma)^*$, where $\Phi_\mu(f) = \int_\Omega f d\mu$ for all $f \in B(\Sigma)$ and $\|\Phi_\mu\| = |\mu|(\Omega)$ (see [5, Chapter 1, Theorem 13, p.6]). Let $ca(\Sigma)$ stand for the Banach space of all countably additive measures $\mu : \Sigma \rightarrow \mathbb{C}$.

It turns out that the basic notions and results from the theory of vector lattices (in particular, locally solid topologies) can be carried over to the complex vector lattices $B(\Sigma)$ and $ba(\Sigma)$.

A subset H of $B(\Sigma)$ is said to be *solid* if it follows from $|f| \leq |g|$ and $f \in B(\Sigma)$, $g \in H$ that $f \in H$. Every subset H of $B(\Sigma)$ is included in the smallest (with respect to the inclusion) solid set, called the *solid hull* of H and denoted by $\text{sol}(H)$. Clearly,

$$\text{sol}(H) = \{g \in B(\Sigma) : |g| \leq |f| \text{ for some } f \in H\}. \quad (1)$$

A linear Hausdorff topology τ on $B(\Sigma)$ is said to be *locally solid* if it has a local base at 0 consisting of solid sets in $B(\Sigma)$.

In Section 2, we study locally solid topologies on $B(\Sigma)$. In Section 3, we develop the duality theory of locally convex-solid topologies on $B(\Sigma)$. In Section 4, it is shown that the natural Mackey topology $\tau(B(\Sigma), ca(\Sigma))$ is the finest locally convex-solid σ -Lebesgue topology on $B(\Sigma)$ (see Theorem 16 below).

2. Locally Solid Topologies on $B(\Sigma)$

Using the dominated decomposition property for $B(\Sigma)$ we can easily obtain the following result (see [6, Theorem 1.11]).

Proposition 1. *The convex hull $\text{conv } H$ of a solid subset H of $B(\Sigma)$ is solid.*

Proposition 2. *Let τ be a locally solid topology on $B(\Sigma)$. Then, the τ -closure $\text{cl}_\tau H$ of a solid subset H of $B(\Sigma)$ is solid.*

Proof. Let $\mathcal{B}_\tau$ be a local base at 0 for τ consisting of solid sets. Then $\text{cl}_\tau H = \bigcap \{H + V : V \in \mathcal{B}_\tau\}$. Assume that $|f| \leq |g|$, where $f \in B(\Sigma)$, $g \in \text{cl}_\tau H$, and let $V_0 \in \mathcal{B}_\tau$. Then, $g = g_1 + g_2$, where $g_1 \in H$ and $g_2 \in V_0$. Since $|f| \leq |g| \leq |g_1| + |g_2|$ (by the dominated decomposition property for $B(\Sigma)$), there exist $f_1, f_2 \in B(\Sigma)$ such that $f = f_1 + f_2$ and $|f_i| \leq |g_i|$ for $i = 1, 2$. Hence, $f_1 \in H$ and $f_2 \in V_0$ because both sets H and V are solid. Thus, $f \in H + V_0$ for every $V_0 \in \mathcal{B}_\tau$, so $f \in \text{cl}_\tau H$. This means that $\text{cl}_\tau H$ is solid, as desired. $\square$

Following the notion of a modulus of an operator between vector lattices (see [7, Definition 1.8 and Theorem 1.10]), for a linear functional $\Phi : B(\Sigma) \rightarrow \mathbb{C}$ let us put

$$|\Phi|(u) := \sup \{|\Phi(h)| : h \in B(\Sigma), |h| \leq u\} \quad (2)$$

for all $u \in B(\Sigma)^+$.

Proposition 3. *For a linear functional Φ on $B(\Sigma)$ the following statements are equivalent:*

- (i) $\Phi \in B(\Sigma)^*$.
- (ii) $|\Phi|(u) < \infty$ for all $u \in B(\Sigma)^+$.

Proof. (i) $\Rightarrow$ (ii) By way of contradiction, assume that there exists $\Phi_0 \in B(\Sigma)^*$ such that $|\Phi_0|(u_0) = \infty$ for some $u_0 \in B(\Sigma)^+$. Hence, there exists a sequence (h_n) in $B(\Sigma)$ such that $|h_n| \leq u_0$ and $|\Phi_0(h_n)| \geq n$ for $n \in \mathbb{N}$. Since $\|(1/n)h_n\| \rightarrow 0$, we get $(1/n)|\Phi_0(h_n)| \rightarrow 0$, which is in contradiction with $|\Phi_0(h_n)| \geq n$.

(ii) $\Rightarrow$ (i) Assume that there exists a linear functional Φ_0 on $B(\Sigma)$ such that $|\Phi_0|(u) < \infty$ for all $u \in B(\Sigma)^+$ and $\Phi_0 \notin B(\Sigma)^*$. Then, there exists a sequence (f_n) in $B(\Sigma)$ such that $\|f_n\| = 1$ and $|\Phi_0(f_n)| > n^3$ for all $n \in \mathbb{N}$. Since $\sum_{n=1}^\infty (1/n^2)\|f_n\| < \infty$ and the space $(B(\Sigma), \|\cdot\|)$ is complete, there exists $u_0 \in B(\Sigma)^+$ such that $\sum_{n=1}^\infty (1/n^2)|f_n| = u_0$. Then, $(1/n^2)|f_n| \leq u_0$, and; hence, $n < |\Phi_0((1/n^2)f_n)| < \infty$ for all $n \in \mathbb{N}$, which is impossible.

Thus, the proof is complete. $\square$

For $\Phi_1, \Phi_2 \in B(\Sigma)^*$ we will write $|\Phi_1| \leq |\Phi_2|$ whenever $|\Phi_1|(u) \leq |\Phi_2|(u)$ for all $u \in B(\Sigma)^+$. A subset $\mathcal{K}$ of $B(\Sigma)^*$ is said to be *solid* if $|\Phi_1| \leq |\Phi_2|$, where $\Phi_1 \in B(\Sigma)^*$, $\Phi_2 \in \mathcal{K}$ implies that $\Phi_1 \in \mathcal{K}$. A linear subspace I of $B(\Sigma)^*$ is said to be an *ideal* if I is a solid set in $B(\Sigma)^*$.

Note that, if $|\Phi_1| \leq |\Phi_2|$, then $\|\Phi_1\| \leq \|\Phi_2\|$.

Proposition 4. *Let τ be a locally solid topology on $B(\Sigma)$. Then, $(B(\Sigma), \tau)'$ is an ideal of $B(\Sigma)^*$.*

Proof. To show that $(B(\Sigma), \tau)' \subset B(\Sigma)^*$, by the way of contradiction, assume that for some $\Phi_0 \in (B(\Sigma), \tau)'$ we have $\Phi_0 \notin B(\Sigma)^*$, so in view of Proposition 3, we obtain that $|\Phi_0|(u_0) = \infty$ for some $u_0 \in B(\Sigma)^+$. Hence, there exists a sequence (h_n) in $B(\Sigma)$ such that $|h_n| \leq u_0$ and $|\Phi_0(h_n)| \geq n$ for all $n \in \mathbb{N}$. Since $(1/n)u_0 \rightarrow 0$ for τ and τ is locally solid, we get $(1/n)h_n \rightarrow 0$ for τ . Hence, $\Phi_0((1/n)h_n) \rightarrow 0$, which is in contradiction with $|\Phi_0(h_n)| \geq n$ for all $n \in \mathbb{N}$.

To see that $(B(\Sigma), \tau)'$ is an ideal of $B(\Sigma)^*$, assume that $|\Phi_1| \leq |\Phi_2|$, where $\Phi_1 \in B(\Sigma)^*$ and $\Phi_2 \in (B(\Sigma), \tau)'$. Let $f_\alpha \rightarrow 0$ for τ and $\varepsilon > 0$ be given. Then, there exists a net (h_α) in $B(\Sigma)$ such that $|h_\alpha| \leq |f_\alpha|$ for each α and $|\Phi_2|(|f_\alpha|) \leq |\Phi_2(h_\alpha)| + \varepsilon$. Clearly, $h_\alpha \rightarrow 0$ for τ because τ is locally solid, so $\Phi_2(h_\alpha) \rightarrow 0$. Since $|\Phi_1(f_\alpha)| \leq |\Phi_1|(|f_\alpha|) \leq |\Phi_2|(|f_\alpha|) \leq |\Phi_2(h_\alpha)| + \varepsilon$, we get $\Phi_1(f_\alpha) \rightarrow 0$, so $\Phi_1 \in (B(\Sigma), \tau)'$, as desired. $\square$

3. Duality of the Space $B(\Sigma)$

A locally convex Hausdorff topology τ on $B(\Sigma)$ is said to be locally convex-solid if it has a local base at 0 consisting of convex and solid sets in $B(\Sigma)$. In view of Propositions 1 and 2, we see that for a locally convex-solid topology τ on $B(\Sigma)$ the collection of all τ -closed convex and solid τ -neighborhoods of 0 forms a local base at 0 for τ .

In this section, we develop the duality theory of locally convex-solid topologies on $B(\Sigma)$. We start with the following lemma.

Lemma 5. *Let $\mu \in ba(\Sigma)$. Then, for each $u \in B(\Sigma)^+$ we have*

$$\begin{aligned} |\Phi_\mu|(u) &= \int_\Omega u d|\mu| \\ &= \sup \left\{ \left| \int_\Omega h d\mu \right| : h \in B(\Sigma), |h| \leq u \right\}. \end{aligned} \quad (3)$$

In particular, we have for $A \in \Sigma$

$$|\mu|(A) = \sup \left\{ \left| \int_A h d\mu \right| : h \in B(\Sigma), \|h\| \leq 1 \right\}. \quad (4)$$

Proof. Let $u \in B(\Sigma)^+$ and $h \in B(\Sigma)$ with $|h| \leq u$. Then, we have

$$\left| \int_\Omega h d\mu \right| \leq \int_\Omega |h| d|\mu| \leq \int_\Omega u d|\mu|, \quad (5)$$

so $|\Phi_\mu|(u) \leq \int_\Omega u d|\mu|$. Now we shall show that $\int_\Omega u d|\mu| \leq |\Phi_\mu|(u)$. Indeed, let $\varepsilon > 0$ be given. Then, there exists a Σ -simple function $s = \sum_{i=1}^k \alpha_i \mathbb{1}_{A_i}$ such that $\alpha_i > 0$, $s \leq u$ and

$$\left| \int_\Omega u d|\mu| - \sum_{i=1}^k \alpha_i |\mu|(A_i) \right| \leq \frac{\varepsilon}{2}. \quad (6)$$

Let $\alpha = \sum_{i=1}^k \alpha_i$. Then for each $i = 1, 2, \dots, k$ one can choose a finite Σ -partition $\{A_{ij} : j = 1, \dots, k_i\}$ of A_i and $\varepsilon_{ij} \in \mathbb{C}$ with $|\varepsilon_{ij}| = 1$, $\varepsilon_{ij}\mu(A_{ij}) = |\mu(A_{ij})|$ for $j = 1, \dots, k_i$ such that

$$|\mu|(A_i) \leq \sum_{j=1}^{k_i} \varepsilon_{ij} \mu(A_{ij}) + \frac{\varepsilon}{2\alpha}. \quad (7)$$

Let $s_0 = \sum_{i=1}^k \sum_{j=1}^{k_i} \alpha_i \varepsilon_{ij} \mathbb{1}_{A_{ij}}$. Then $|s_0| = s \leq u$ and we have

$$\begin{aligned} & \left| \int_{\Omega} u d|\mu| - \int_{\Omega} s_0 d\mu \right| \\ & \leq \left| \int_{\Omega} u d|\mu| - \sum_{i=1}^k \alpha_i |\mu|(A_i) \right| \\ & \quad + \left| \sum_{i=1}^k \alpha_i |\mu|(A_i) - \sum_{i=1}^k \sum_{j=1}^{k_i} \alpha_i \varepsilon_{ij} \mu(A_{ij}) \right| \quad (8) \\ & \leq \frac{\varepsilon}{2} + \sum_{i=1}^k \alpha_i \left| |\mu|(A_i) - \sum_{j=1}^{k_i} \varepsilon_{ij} \mu(A_{ij}) \right| \\ & \leq \frac{\varepsilon}{2} + \sum_{i=1}^k \alpha_i \frac{\varepsilon}{2\alpha_i} = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Hence, $\int_{\Omega} u d|\mu| \leq \int_{\Omega} s_0 d\mu + \varepsilon$, and it follows that $\int_{\Omega} u d|\mu| \leq |\Phi_{\mu}|(u)$. $\square$

Since $|\Phi_{\mu}| : B(\Sigma)^+ \rightarrow \mathbb{R}^+$ is additive, $|\Phi_{\mu}|$ can be uniquely extended to a positive linear functional (denoted by $|\Phi_{\mu}|$ again) $|\Phi_{\mu}| : B(\Sigma) \rightarrow \mathbb{C}$ (see [2, Theorem 16-4.7]). Then, for $f \in B(\Sigma)$ we have

$$\begin{aligned} |\Phi_{\mu}|(f) &= |\Phi_{\mu}|((\operatorname{Re} f)^+) - |\Phi_{\mu}|((\operatorname{Re} f)^-) \\ &\quad + i |\Phi_{\mu}|((\operatorname{Im} f)^+) - i |\Phi_{\mu}|((\operatorname{Im} f)^-) \\ &= \int_{\Omega} (\operatorname{Re} f)^+ d|\mu| - \int_{\Omega} (\operatorname{Re} f)^- d|\mu| \quad (9) \\ &\quad + i \int_{\Omega} (\operatorname{Im} f)^+ d|\mu| - i \int_{\Omega} (\operatorname{Im} f)^- d|\mu| \\ &= \int_{\Omega} f d|\mu| = \Phi_{|\mu|}(f). \end{aligned}$$

Hence, $|\Phi_{\mu}| = \Phi_{|\mu|}$.

Proposition 6. For $\mu, \nu \in ba(\Sigma)$ the following statements are equivalent:

- (i) $|\mu| \leq |\nu|$.
- (ii) $|\Phi_{\mu}| \leq |\Phi_{\nu}|$.

Proof. (i) $\Rightarrow$ (ii) Assume that $|\mu| \leq |\nu|$. Then, by Lemma 5, for $u \in B(\Sigma)^+$, $|\Phi_{\mu}|(u) = \int_{\Omega} u d|\mu| \leq \int_{\Omega} u d|\nu| = |\Phi_{\nu}|(u)$, that is, $|\Phi_{\mu}| \leq |\Phi_{\nu}|$.

(ii) $\Rightarrow$ (i) Assume that $|\Phi_{\mu}| \leq |\Phi_{\nu}|$. Then, for $A \in \Sigma$ we have $|\mu|(A) = |\Phi_{\mu}|(\mathbb{1}_A) \leq |\Phi_{\nu}|(\mathbb{1}_A) = |\nu|(A)$. $\square$

Now we consider the concept of solidness in the space $ba(\Sigma)$. A subset $\mathcal{M}$ of $ba(\Sigma)$ is said to be *solid* if $|\nu| \leq |\mu|$, where $\nu \in ba(\Sigma)$, $\mu \in \mathcal{M}$ implies that $\nu \in \mathcal{M}$.

For a subset $\mathcal{M}$ of $ba(\Sigma)$ let

$$\operatorname{sol}(\mathcal{M}) = \{\nu \in ba(\Sigma) : |\nu| \leq |\mu| \text{ for some } \mu \in \mathcal{M}\}. \quad (10)$$

Then, $\operatorname{sol}(\mathcal{M})$ is the solid hull of $\mathcal{M}$, that is, $\operatorname{sol}(\mathcal{M})$ is the smallest solid set in $\mathcal{M}$ that contains $\mathcal{M}$.

A linear subspace I of $ba(\Sigma)$ is called an *ideal* in $ba(\Sigma)$ if I is solid. Note that $ca(\Sigma)$ is an ideal of $ba(\Sigma)$. In view of Proposition 6, we see that a subset $\mathcal{M}$ of $ba(\Sigma)$ is solid if and only if the set $\{\Phi_{\mu} : \mu \in \mathcal{M}\}$ is solid in $B(\Sigma)^*$.

Corollary 7. Let $\mu \in ba(\Sigma)$. Then, for $f \in B(\Sigma)$ we have

$$\begin{aligned} |\Phi_{\mu}|(|f|) &= \int_{\Omega} |f| d|\mu| \\ &= \sup \left\{ \left| \int_{\Omega} f d\nu \right| : \nu \in ba(\Sigma), |\nu| \leq |\mu| \right\}. \quad (*) \end{aligned}$$

Moreover, for a subset $\mathcal{M}$ of $ba(\Sigma)$ and each $f \in B(\Sigma)$ we have

$$\begin{aligned} & \sup \{ |\Phi_{\mu}|(|f|) : \mu \in \mathcal{M} \} \\ &= \sup \left\{ \int_{\Omega} |f| d|\mu| : \mu \in \mathcal{M} \right\} \\ &= \sup \left\{ \left| \int_{\Omega} f d\nu \right| : \nu \in \operatorname{sol}(\mathcal{M}) \right\} \quad (**) \\ &= \sup \left\{ \left| \int_{\Omega} f d\nu \right| : \nu \in \operatorname{conv}(\operatorname{sol}(\mathcal{M})) \right\}. \end{aligned}$$

Proof. In view of Proposition 6 and [2, Theorem 16-5.6] we get

$$\begin{aligned} |\Phi_{\mu}|(|f|) &= \sup \left\{ \left| \int_{\Omega} f d\nu \right| : \nu \in ba(\Sigma), |\Phi_{\nu}| \leq |\Phi_{\mu}| \right\} \\ &= \sup \left\{ \left| \int_{\Omega} f d\nu \right| : \nu \in ba(\Sigma), |\nu| \leq |\mu| \right\}. \quad (11) \end{aligned}$$

Next, we shall show that for each $f \in B(\Sigma)$,

$$\sup \{ |\Phi_{\mu}|(|f|) : \mu \in \mathcal{M} \} \leq \sup \left\{ \left| \int_{\Omega} f d\nu \right| : \nu \in \operatorname{sol}(\mathcal{M}) \right\}. \quad (12)$$

Indeed, let $f \in B(\Sigma)$ be given. Then, for $\mu \in \mathcal{M}$ and $\varepsilon > 0$ by (*) there exists $\nu \in ba(\Sigma)$ with $|\nu| \leq |\mu|$ and such that $|\Phi_{\mu}|(|f|) \leq \left| \int_{\Omega} f d\nu \right| + \varepsilon$. Since $\nu \in \operatorname{sol}(\mathcal{M})$, we get $|\Phi_{\mu}|(|f|) \leq \sup \{ \left| \int_{\Omega} f d\nu \right| : \nu \in \operatorname{sol}(\mathcal{M}) \} + \varepsilon$, and it follows that (12) holds.

It is enough to show that for each $f \in B(\Sigma)$ we have

$$\begin{aligned} & \sup \left\{ \left| \int_{\Omega} f d\nu \right| : \nu \in \operatorname{conv}(\operatorname{sol}(\mathcal{M})) \right\} \\ & \leq \sup \{ |\Phi_{\mu}|(|f|) : \mu \in \mathcal{M} \}. \quad (13) \end{aligned}$$

Indeed, let $\nu \in \text{conv}(\text{sol}(\mathcal{M}))$. Then, $\nu = \sum_{i=1}^n \alpha_i \nu_i$ with $\nu_i \in \text{sol}(\mathcal{M})$, $\alpha_i \geq 0$, $i = 1, \dots, n$ and $\sum_{i=1}^n \alpha_i = 1$. Hence, $|\nu_i| \leq |\mu_i|$ for some $\mu_i \in \mathcal{M}$, $i = 1, 2, \dots, n$. Hence, for $f \in B(\Sigma)$ we get

$$\begin{aligned} \left| \int_{\Omega} f d\nu \right| &\leq \int_{\Omega} |f| d|\nu| \\ &\leq \sum_{i=1}^n \alpha_i \int_{\Omega} |f| d|\nu_i| \leq \sum_{i=1}^n \alpha_i \int_{\Omega} |f| d|\mu_i| \\ &\leq \left(\sum_{i=1}^n \alpha_i \right) \sup \left\{ \int_{\Omega} |f| d|\mu| : \mu \in \mathcal{M} \right\} \\ &= \sup \left\{ |\Phi_{\mu}|(|f|) : \mu \in \mathcal{M} \right\}. \end{aligned} \quad (14)$$

It follows that (13) holds, as desired. $\square$

Let I be an ideal of $ba(\Sigma)$ separating the points of $B(\Sigma)$. Then, the pair $\langle B(\Sigma), I \rangle$, under its natural duality

$$\langle f, \mu \rangle := \int_{\Omega} f d\mu \quad \text{for } f \in B(\Sigma), \mu \in I \quad (15)$$

will be referred to as a solid dual system (see [7, Definition 11.7]). For subsets H of $B(\Sigma)$ and $\mathcal{M}$ of I , let us put

$$\begin{aligned} H^0 &= \left\{ \mu \in I : \left| \int_{\Omega} f d\mu \right| \leq 1 \forall f \in H \right\}, \\ {}^0\mathcal{M} &= \left\{ f \in B(\Sigma) : \left| \int_{\Omega} f d\mu \right| \leq 1 \forall \mu \in \mathcal{M} \right\}. \end{aligned} \quad (16)$$

Proposition 8. *Let $\langle B(\Sigma), I \rangle$ be a solid dual system.*

- (i) *If H is a solid subset of $B(\Sigma)$, then H^0 is a solid subset of I .*
- (ii) *If $\mathcal{M}$ is a solid subset of I , then ${}^0\mathcal{M}$ is a solid subset of $B(\Sigma)$.*

Proof. (i) Let $|\nu| \leq |\mu|$ with $\nu \in I$ and $\mu \in H^0$. Assume that $f \in H$ and let $h \in B(\Sigma)$ with $|h| \leq |f|$. Then, $h \in H$ because H is solid, so $|\int_{\Omega} h d\mu| \leq 1$. Hence $\int_{\Omega} |f| d|\mu| = |\Phi_{\mu}|(|f|) \leq 1$ (see Lemma 5). Thus $|\int_{\Omega} f d\nu| \leq \int_{\Omega} |f| d|\nu| \leq \int_{\Omega} |f| d|\mu| \leq 1$, that is, $\nu \in H^0$. This means that H^0 is a solid subset of I .

(ii) Let $|f| \leq |g|$, where $f \in B(\Sigma)$ and $g \in {}^0\mathcal{M}$. Since $\mathcal{M}$ is a solid set in $ba(\Sigma)$, by Corollary 7 for $\mu \in \mathcal{M}$, we have

$$\int_{\Omega} |g| d|\mu| = \sup \left\{ \left| \int_{\Omega} g d\nu \right| : \nu \in \mathcal{M}, |\nu| \leq |\mu| \right\}. \quad (17)$$

Then, for every $\nu \in \mathcal{M}$ with $|\nu| \leq |\mu|$ we have $|\int_{\Omega} g d\nu| \leq 1$, so $\int_{\Omega} |g| d|\mu| \leq 1$. It follows that

$$\left| \int_{\Omega} f d\mu \right| \leq \int_{\Omega} |f| d|\mu| \leq \int_{\Omega} |g| d|\mu| \leq 1, \quad (18)$$

that is, $f \in {}^0\mathcal{M}$. $\square$

Observe that for each $\mu \in ba(\Sigma)$, by Corollary 7 the functional $\varrho_{\mu} : B(\Sigma) \rightarrow [0, \infty)$ defined by

$$\begin{aligned} \varrho_{\mu}(f) &= |\Phi_{\mu}|(|f|) = \sup \left\{ \left| \int_{\Omega} h d\mu \right| : h \in B(\Sigma), |h| \leq |f| \right\} \\ &= \sup \left\{ \left| \int_{\Omega} f d\nu \right| : \nu \in ba(\Sigma), |\nu| \leq |\mu| \right\}, \end{aligned} \quad (19)$$

is a solid seminorm on $B(\Sigma)$, that is, $\varrho_{\mu}(f_1) \leq \varrho_{\mu}(f_2)$ if $|f_1| \leq |f_2|$.

Let I be an ideal of $ba(\Sigma)$ separating the points of $B(\Sigma)$. The absolute weak topology $|\sigma|(B(\Sigma), I)$ generated by I on $B(\Sigma)$ is the locally convex-solid topology on $B(\Sigma)$ generated by the family of solid seminorms $\{\varrho_{\mu} : \mu \in I\}$. Note that $|\sigma|(B(\Sigma), I)$ is the topology of uniform convergence on all sets $I_{\mu} = \{\nu \in I : |\nu| \leq |\mu|\}$, where $\mu \in I$.

Proposition 9. *For each $\mu \in I$, the set I_{μ} is convex, solid, and $\sigma(I, B(\Sigma))$ -compact in I .*

Proof. Clearly, I_{μ} is a convex and solid subset of I . We shall show that I_{μ} is $\sigma(ba(\Sigma), B(\Sigma))$ -closed subset of $ba(\Sigma)$. To this end, assume that (ν_{α}) is a net in I_{μ} such that $\int_{\Omega} f d\nu_{\alpha} \rightarrow \int_{\Omega} f d\nu$ for all $f \in B(\Sigma)$, where $\nu \in ba(\Sigma)$. Then

$$\begin{aligned} \left| \int_{\Omega} f d\nu_{\alpha} \right| &\rightarrow \left| \int_{\Omega} f d\nu \right|, \\ \left| \int_{\Omega} f d\nu_{\alpha} \right| &\leq \int_{\Omega} |f| d|\nu_{\alpha}| \leq \int_{\Omega} |f| d|\mu|. \end{aligned} \quad (20)$$

Hence, for each $A \in \Sigma$, we get $|\int_A f d\nu| \leq \int_A |f| d|\mu|$ for all $f \in B(\Sigma)$, and by Lemma 5,

$$\begin{aligned} |\nu|(A) &= \sup \left\{ \left| \int_A f d\nu \right| : f \in B(\Sigma), \|f\| \leq 1 \right\} \\ &\leq \int_A 1_{\Omega} d|\mu| = |\mu|(A), \end{aligned} \quad (21)$$

that is, $|\nu| \leq |\mu|$. Thus, $\nu \in I_{\mu}$, as desired.

In view of the Banach-Alaoglu theorem, I_{μ} is a relatively $\sigma(ba(\Sigma), B(\Sigma))$ -compact subset of $ba(\Sigma)$. Since I_{μ} is $\sigma(ba(\Sigma), B(\Sigma))$ -closed in $ba(\Sigma)$, I_{μ} is $\sigma(ba(\Sigma), B(\Sigma))$ -compact, and it follows that I_{μ} is $\sigma(I, B(\Sigma))$ -compact. $\square$

We define the absolute Mackey topology $|\tau|(B(\Sigma), I)$ as the topology of uniform convergence on all convex, solid and $\sigma(I, B(\Sigma))$ -compact subsets of $ba(\Sigma)$. By Proposition 8 $|\tau|(B(\Sigma), I)$ is a locally convex-solid topology on $B(\Sigma)$. Thus, in view of Proposition 9 we have

$$\begin{aligned} \sigma(B(\Sigma), I) &\subset |\sigma|(B(\Sigma), I) \\ &\subset |\tau|(B(\Sigma), I) \subset \tau(B(\Sigma), I). \end{aligned} \quad (22)$$

As a consequence, we obtain the following Mackey-Arens type theorem for locally convex-solid topologies on $B(\Sigma)$.

Proposition 10. *Let τ be a locally convex-solid topology on $B(\Sigma)$. Then, $|\sigma|(B(\Sigma), B(\Sigma)'_{\tau}) \subset \tau \subset |\tau|(B(\Sigma), B(\Sigma)'_{\tau})$.*

Proof. Note that the family $\mathcal{B}_\tau$ of all convex, solid and τ -closed τ -neighborhoods of 0 forms a local base at 0 for τ . Then, τ is the topology of uniform convergence on the family $\{V^0 : V \in \mathcal{B}_\tau\}$. By Proposition 8 and the Banach-Alaoglu theorem, for each $V \in \mathcal{B}_\tau$, V^0 is convex, solid, and $\sigma(B(\Sigma)'_\tau, B(\Sigma))$ -compact in $ba(\Sigma)$. It follows that $\tau \subset |\tau|(B(\Sigma), B(\Sigma)'_\tau)$.

To show that $|\sigma|(B(\Sigma), B(\Sigma)'_\tau) \subset \tau$, assume that $f_\alpha \rightarrow 0$ in $B(\Sigma)$ for τ . Then, $|f_\alpha| \rightarrow 0$ for τ because τ is locally solid. Let $\mu \in ba(\Sigma)$ be such that $\Phi_\mu \in B(\Sigma)'_\tau$. Then, $|\Phi_\mu| \in B(\Sigma)'_\tau$ because $B(\Sigma)'_\tau$ is an ideal of $B(\Sigma)^*$ (see Proposition 4). Hence, $\rho_\mu(f_\alpha) = |\Phi_\mu|(|f_\alpha|) \rightarrow 0$, and it follows that $f_\alpha \rightarrow 0$ for $|\sigma|(B(\Sigma), B(\Sigma)'_\tau)$, as desired. $\square$

4. The Mackey Topology $\tau(B(\Sigma), ca(\Sigma))$

In this section we study the Mackey topology $\tau(B(\Sigma), ca(\Sigma))$ on $B(\Sigma)$. It is known that $(B(\Sigma), \tau(B(\Sigma), ca(\Sigma)))$ is a generalized DF-space, that is, $\tau(B(\Sigma), ca(\Sigma))$ is the finest locally convex topology agreeing with itself on norm-bounded sets in $B(\Sigma)$ (see [8, Corollary 11.8], [9, Section 4], [10, Theorem 2]).

Let $ca_{\mathbb{R}}(\Sigma)$ stand for the Dedekind complete Banach lattice of all countably additive measures $\mu : \Sigma \rightarrow \mathbb{R}$. Then, $ca_{\mathbb{R}}(\Sigma)$ is a band of $ba_{\mathbb{R}}(\Sigma)$. Moreover, the σ -order continuous dual $B_{\mathbb{R}}(\Sigma)_c^*$ of $B_{\mathbb{R}}(\Sigma)$ can be identified with $ca_{\mathbb{R}}(\Sigma)$ through the integration mapping $ca_{\mathbb{R}}(\Sigma) \ni \mu \mapsto \varphi_\mu \in B_{\mathbb{R}}(\Sigma)_c^*$, where $\varphi_\mu(u) = \int_\Omega u d\mu$ for all $u \in B_{\mathbb{R}}(\Sigma)$ (see [4, Theorem 13.5]).

Recall that a sequence (u_n) in $B_{\mathbb{R}}(\Sigma)$ is said to be *order convergent* to $u \in B_{\mathbb{R}}(\Sigma)$ (in symbols, $u_n \xrightarrow{(o)} u$) if there exists a sequence (v_n) in $B_{\mathbb{R}}(\Sigma)^+$ such that $|u_n - u| \leq v_n \downarrow 0$ (in the vector lattice sense; see [6, Definition 1.12]).

Definition 11. A linear functional Φ on $B(\Sigma)$ is said to be *σ -order continuous* if for each sequence (f_n) in $B(\Sigma)$, $|f_n| \xrightarrow{(o)} 0$ in $B_{\mathbb{R}}(\Sigma)$ implies $\Phi(f_n) \rightarrow 0$.

Proposition 12. For $\mu \in ba(\Sigma)$ the following statements are equivalent:

- (i) Φ_μ is σ -order continuous.
- (ii) $\Phi_\mu(f_n) \rightarrow 0$ for each uniformly bounded sequence (f_n) in $B(\Sigma)$ such that $f_n(\omega) \rightarrow 0$ for all $\omega \in \Omega$.
- (iii) $\Phi_\mu(u_n) \rightarrow 0$ for each sequence (u_n) in $B(\Sigma)^+$ such that $u_n(\omega) \downarrow 0$ for all $\omega \in \Omega$.
- (iv) $\mu \in ca(\Sigma)$.

Proof. (i) $\Rightarrow$ (ii) Assume that Φ_μ is σ -order continuous and let $f_n(\omega) \rightarrow 0$ for all $\omega \in \Omega$ and $\sup_n \|f_n\| < \infty$. Let $u_n(\omega) = \sup_{k \geq n} |f_k(\omega)|$ for $\omega \in \Omega$, $n \in \mathbb{N}$. Then, $u_n \in B_{\mathbb{R}}(\Sigma)$ and $|f_n(\omega)| \leq u_n(\omega) \downarrow 0$ for all $\omega \in \Omega$ and $n \in \mathbb{N}$. It follows that $|f_n| \xrightarrow{(o)} 0$ in $B_{\mathbb{R}}(\Sigma)$, and hence, $\int_\Omega f_n d\mu \rightarrow 0$.

(ii) $\Rightarrow$ (iii) It is obvious.

(iii) $\Rightarrow$ (iv) Assume that (iii) holds, and let $A_n \downarrow \emptyset$, $(A_n) \subset \Sigma$. Then, $\mathbb{1}_{A_n}(\omega) \downarrow 0$ for all $\omega \in \Omega$. Hence, $\mu(A_n) = \int_\Omega \mathbb{1}_{A_n} d\mu \rightarrow 0$.

(iv) $\Rightarrow$ (i) Assume that $\mu \in ca(\Sigma)$. Let (f_n) be a sequence in $B(\Sigma)$ such that $|f_n| \xrightarrow{(o)} 0$ in $B_{\mathbb{R}}(\Sigma)$, that is, $|f_n| \leq u_n \downarrow 0$ for some sequence (u_n) in $B(\Sigma)^+$. Then

$$|\Phi_\mu(f_n)| = \left| \int_\Omega f_n d\mu \right| \leq \int_\Omega |f_n| d|\mu| \leq \int_\Omega u_n d|\mu|. \quad (23)$$

Note that $u_n(\omega) \downarrow 0$ for $|\mu|$ -almost all $\omega \in \Omega$. Indeed, let $u_n(\omega) \downarrow u(\omega)$ for each $\omega \in \Omega$. Then, $u \in B_{\mathbb{R}}(\Sigma)$, so $A = \{\omega \in \Omega : u(\omega) > 0\} \in \Sigma$. If $\mu(A) > 0$, then there exists $\varepsilon > 0$ such that $A_\varepsilon = \{\omega \in \Omega : u(\omega) \geq \varepsilon\} \in \Sigma$ satisfies $|\mu|(A_\varepsilon) > 0$. But then $u_n \geq \varepsilon \mathbb{1}_{A_\varepsilon}$ for each $n \in \mathbb{N}$, contradicting $u_n \downarrow 0$ in $B_{\mathbb{R}}(\Sigma)$. Hence, by Lebesgue Dominated Convergence theorem, we get $\int_\Omega u_n d|\mu| \rightarrow 0$, so $\int_\Omega f_n d\mu \rightarrow 0$. $\square$

By $B(\Sigma)_c^*$ we will denote the space of all σ -order continuous functionals on $B(\Sigma)$. For $\omega \in \Omega$ let δ_ω denote the evaluation functional on $B(\Sigma)$, that is, $\delta_\omega(f) = f(\omega)$ for all $f \in B(\Sigma)$. Then $\delta_\omega \in B(\Sigma)_c^*$ and it follows that $B(\Sigma)_c^*$ separates the points of $B(\Sigma)$.

From Proposition 12 and Lemma 5 it follows that for $\mu \in ba(\Sigma)$, $\Phi_\mu \in B(\Sigma)_c^*$ if and only if $|\Phi_\mu| \in B(\Sigma)_c^*$. Since $ca(\Sigma)$ is an ideal of $ba(\Sigma)$, we see that $B(\Sigma)_c^*$ is an ideal of $B(\Sigma)^*$. For a subset $\mathcal{M}$ of $ba(\Sigma)$ let

$$|\mathcal{M}| = \{|\mu| : \mu \in \mathcal{M}\}. \quad (24)$$

Proposition 13. For a subset $\mathcal{M}$ of $ca(\Sigma)$ the following statements are equivalent:

- (i) $\mathcal{M}$ is relatively $\sigma(ca(\Sigma), B(\Sigma))$ -compact.
- (ii) $\mathcal{M}$ is relatively $\sigma(ca(\Sigma), B(\Sigma))$ -countably compact.
- (iii) $\sup_{\mu \in \mathcal{M}} \|\mu\| < \infty$ and $\mathcal{M}$ is uniformly countably additive.
- (iv) $\sup_{\gamma \in \text{conv}(\text{sol}(\mathcal{M}))} \|\gamma\| < \infty$ and $\text{conv}(\text{sol}(\mathcal{M}))$ is uniformly countably additive.
- (v) $\text{conv}(\text{sol}(\mathcal{M}))$ is relatively weakly compact.
- (vi) $\mathcal{M}$ is relatively weakly compact.

Proof. (i) $\Rightarrow$ (ii) It is obvious.

(ii) $\Rightarrow$ (iii) Assume that $\mathcal{M}$ is relatively $\sigma(ca(\Sigma), B(\Sigma))$ -countably compact. It follows that $\mathcal{M}$ is $\sigma(ca(\Sigma), B(\Sigma))$ -bounded, and since $\{\Phi_\mu : \mu \in \mathcal{M}\} \subset B(\Sigma)^*$, by the uniform boundedness theorem, $\sup_{\mu \in \mathcal{M}} \|\Phi_\mu\| < \infty$, that is, $\sup_{\mu \in \mathcal{M}} \|\mu\| < \infty$.

Assume, on the contrary, that $\mathcal{M}$ is not uniformly countably additive. Then there exist a sequence (A_n) in Σ , a sequence (μ_n) in $\mathcal{M}$ and $\delta > 0$ such that $A_n \downarrow \emptyset$ and $|\mu_n(A_n)| \geq \delta$ for all $n \in \mathbb{N}$. Let $\mathcal{C} = \{\mu_n : n \in \mathbb{N}\}$. Putting $\lambda = \sum_{n=1}^\infty (1/2^n)(|\mu_n|/(1 + \|\mu_n\|))$, we see that $\lambda \in ca(\Sigma)^+$ and $\mu_n \ll \lambda$ for all $n \in \mathbb{N}$. Hence, $\mathcal{C}$ is a relatively $\sigma(ca_\lambda(\Sigma), B(\Sigma))$ -countably compact subset of $ca_\lambda(\Sigma)$ (= the space of all λ -absolutely continuous measures in $ca(\Sigma)$). In view of the Radon-Nikodym theorem, for each $\mu \in ca_\lambda(\Sigma)$ there exists uniquely $f_\mu \in L^1(\lambda)$ such that $\mu(A) = \int_A f_\mu d\lambda$ for each $A \in \Sigma$. It follows that for every $g \in L^\infty(\lambda)$ we have $\int_\Omega f_\mu g d\lambda = \int_\Omega g d\mu$. Hence, the linear mapping $F_\lambda : ca_\lambda(\Sigma) \ni$

$\mu \mapsto f_\mu \in L^1(\lambda)$ is $(\sigma(ca_\lambda(\Sigma), B(\Sigma)), \sigma(L^1(\lambda), L^\infty(\lambda)))$ -continuous. Thus, the set $F_\lambda(\mathcal{E}) = \{f_{\mu_n} : n \in \mathbb{N}\}$ is relatively $\sigma(L^1(\lambda), L^\infty(\lambda))$ -countably compact in $L^1(\lambda)$. Hence, by the Eberlein-Smulian theorem $\{f_{\mu_n} : n \in \mathbb{N}\}$ is relatively $\sigma(L^1(\lambda), L^\infty(\lambda))$ -compact in $L^1(\lambda)$. According to the Dunford-Pettis theorem (see [5, Chapter 3, Theorem 15]), the set $\{f_{\mu_n} : n \in \mathbb{N}\}$ in $L^1(\lambda)$ is uniformly integrable. Since $A_n \downarrow \emptyset$, we have that $\lambda(A_n) \rightarrow 0$, and hence $\mu(A_n) = \int_{A_n} f_{\mu_n} d\lambda \rightarrow 0$, contradicting $|\mu_n(A_n)| \geq \delta$ for $n \in \mathbb{N}$.

(iii) $\Rightarrow$ (iv) Assume that $\sup_{\mu \in \mathcal{M}} \|\mu\| < \infty$ and $\mathcal{M}$ is uniformly countably additive. Then, $|\mathcal{M}|$ is uniformly countably additive (see [5, Chapter 1, Proposition 17]). Note that for $A \in \Sigma$, $\sup_{\nu \in \text{sol}(\mathcal{M})} |\nu|(A) = \sup_{\mu \in \mathcal{M}} |\mu|(A)$. It follows that $\sup_{\nu \in \text{sol}(\mathcal{M})} \|\nu\| < \infty$ and $\text{sol}(\mathcal{M})$ is uniformly countably additive. Moreover, one can easily observe that for $A \in \Sigma$, $\sup_{\nu \in \text{conv}(\text{sol}(\mathcal{M}))} |\nu|(A) = \sup_{\nu \in \text{sol}(\mathcal{M})} |\nu|(A)$ and $\sup_{\nu \in \text{conv}(\text{sol}(\mathcal{M}))} \|\nu\| = \sup_{\nu \in \text{sol}(\mathcal{M})} \|\nu\|$. It follows that $\text{conv}(\text{sol}(\mathcal{M}))$ is uniformly bounded and uniformly countably additive.

(iv) $\Rightarrow$ (v) It is well known (see [11, Chapter 7, Theorem 13]).

(v) $\Rightarrow$ (vi) $\Rightarrow$ (i) It is obvious. $\square$

Recall that a locally convex space (L, η) is said to be strongly Mackey if every relatively $\sigma(L'_\eta, L)$ -countably compact set in L'_η is η -equicontinuous.

Corollary 14. *The space $(B(\Sigma), \tau(B(\Sigma), ca(\Sigma)))$ is strongly Mackey.*

Proof. Let $\mathcal{M}$ be a relatively $\sigma(ca(\Sigma), B(\Sigma))$ -countably compact set in $ca(\Sigma)$. Then, by Proposition 13, the set $\text{cl}_\sigma(\text{abs conv } \mathcal{M})$ is absolutely convex and $\sigma(ca(\Sigma), B(\Sigma))$ -compact. Hence, ${}^0(\text{cl}_\sigma(\text{abs conv } \mathcal{M}))$ is a $\tau(B(\Sigma), ca(\Sigma))$ -neighborhood of 0. It follows that $\mathcal{M}$ is $\tau(B(\Sigma), ca(\Sigma))$ -equicontinuous. $\square$

Now we distinguish three natural classes of linear topologies on $B(\Sigma)$.

Definition 15. A locally solid topology τ on $B(\Sigma)$ is said to be

- (i) σ -Lebesgue if $u_n \rightarrow 0$ for τ whenever (u_n) is a sequence in $B(\Sigma)^+$ such that $u_n \downarrow 0$ in $B_\mathbb{R}(\Sigma)$ (in the vector lattice sense).
- (ii) a σ -smooth topology if $f_n \rightarrow 0$ for τ whenever (f_n) is uniformly bounded sequence in $B(\Sigma)$ such that $f_n(\omega) \rightarrow 0$ for all $\omega \in \Omega$.
- (iii) a σ -Dini topology if $u_n \rightarrow 0$ for τ whenever (u_n) is a sequence in $B(\Sigma)^+$ such that $u_n(\omega) \downarrow 0$ for all $\omega \in \Omega$.

Note that every σ -Lebesgue topology on $B(\Sigma)$ is σ -smooth, and every σ -smooth topology on $B(\Sigma)$ is σ -Dini.

Now, we characterize the Mackey topology $\tau(B(\Sigma), ca(\Sigma))$ on $B(\Sigma)$.

Theorem 16. (i) $\tau(B(\Sigma), ca(\Sigma))$ is a locally convex-solid topology.

(ii) $\tau(B(\Sigma), ca(\Sigma))$ is the finest locally convex σ -Lebesgue topology on $B(\Sigma)$.

(iii) $\tau(B(\Sigma), ca(\Sigma))$ is the finest locally convex σ -smooth topology on $B(\Sigma)$.

(iv) $\tau(B(\Sigma), ca(\Sigma))$ is the finest locally convex σ -Dini topology on $B(\Sigma)$.

Proof. (i) Let V be an absolutely convex and closed neighborhood of 0 for $\tau(B(\Sigma), ca(\Sigma))$. Then, the polar V^0 of V (with respect to the solid dual system $\langle B(\Sigma), ca(\Sigma) \rangle$) is $\sigma(ca(\Sigma), B(\Sigma))$ -compact. In view of Proposition 13 $\text{conv}(\text{sol}(V^0))$ is relatively $\sigma(ca(\Sigma), B(\Sigma))$ -compact and hence $\text{cl}_\sigma(\text{conv}(\text{sol}(V^0)))$ (the closure is taken in $\sigma(ca(\Sigma), B(\Sigma))$) is absolutely convex and $\sigma(ca(\Sigma), B(\Sigma))$ -compact. Hence,

$${}^0(\text{cl}_\sigma(\text{conv}(\text{sol}(V^0)))) \subset {}^0(\text{sol}(V^0)) \subset (V^0)^\circ = V. \quad (25)$$

It follows that ${}^0(\text{sol}(V^0))$ is a convex and solid neighborhood of 0 for $\tau(B(\Sigma), ca(\Sigma))$ (see Proposition 8). This means that $\tau(B(\Sigma), ca(\Sigma))$ is a locally convex-solid topology on $B(\Sigma)$.

(ii) Now we shall show that $\tau(B(\Sigma), ca(\Sigma))$ is a σ -Lebesgue topology. Assume that (u_n) is a sequence in $B(\Sigma)^+$ such that $u_n \downarrow 0$ in the vector lattice $B_\mathbb{R}(\Sigma)$. Let $\mathcal{M}$ be a relatively $\sigma(ca(\Sigma), B(\Sigma))$ -compact set in $ca(\Sigma)$. Then $\mathcal{M}$ is uniformly bounded and uniformly countably additive (see Proposition 13), and it follows that $|\mathcal{M}|$ is uniformly countably additive. Hence, by [10, Theorem 3] $|\mathcal{M}|$ is $\tau(B_\mathbb{R}(\Sigma), ca_\mathbb{R}(\Sigma))$ -equicontinuous. Hence, the polar ${}^0_\mathbb{R}|\mathcal{M}|$ of $|\mathcal{M}|$ (with respect to the dual pair $\langle B_\mathbb{R}(\Sigma), ca_\mathbb{R}(\Sigma) \rangle$) is a $\tau(B_\mathbb{R}(\Sigma), ca_\mathbb{R}(\Sigma))$ -neighborhood of 0 in $B_\mathbb{R}(\Sigma)$. Let $\varepsilon > 0$ be given. Since $\tau(B_\mathbb{R}(\Sigma), ca_\mathbb{R}(\Sigma))$ is a σ -Lebesgue topology on $B_\mathbb{R}(\Sigma)$ (see [6, Example 18, page 178], [6, Corollary 6.29 and Theorem 3.20]), there exists $n_\varepsilon \in \mathbb{N}$ such that $|u_n| \in \varepsilon({}^0_\mathbb{R}|\mathcal{M}|)$ for $n \geq n_\varepsilon$. Hence for $n \geq n_\varepsilon$ we get

$$\begin{aligned} p_\mathcal{M}(u_n) &= \sup \left\{ \left| \int_\Omega u_n d\mu \right| : \mu \in \mathcal{M} \right\} \\ &\leq \sup \left\{ \int_\Omega u_n d|\mu| : \mu \in \mathcal{M} \right\} \leq \varepsilon. \end{aligned} \quad (26)$$

It follows that $f_n \rightarrow 0$ for $\tau(B(\Sigma), ca(\Sigma))$, and this means that $\tau(B(\Sigma), ca(\Sigma))$ is a σ -Lebesgue topology on $B(\Sigma)$.

Now assume that τ is a locally convex-solid σ -Lebesgue topology on $B(\Sigma)$. Then $B(\Sigma)'_\tau \subset B(\Sigma)^*_c$, and hence, $\tau \subset \tau(B(\Sigma), B(\Sigma)'_\tau) \subset \tau(B(\Sigma), B(\Sigma)^*_c) = \tau(B(\Sigma), ca(\Sigma))$.

(iii) From (ii) it follows that $\tau(B(\Sigma), ca(\Sigma))$ is a σ -smooth topology.

Now let τ be a locally convex σ -smooth topology on $B(\Sigma)$. Then by Proposition 12 we get $B(\Sigma)'_\tau \subset B(\Sigma)^*_c$. Hence, $\tau \subset \tau(B(\Sigma), B(\Sigma)'_\tau) \subset \tau(B(\Sigma), B(\Sigma)^*_c) = \tau(B(\Sigma), ca(\Sigma))$.

(iv) Clearly $\tau(B(\Sigma), ca(\Sigma))$ is a σ -Dini topology on $B(\Sigma)$. Assume that τ is a locally convex σ -Dini topology on $B(\Sigma)$. In view of Proposition 12, $B(\Sigma)'_\tau \subset B(\Sigma)^*_c$ and it follows that $\tau \subset \tau(B(\Sigma), ca(\Sigma))$. $\square$

A characterization of relative weak compactness in the Banach space $ca(\Sigma)$ in terms of uniform boundedness and

uniform countable additivity of measures belongs to the classical results in measure theory (see [11, Chapter 7, Theorem 13], [8, Theorem 11.6], [12, Theorem 1.1]). Now, using the properties of the Mackey topology $\tau(B(\Sigma), ca(\Sigma))$ we are ready to add Grothendieck type conditions to this characterization.

Corollary 17. *For a subset $\mathcal{M}$ of $ca(\Sigma)$ the following statements are equivalent:*

- (i) $\mathcal{M}$ is a relatively weakly compact.
- (ii) $\sup_{\mu \in \mathcal{M}} \|\mu\| < \infty$ and $\int_{\Omega} f_n d\mu \rightarrow 0$ uniformly for $\mu \in \mathcal{M}$ whenever (f_n) is a uniformly bounded sequence in $B(\Sigma)$ such that $f_n(\omega) \rightarrow 0$ for all $\omega \in \Omega$.
- (iii) $\sup_{\mu \in \mathcal{M}} \|\mu\| < \infty$ and $\int_{\Omega} u_n d\mu \rightarrow 0$ uniformly for $\mu \in \mathcal{M}$ whenever (u_n) is a sequence in $B(\Sigma)^+$ such that $u_n(\omega) \downarrow 0$ for all $\omega \in \Omega$.

Proof. (i) $\Rightarrow$ (ii) Assume that $\mathcal{M}$ is relatively weakly compact. Then, in view of Corollary 14 $\mathcal{M}$ is bounded and $\tau(B(\Sigma), ca(\Sigma))$ -equicontinuous. Let (f_n) be a uniformly bounded sequence in $B(\Sigma)$ such that $f_n(\omega) \rightarrow 0$ for all $\omega \in \Omega$. Since $\tau(B(\Sigma), ca(\Sigma))$ is a σ -smooth topology on $B(\Sigma)$ (see Theorem 16), we obtain that $f_n \rightarrow 0$ for $\tau(B(\Sigma), ca(\Sigma))$. Now let $\varepsilon > 0$ be given. Then, there exists a $\tau(B(\Sigma), ca(\Sigma))$ -neighborhood V of 0 such that $\sup_{\mu \in \mathcal{M}} |\int_{\Omega} f d\mu| \leq \varepsilon$ for all $f \in V$. Hence, there exists $n_{\varepsilon} \in \mathbb{N}$ such that $f_n \in V$ for $n \geq n_{\varepsilon}$, and it follows that $\sup_{\mu \in \mathcal{M}} |\int_{\Omega} f_n d\mu| \leq \varepsilon$ for $n \geq n_{\varepsilon}$, as desired.

(ii) $\Rightarrow$ (iii) It is obvious.

(iii) $\Rightarrow$ (i) Assume that (iv) holds and let $A_n \downarrow \emptyset$, $(A_n) \subset \Sigma$. Then, $\mathbb{1}_{A_n}(\omega) \downarrow 0$ for all $\omega \in \Omega$. Hence $\sup_{\mu \in \mathcal{M}} |\mu(A_n)| = \sup_{\mu \in \mathcal{M}} |\int_{\Omega} \mathbb{1}_{A_n} d\mu| \rightarrow 0$, that is, $\mathcal{M}$ is uniformly countably additive. Hence, $\mathcal{M}$ is relatively weakly compact (see [11, Chapter 7, Theorem 13]). $\square$

Remark 18. One can observe that Corollary 17 is closely related to a Grothendieck's characterization of relative weak compactness in the space of bounded complex Radon measures on a locally compact space (see [13, Theorem 2]).

Now we define a class of linear operators on $B(\Sigma)$.

Definition 19. Let (E, ξ) be a locally convex Hausdorff space (briefly, lchS). A linear operator $T : B(\Sigma) \rightarrow E$ is said to be σ -order continuous if for each sequence (f_n) in $B(\Sigma)$, $|f_n| \xrightarrow{(o)} 0$ in $B_{\mathbb{R}}(\Sigma)$ implies $T(f_n) \rightarrow 0$ for ξ .

For terminology and basic results concerning the integration with respect to vector measures, we refer the reader to [14–17].

Let (E, ξ) be a quasicomplete lchS. Let $m : \Sigma \rightarrow E$ be a ξ -bounded measure (i.e., the range of m is ξ -bounded in E). Given $f \in B(\Sigma)$, let (s_n) be a sequence of Σ -simple scalar functions that converges uniformly to f on Ω . Following [14, Definition 1] we say that f is m -integrable and define

$$\int_{\Omega} f dm := \xi - \lim \int_{\Omega} s_n dm. \quad (27)$$

The $\int_{\Omega} f dm$ is well defined (see [14, Lemma 5]) and the map $T_m : B(\Sigma) \rightarrow E$ given by $T_m(f) = \int_{\Omega} f dm$ is $(\|\cdot\|, \xi)$ -continuous and linear, and for each $e' \in E'_{\xi}$

$$e' \left(\int_{\Omega} f dm \right) = \int_{\Omega} f d(e' \circ m) \quad \text{for } f \in B(\Sigma) \quad (28)$$

(see [14, Lemma 5]). Conversely, let $T : B(\Sigma) \rightarrow E$ be a $(\|\cdot\|, \xi)$ -continuous linear operator, and let $m(A) = T(\mathbb{1}_A)$ for $A \in \Sigma$. Then, $m : \Sigma \rightarrow E$ is a ξ -bounded vector measure, called the *representing measure* of T and $T_m(f) = T(f)$ for $f \in B(\Sigma)$ (see [14, Definition 2]).

Recall that a measure $m : \Sigma \rightarrow E$ is said to be ξ -countably additive if $m(A_n) \rightarrow 0$ for ξ whenever $A_n \downarrow \emptyset$, $(A_n) \subset \Sigma$.

The following characterization of σ -order continuous operators from $B(\Sigma)$ into a quasicomplete lchS (E, ξ) displays the close connection between the Mackey topology $\tau(B(\Sigma), ca(\Sigma))$ on $B(\Sigma)$ and E -valued ξ -countably additive measures.

Proposition 20. *Assume that (E, ξ) is a quasicomplete lchS. Then for a ξ -bounded measure $m : \Sigma \rightarrow E$ the following statements are equivalent:*

- (i) T_m is $(\tau(B(\Sigma), ca(\Sigma)), \tau(E, E'_{\xi}))$ -continuous.
- (ii) T_m is $(\tau(B(\Sigma), ca(\Sigma)), \xi)$ -continuous.
- (iii) T_m is sequentially $(\tau(B(\Sigma), ca(\Sigma)), \xi)$ -continuous.
- (iv) T_m is σ -order continuous.
- (v) $T_m(f_n) \rightarrow 0$ for ξ whenever (f_n) is a uniformly bounded sequence in $B(\Sigma)$ such that $f_n(\omega) \rightarrow 0$ for all $\omega \in \Omega$.
- (vi) $T_m(u_n) \rightarrow 0$ for ξ for each sequence (u_n) in $B(\Sigma)^+$ such that $u_n(\omega) \rightarrow 0$ for all $\omega \in \Omega$.
- (vii) m is ξ -countably additive.

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) It is obvious.

(iii) $\Rightarrow$ (iv) Assume that T_m is sequentially $(\tau(B(\Sigma), ca(\Sigma)), \xi)$ -continuous, and let (f_n) be a sequence in $B(\Sigma)$ such that $|f_n| \xrightarrow{(o)} 0$ in $B_{\mathbb{R}}(\Sigma)$. Then, by Theorem 16, $f_n \rightarrow 0$ for $\tau(B(\Sigma), ca(\Sigma))$. Hence, $T_m(f_n) \rightarrow 0$ for ξ , that is, T is σ -order continuous.

(iv) $\Rightarrow$ (v) Assume that T_m is σ -order continuous, and assume that (f_n) is a uniformly bounded sequence in $B(\Sigma)$ such that $f_n(\omega) \rightarrow 0$ for all $\omega \in \Omega$. Then $|f_n| \xrightarrow{(o)} 0$ in $B_{\mathbb{R}}(\Sigma)$ (see the proof of Proposition 12). Hence, $T_m(f_n) \rightarrow 0$ for ξ , as desired.

(v) $\Rightarrow$ (vi) It is obvious.

(vi) $\Rightarrow$ (vii) Assume that (vi) holds, and let $A_n \downarrow \emptyset$, $(A_n) \subset \Sigma$. Then, $\mathbb{1}_{A_n}(\omega) \downarrow 0$ for all $\omega \in \Omega$, and it follows that $\mu(A_n) = T_m(\mathbb{1}_{A_n}) \rightarrow 0$ for ξ . This means that m is ξ -countably additive.

(vii) $\Rightarrow$ (i) Assume that m is ξ -countably additive. Then $e' \circ m \in ca(\Sigma)$ for each $e' \in E'_{\xi}$, and in view of (28) and Proposition 12 we get $e' \circ T_m \in B(\Sigma)^*_c$ for each $e' \in E'_{\xi}$. Hence, T_m is $(\sigma(B(\Sigma), B(\Sigma)^*_c), \sigma(E, E'_{\xi}))$ -continuous (see [7, Theorem 9.26]). It follows that T_m is $(\tau(B(\Sigma), B(\Sigma)^*_c), \tau(E, E'_{\xi}))$ -continuous (see [7, Example 11, page 149]). $\square$

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