Research Article

Boundedness of p -Adic Hardy Operators and Their Commutators on p -Adic Central Morrey and BMO Spaces

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We obtain the sharp bounds of p -adic Hardy operators on p -adic central Morrey spaces and p -adic λ -central BMO spaces, respectively. We also establish the λ -central BMO estimates for commutators of p -adic Hardy operators on p -adic central Morrey spaces.

1. Introduction

In the past decades, the field of p -adic numbers has been intensively used in *theoretical and mathematical physics* (see [1–9] and references therein). As a consequence, new mathematical problems have emerged, among which we refer to [10, 11] for Riesz potentials [12–16], for p -adic pseudodifferential equations, and so forth. In the past few years, there is an increasing interest in the study of *harmonic analysis on p -adic field* and their various generalizations and the related theory of operators and spaces; see, for example [17–27].

For a prime number p , let $\mathbb{Q}_p$ be the field of p -adic numbers. It is defined as the completion of the field of rational numbers $\mathbb{Q}$ with respect to the non-Archimedean p -adic norm $|\cdot|_p$. This norm is defined as follows: $|0|_p = 0$; if any nonzero rational number x is represented as $x = p^\gamma(m/n)$, where γ is an integer and integers m, n are indivisible by p , then $|x|_p = p^{-\gamma}$. It is easy to see that the norm satisfies the following properties:

$$|xy|_p = |x|_p |y|_p, \quad |x + y|_p \leq \max\{|x|_p, |y|_p\}. \quad (1)$$

Moreover, if $|x|_p \neq |y|_p$, then $|x + y|_p = \max\{|x|_p, |y|_p\}$. It is well known that $\mathbb{Q}_p$ is a typical model of non-Archimedean local fields. From the standard p -adic analysis [7], we see

that any nonzero p -adic number $x \in \mathbb{Q}_p$ can be uniquely represented in the canonical series as follows:

$$x = p^\gamma \sum_{j=0}^{\infty} a_j p^j, \quad \gamma = \gamma(x) \in \mathbb{Z}, \quad (2)$$

where a_j are integers, $0 \leq a_j \leq p-1$, and $a_0 \neq 0$. The series (2) converges in the p -adic norm since $|a_j p^j|_p = p^{-j}$.

The space $\mathbb{Q}_p^n$ consists of points $x = (x_1, x_2, \dots, x_n)$, where $x_j \in \mathbb{Q}_p$, $j = 1, 2, \dots, n$. The p -adic norm on $\mathbb{Q}_p^n$ is

$$|x|_p := \max_{1 \leq j \leq n} |x_j|_p, \quad x \in \mathbb{Q}_p^n. \quad (3)$$

Denote by $B_\gamma(a) = \{x \in \mathbb{Q}_p^n : |x - a|_p \leq p^\gamma\}$ the ball with center at $a \in \mathbb{Q}_p^n$ and radius p^γ and by $S_\gamma(a) := \{x \in \mathbb{Q}_p^n : |x - a|_p = p^\gamma\}$ the sphere with center at $a \in \mathbb{Q}_p^n$ and radius p^γ , $\gamma \in \mathbb{Z}$. It is clear that $S_\gamma(a) = B_\gamma(a) \setminus B_{\gamma-1}(a)$, and

$$B_\gamma(a) = \bigcup_{k \leq \gamma} S_k(a), \quad (4)$$

$$\bigcup_{\gamma=-\infty}^{+\infty} B_\gamma(a) = \bigcup_{\gamma=-\infty}^{+\infty} S_\gamma(a) = \mathbb{Q}_p^n.$$

We set $B_\gamma(0) = B_\gamma$ and $S_\gamma(0) = S_\gamma$.

Since $\mathbb{Q}_p^n$ is a locally compact commutative group under addition, it follows from the standard analysis that there exists

a Haar measure dx on $\mathbb{Q}_p^n$, which is unique up to positive constant multiple and is translation invariant. We normalize the measure dx by the equality

$$\int_{B_0(0)} dx = |B_0(0)|_H = 1, \quad (5)$$

where $|E|_H$ denotes the Haar measure of a measurable subset E of $\mathbb{Q}_p^n$. By simple calculation, we can obtain that

$$|B_\gamma(a)|_H = p^{\gamma n}, \quad |S_\gamma(a)|_H = p^{\gamma n} (1 - p^{-n}) \quad (6)$$

for any $a \in \mathbb{Q}_p^n$. For a more complete introduction to the p -adic field, see [27] or [7].

The well-known Hardy's integral inequality [28] tells us that, for $1 < q < \infty$,

$$\|Hf\|_{L^q(\mathbb{R}^+)} \leq \frac{q}{q-1} \|f\|_{L^q(\mathbb{R}^+)}, \quad (7)$$

where the classical Hardy operator is defined by

$$Hf(x) := \frac{1}{x} \int_0^x f(t) dt \quad (8)$$

for nonnegative integral function f on $\mathbb{R}^+$, and the constant $q/(q-1)$ is the best possible. Thus the norm of Hardy operator on $L^q(\mathbb{R}^+)$ is

$$\|H\|_{L^q(\mathbb{R}^+) \rightarrow L^q(\mathbb{R}^+)} = \frac{q}{q-1}. \quad (9)$$

Faris [29] introduced the following n -dimensional Hardy operator, for nonnegative function f on $\mathbb{R}^n$,

$$\mathcal{H}f(x) := \frac{1}{\Omega_n |x|^n} \int_{|y| < |x|} f(y) dy, \quad (10)$$

where Ω_n is the volume of the unit ball in $\mathbb{R}^n$. Christ and Grafakos [30] obtained that the norm of $\mathcal{H}$ on $L^q(\mathbb{R}^n)$ is

$$\|\mathcal{H}\|_{L^q(\mathbb{R}^n) \rightarrow L^q(\mathbb{R}^n)} = \frac{q}{q-1}, \quad (11)$$

which is the same as that of the 1-dimensional Hardy operator.

In [31], Fu et al. obtained the precise norm of m -linear Hardy operators on weighted Lebesgue spaces and central Morrey spaces. Fu et al. [32] introduced p -adic Hardy operators and got the sharp estimates of p -adic Hardy operators on p -adic weighted Lebesgue spaces. Moreover, they proved that the commutators generated by the p -adic Hardy operators and the central BMO functions are bounded on p -adic weighted Lebesgue spaces and p -adic Herz spaces see; [33] for more information about Herz spaces. Ren and Tao [34] Yu and Lu [35] studied the boundedness of commutators of Hardy type on some spaces.

Inspired by these results, in this paper we will establish the sharp estimates of p -adic Hardy operators on p -adic central Morrey and λ -central BMO spaces. Furthermore, we will discuss the boundedness for commutators of p -adic Hardy operators and λ -central BMO functions on p -adic central Morrey spaces.

Definition 1. For a function f on $\mathbb{Q}_p^n$, we define the p -adic Hardy operator as follows:

$$\mathcal{H}^p f(x) = \frac{1}{|x|_p^n} \int_{B(0, |x|_p)} f(t) dt, \quad x \in \mathbb{Q}_p^n \setminus \{0\}, \quad (12)$$

where $B(0, |x|_p)$ is a ball in $\mathbb{Q}_p^n$ with center at $0 \in \mathbb{Q}_p^n$ and radius $|x|_p$.

Morrey [36] introduced the $L^{q,\lambda}(\mathbb{R}^n)$ spaces to study the local behavior of solutions to second-order elliptic partial differential equations. The p -adic Morrey space is defined as follows.

Definition 2. Let $1 \leq q < \infty$ and let $\lambda \geq -1/q$. The p -adic Morrey space $L^{q,\lambda}(\mathbb{Q}_p^n)$ is defined by

$$L^{q,\lambda}(\mathbb{Q}_p^n) = \left\{ f \in L^q_{\text{loc}}(\mathbb{Q}_p^n) : \|f\|_{L^{q,\lambda}(\mathbb{Q}_p^n)} < \infty \right\}, \quad (13)$$

where

$$\|f\|_{L^{q,\lambda}(\mathbb{Q}_p^n)} = \sup_{a \in \mathbb{Q}_p^n, \gamma \in \mathbb{Z}} \left(\frac{1}{|B_\gamma(a)|_H^{1+\lambda q}} \int_{B_\gamma(a)} |f(x)|^q dx \right)^{1/q}. \quad (14)$$

Remark 3. It is clear that $L^{q,-1/q}(\mathbb{Q}_p^n) = L^q(\mathbb{Q}_p^n)$, $L^{q,0}(\mathbb{Q}_p^n) = L^\infty(\mathbb{Q}_p^n)$.

For some recent developments of Morrey spaces and their related function spaces on $\mathbb{R}^n$, we refer the reader to [37]. In 2000, Alvarez et al. [38] studied the relationship between central BMO spaces and Morrey spaces. Furthermore, they introduced λ -central bounded mean oscillation spaces and central Morrey spaces, respectively. Next, we introduce their p -adic versions.

Definition 4. Let $\lambda \in \mathbb{R}$ and let $1 < q < \infty$. The p -adic central Morrey space $\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)$ is defined by

$$\|f\|_{\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)} := \sup_{\gamma \in \mathbb{Z}} \left(\frac{1}{|B_\gamma|_H^{1+\lambda q}} \int_{B_\gamma} |f(x)|^q dx \right)^{1/q} < \infty, \quad (15)$$

where $B_\gamma = B_\gamma(0)$.

Remark 5. It is clear that

$$\dot{B}^{q,-1/q}(\mathbb{Q}_p^n) = L^q(\mathbb{Q}_p^n). \quad (16)$$

When $\lambda < -1/q$, the space $\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)$ reduces to $\{0\}$; therefore, we can only consider the case $\lambda \geq -1/q$. If $1 \leq q_1 < q_2 < \infty$, by Hölder's inequality,

$$\dot{B}^{q_2,\lambda}(\mathbb{Q}_p^n) \subset \dot{B}^{q_1,\lambda}(\mathbb{Q}_p^n) \quad (17)$$

for $\lambda \in \mathbb{R}$.

Definition 6. Let $\lambda < 1/n$ and let $1 < q < \infty$. The space $\text{CBMO}^{q,\lambda}(\mathbb{Q}_p^n)$ is defined by the condition

$$\|f\|_{\text{CBMO}^{q,\lambda}(\mathbb{Q}_p^n)} := \sup_{\gamma \in \mathbb{Z}} \left(\frac{1}{|B_\gamma|_H^{1+\lambda q}} \int_{B_\gamma} |f(x) - f_{B_\gamma}|^q dx \right)^{1/q} < \infty, \quad (18)$$

where $f_{B_\gamma} = (1/|B_\gamma|_H) \int_{B_\gamma} f(x) dx$.

Remark 7. When $\lambda = 0$, the space $\text{CBMO}^{q,\lambda}(\mathbb{Q}_p^n)$ is just $\text{CBMO}^q(\mathbb{Q}_p^n)$, which is defined in [32]. If $1 \leq q_1 < q_2 < \infty$, by Hölder's inequality,

$$\text{CBMO}^{q_2,\lambda}(\mathbb{Q}_p^n) \subset \text{CBMO}^{q_1,\lambda}(\mathbb{Q}_p^n) \quad (19)$$

for $\lambda \in \mathbb{R}$. By the standard proof as that in $\mathbb{R}^n$, we can see that

$$\|f\|_{\text{CBMO}^{q,\lambda}(\mathbb{Q}_p^n)} \sim \sup_{\gamma \in \mathbb{Z}} \inf_{c \in \mathbb{C}} \left(\frac{1}{|B_\gamma|_H^{1+\lambda q}} \int_{B_\gamma} |f(x) - c|^q dx \right)^{1/q}. \quad (20)$$

Remark 8. The formulas (18) and (15) yield that $\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)$ is a Banach space continuously included in $\text{CBMO}^{q,\lambda}(\mathbb{Q}_p^n)$.

In Section 2, we obtain the sharp estimates of p -adic Hardy operators on p -adic central Morrey spaces and p -adic λ -central BMO spaces. Analogous result is also established for p -adic Morrey spaces. In Section 3, we discuss the boundedness of commutators generated by p -adic Hardy operators and p -adic λ -central BMO functions on p -adic central Morrey spaces.

We should note that in Euclidean space, when estimating the Hardy operator, one usually discusses its restriction on radical functions. However, on p -adic field, we will consider its restriction on the functions f with $f(x) = f(|x|_p^{-1})$ instead.

Throughout this paper the letter C will be used to denote various constants, and the various uses of the letter do not, however, denote the same constant.

2. Sharp Estimates of p -Adic Hardy Operator

We get the following precise norms of p -adic Hardy operators on p -adic central Morrey spaces and p -adic λ -central BMO spaces.

Theorem 9. Let $1 < q < \infty$ and let $\lambda \geq -1/q$. Then

$$\|\mathcal{H}^p\|_{\dot{B}^{q,\lambda}(\mathbb{Q}_p^n) \rightarrow \dot{B}^{q,\lambda}(\mathbb{Q}_p^n)} = \frac{1 - p^{-n}}{1 - p^{-n(1+\lambda)}}. \quad (21)$$

Theorem 10. Let $1 < q < \infty$ and let $-1/q < \lambda < 1/n$. Then

$$\|\mathcal{H}^p\|_{\text{CBMO}^{q,\lambda}(\mathbb{Q}_p^n) \rightarrow \text{CBMO}^{q,\lambda}(\mathbb{Q}_p^n)} = \frac{1 - p^{-n}}{1 - p^{-n(1+\lambda)}}. \quad (22)$$

Corollary 11. Let $1 < q < \infty$. Then

$$\|\mathcal{H}^p\|_{\text{CBMO}^q(\mathbb{Q}_p^n) \rightarrow \text{CBMO}^q(\mathbb{Q}_p^n)} = 1. \quad (23)$$

Let $\mathcal{L}^{q,\lambda}(\mathbb{Q}_p^n)$ denote the subspace of $L^{q,\lambda}(\mathbb{Q}_p^n)$ consisting of all functions $f \in L^{q,\lambda}(\mathbb{Q}_p^n)$ with $f(x) = f(|x|_p^{-1})$. We obtain the sharp estimate of p -adic Hardy operator from $\mathcal{L}^{q,\lambda}(\mathbb{Q}_p^n)$ to $L^{q,\lambda}(\mathbb{Q}_p^n)$.

Theorem 12. Suppose that $1 < q < \infty$, $-1/q < \lambda < 0$. Then $\mathcal{H}^p$ maps $\mathcal{L}^{q,\lambda}(\mathbb{Q}_p^n)$ to $L^{q,\lambda}(\mathbb{Q}_p^n)$ with norm

$$\|\mathcal{H}^p\|_{\mathcal{L}^{q,\lambda}(\mathbb{Q}_p^n) \rightarrow L^{q,\lambda}(\mathbb{Q}_p^n)} = \frac{1 - p^{-n}}{1 - p^{-n(1+\lambda)}}. \quad (24)$$

Proof of Theorem 9. When $\lambda = -1/q$, $\dot{B}^{q,-1/q}(\mathbb{Q}_p^n) = L^q(\mathbb{Q}_p^n)$, by Corollary 2.2 in [32],

$$\|\mathcal{H}^p\|_{\dot{B}^{q,\lambda}(\mathbb{Q}_p^n) \rightarrow \dot{B}^{q,\lambda}(\mathbb{Q}_p^n)} = \frac{1 - p^{-n}}{1 - p^{-n(1-1/q)}}. \quad (25)$$

When $\lambda > -1/q$, we first claim that the operator $\mathcal{H}^p$ and its restriction to the subset of $\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)$, which consist of functions g satisfying $g(x) = g(|x|_p^{-1})$, have the same operator norm on $\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)$.

In fact, for $f \in \dot{B}^{q,\lambda}(\mathbb{Q}_p^n)$, set

$$g(x) = \frac{1}{1 - p^{-n}} \int_{|\xi|_p=1} f(|x|_p^{-1}\xi) d\xi, \quad x \in \mathbb{Q}_p^n. \quad (26)$$

It is easy to see that g satisfies that $g(x) = g(|x|_p^{-1})$ and $\mathcal{H}^p g = \mathcal{H}^p f$. By Hölder's inequality, for $\gamma \in \mathbb{Z}$, we have

$$\begin{aligned} & \frac{1}{|B_\gamma|_H^{1+\lambda q}} \int_{B_\gamma} |g|^q dx \\ &= \frac{1}{|B_\gamma|_H^{1+\lambda q}} \int_{B_\gamma} \left| \frac{1}{1 - p^{-n}} \int_{|\xi|_p=1} f(|x|_p^{-1}\xi) d\xi \right|^q dx \\ &\leq \frac{1}{|B_\gamma|_H^{1+\lambda q}} \\ &\quad \times \int_{B_\gamma} \frac{1}{(1 - p^{-n})^q} \left(\int_{|\xi|_p=1} |f(|x|_p^{-1}\xi)|^q d\xi \right) \\ &\quad \times \left(\int_{|\xi|_p=1} d\xi \right)^{q/q'} dx \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{|B_\gamma|_H^{1+\lambda q}} \int_{B_\gamma} \frac{1}{1-p^{-n}} \left(\int_{|\xi|_p=1} |f(|x|_p^{-1}\xi)|^q d\xi \right) dx \\
&= \frac{1}{|B_\gamma|_H^{1+\lambda q} (1-p^{-n})} \int_{B_\gamma} |x|_p^{-n} dx \int_{|y|_p=|x|_p} |f(y)|^q dy \\
&= \frac{1}{|B_\gamma|_H^{1+\lambda q} (1-p^{-n})} \int_{B_\gamma} |f(y)|^q dy \int_{|x|_p=|y|_p} |x|_p^{-n} dx \\
&= \frac{1}{|B_\gamma|_H^{1+\lambda q}} \int_{B_\gamma} |f(y)|^q dy.
\end{aligned} \tag{27}$$

Therefore,

$$\|g\|_{\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)} \leq \|f\|_{\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)}. \tag{28}$$

Consequently,

$$\frac{\|\mathcal{H}^P f\|_{\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)}}{\|f\|_{\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)}} \leq \frac{\|\mathcal{H}^P g\|_{\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)}}{\|g\|_{\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)}}, \tag{29}$$

which implies the claim. In the following, without loss of generality, we may assume that $f \in \dot{B}^{q,\lambda}(\mathbb{Q}_p^n)$ satisfies $f(x) = f(|x|_p^{-1})$. Then by Minkowski's inequality, we have

$$\begin{aligned}
&\left(\frac{1}{|B_\gamma|_H^{1+\lambda q}} \int_{B_\gamma} |\mathcal{H}^P f(x)|^q dx \right)^{1/q} \\
&= \left(\frac{1}{|B_\gamma|_H^{1+\lambda q}} \int_{B_\gamma} \left| \frac{1}{|x|_p^n} \int_{B(0,|x|_p)} f(t) dt \right|^q dx \right)^{1/q} \\
&= \left(\frac{1}{|B_\gamma|_H^{1+\lambda q}} \int_{B_\gamma} \left| \int_{B_0} f(|y|_p^{-1}y) dy \right|^q dx \right)^{1/q} \\
&\leq \left[\frac{1}{|B_\gamma|_H^{1+\lambda q}} \int_{B_\gamma} \left(\int_{B_0} |f(|y|_p^{-1}x)| dy \right)^q dx \right]^{1/q} \\
&\leq \int_{B_0} \left[\frac{1}{|B_\gamma|_H^{1+\lambda q}} \int_{B_\gamma} |f(|y|_p^{-1}x)|^q dx \right]^{1/q} dy
\end{aligned}$$

$$\begin{aligned}
&\leq \int_{B_0} \left[\frac{1}{|B_\gamma|_H^{1+\lambda q}} \int_{B(0,p^\gamma|y|_p)} |f(z)|^q |y|_p^{-n} dz \right]^{1/q} dy \\
&\leq \int_{B_0} \left[\frac{|y|_p^{n\lambda q}}{|B(0,p^\gamma|y|_p)|_H^{1+\lambda q}} \int_{B(0,p^\gamma|y|_p)} |f(z)|^q dz \right]^{1/q} dy \\
&\leq \|f\|_{\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)} \int_{B_0} |y|_p^{n\lambda} dy \\
&= \|f\|_{\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)} \left(\sum_{k=0}^{\infty} \int_{|y|_p=p^{-k}} p^{-kn\lambda} dy \right) \\
&= \frac{1-p^{-n}}{1-p^{-n(1+\lambda)}} \|f\|_{\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)}.
\end{aligned} \tag{30}$$

Thus,

$$\|\mathcal{H}^P f\|_{\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)} \leq \frac{1-p^{-n}}{1-p^{-n(1+\lambda)}} \|f\|_{\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)}. \tag{31}$$

On the other hand, take $f_0(x) = |x|_p^{n\lambda}$. Then

$$\begin{aligned}
&\frac{1}{|B_\gamma|_H^{1+\lambda q}} \int_{B_\gamma} |x|_p^{n\lambda q} dx \\
&= p^{-n\gamma(1+\lambda q)} \sum_{k=-\infty}^{\gamma} \int_{S_k} p^{nk\lambda q} dx \\
&= (1-p^{-n}) p^{-n\gamma(1+\lambda q)} \sum_{k=-\infty}^{\gamma} p^{nk(1+\lambda q)} \\
&= \frac{1-p^{-n}}{1-p^{-n(1+\lambda q)}},
\end{aligned} \tag{32}$$

where the series converges due to $\lambda > -1/q$. Thus, $f_0 \in \dot{B}^{q,\lambda}(\mathbb{Q}_p^n)$ since

$$\begin{aligned}
\mathcal{H}^P f_0(x) &= \frac{1}{|x|_p^n} \int_{B(0,|x|_p)} |t|_p^{n\lambda} dt = |x|_p^{n\lambda} \int_{B_0} |y|_p^{n\lambda} dy \\
&= \frac{1-p^{-n}}{1-p^{-n(1+\lambda)}} f_0(x).
\end{aligned} \tag{33}$$

Therefore,

$$\|\mathcal{H}^P\|_{\dot{B}^{q,\lambda}(\mathbb{Q}_p^n) \rightarrow \dot{B}^{q,\lambda}(\mathbb{Q}_p^n)} \geq \frac{1-p^{-n}}{1-p^{-n(1+\lambda)}}. \tag{34}$$

Then (31) and (34) imply that

$$\|\mathcal{H}^P\|_{\dot{B}^{q,\lambda}(\mathbb{Q}_p^n) \rightarrow \dot{B}^{q,\lambda}(\mathbb{Q}_p^n)} = \frac{1-p^{-n}}{1-p^{-n(1+\lambda)}}. \tag{35}$$

□

Proof of Theorem 10. As in the proof of Theorem 9, we first show that the operator $\mathcal{H}^p$ and its restriction to the subset of $\text{CBMO}^{q,\lambda}(\mathbb{Q}_p^n)$ consisting of functions g with $g(x) = g(|x|_p^{-1})$ have the same operator norm on $\text{CBMO}^{q,\lambda}(\mathbb{Q}_p^n)$.

In fact, set

$$g(x) = \frac{1}{1-p^{-n}} \int_{|\xi|_p=1} f(|x|_p^{-1}\xi) d\xi, \quad x \in \mathbb{Q}_p^n. \quad (36)$$

Then $g(x) = g(|x|_p^{-1})$ and $\mathcal{H}^p g = \mathcal{H}^p f$. By change of variable, we get

$$\begin{aligned} g_{B_\gamma} &= \frac{1}{|B_\gamma|_H} \int_{B_\gamma} g(x) dx \\ &= \frac{1}{|B_\gamma|_H} \int_{B_\gamma} \frac{1}{1-p^{-n}} \int_{|\xi|_p=1} f(|x|_p^{-1}\xi) d\xi dx \\ &= \frac{1}{|B_\gamma|_H (1-p^{-n})} \int_{B_\gamma} dx \int_{|y|_p=|x|_p} f(y) |x|_p^{-n} dy \quad (37) \\ &= \frac{1}{|B_\gamma|_H (1-p^{-n})} \int_{B_\gamma} f(y) dy \int_{|x|_p=|y|_p} |x|_p^{-n} dx \\ &= \frac{1}{|B_\gamma|_H} \int_{B_\gamma} f(y) dy = f_{B_\gamma}. \end{aligned}$$

Using Minkowski's inequality and (37), we have

$$\begin{aligned} &\left(\frac{1}{|B_\gamma|_H^{1+\lambda q}} \int_{B_\gamma} |g - g_{B_\gamma}|^q dx \right)^{1/q} \\ &= \left(\frac{1}{|B_\gamma|_H^{1+\lambda q}} \int_{B_\gamma} |g - f_{B_\gamma}|^q dx \right)^{1/q} \\ &= \left(\frac{1}{|B_\gamma|_H^{1+\lambda q}} \right. \\ &\quad \times \left. \int_{B_\gamma} \left| \frac{1}{1-p^{-n}} \int_{|\xi|_p=1} f(|x|_p^{-1}\xi) d\xi - f_{B_\gamma} \right|^q dx \right)^{1/q} \\ &\leq \frac{1}{1-p^{-n}} \left[\frac{1}{|B_\gamma|_H^{1+\lambda q}} \right. \\ &\quad \times \left. \int_{B_\gamma} \left(\int_{|\xi|_p=1} |f(|x|_p^{-1}\xi) - f_{B_\gamma}|^q d\xi \right) dx \right]^{1/q} \end{aligned}$$

$$\begin{aligned} &\leq \frac{1}{1-p^{-n}} \\ &\quad \times \int_{|\xi|_p=1} \left(\frac{1}{|B_\gamma|_H^{1+\lambda q}} \int_{B_\gamma} |f(|x|_p^{-1}\xi) - f_{B_\gamma}|^q dx \right)^{1/q} d\xi \\ &= \frac{1}{1-p^{-n}} \int_{|\xi|_p=1} \left(\frac{1}{|B_\gamma|_H^{1+\lambda q}} \int_{B_\gamma} |f(x) - f_{B_\gamma}|^q dx \right)^{1/q} d\xi \\ &\leq \|f\|_{\text{CBMO}^{q,\lambda}(\mathbb{Q}_p^n)}. \end{aligned} \quad (38)$$

Therefore,

$$\|g\|_{\text{CBMO}^{q,\lambda}(\mathbb{Q}_p^n)} \leq \|f\|_{\text{CBMO}^{q,\lambda}(\mathbb{Q}_p^n)}. \quad (39)$$

We conclude that

$$\frac{\|\mathcal{H}^p f\|_{\text{CBMO}^{q,\lambda}(\mathbb{Q}_p^n)}}{\|f\|_{\text{CBMO}^{q,\lambda}(\mathbb{Q}_p^n)}} \leq \frac{\|\mathcal{H}^p g\|_{\text{CBMO}^{q,\lambda}(\mathbb{Q}_p^n)}}{\|g\|_{\text{CBMO}^{q,\lambda}(\mathbb{Q}_p^n)}}. \quad (40)$$

In the following, without loss of generality, we may assume that $f \in \text{CBMO}^{q,\lambda}(\mathbb{Q}_p^n)$ with $f(x) = f(|x|_p^{-1})$. By Fubini theorem, we have

$$\begin{aligned} &(\mathcal{H}^p f)_{B_\gamma} \\ &= \frac{1}{|B_\gamma|_H} \int_{B_\gamma} \mathcal{H}^p f(x) dx \\ &= \frac{1}{|B_\gamma|_H} \int_{B_\gamma} \int_{B_0} f(|x|_p^{-1}y) dy dx \\ &= \int_{B_0} \left(\frac{1}{|B_\gamma|_H} \int_{B_\gamma} f(|x|_p^{-1}y) dx \right) dy \\ &= \int_{B_0} \left(\frac{1}{|B_\gamma|_H} \int_{B_\gamma} f(|y|_p^{-1}x) dx \right) dy \\ &= \int_{B_0} \left(\frac{1}{|B(0, |y|_p p^\gamma)|_H} \int_{B(0, |y|_p p^\gamma)} f(z) dz \right) dy \\ &= \int_{B_0} f_{B(0, |y|_p p^\gamma)} dy. \end{aligned} \quad (41)$$

Then by Minkowski's inequality, we get

$$\begin{aligned}
& \left(\frac{1}{|B_\gamma|_H^{1+\lambda q}} \int_{B_\gamma} |\mathcal{H}^p f(x) - (\mathcal{H}^p f)_{B_\gamma}|^q dx \right)^{1/q} \\
&= \left(\frac{1}{|B_\gamma|_H^{1+\lambda q}} \int_{B_\gamma} \left| \int_{B_0} f(|x|_p^{-1} y) - f_{B(0, |y|_p p^\gamma)} dy \right|^q dx \right)^{1/q} \\
&\leq \int_{B_0} \left(\frac{1}{|B_\gamma|_H^{1+\lambda q}} \int_{B_\gamma} |f(|x|_p^{-1} y) - f_{B(0, |y|_p p^\gamma)}|^q dx \right)^{1/q} dy \\
&= \int_{B_0} \left(\frac{1}{|B(0, |y|_p p^\gamma)|_H^{1+\lambda q}} \right. \\
&\quad \times \left. \int_{B(0, |y|_p p^\gamma)} |f(z) - f_{B(0, |y|_p p^\gamma)}|^q dz \right)^{1/q} |y|_p^{n\lambda} dy \\
&\leq \|f\|_{\text{CBMO}^{q,\lambda}(\mathbb{Q}_p^n)} \int_{B_0} |y|_p^{n\lambda} dy \\
&= \frac{1 - p^{-n}}{1 - p^{-n(1+\lambda)}} \|f\|_{\text{CBMO}^{q,\lambda}(\mathbb{Q}_p^n)}. \tag{42}
\end{aligned}$$

Namely,

$$\|\mathcal{H}^p f\|_{\text{CBMO}^{q,\lambda}(\mathbb{Q}_p^n)} \leq \frac{1 - p^{-n}}{1 - p^{-n(1+\lambda)}} \|f\|_{\text{CBMO}^{q,\lambda}(\mathbb{Q}_p^n)}. \tag{43}$$

On the other hand, take $f_0(x) = |x|_p^{n\lambda}$. By Remark 8 and (32), for $\lambda > -1/q$, we have $f_0 \in \dot{B}^{q,\lambda}(\mathbb{Q}_p^n) \subset \text{CBMO}^{q,\lambda}(\mathbb{Q}_p^n)$. Then by (33), we get

$$\begin{aligned}
& (\mathcal{H}^p f_0)_{B_\gamma} \\
&= \frac{1}{|B_\gamma|_H} \int_{B_\gamma} \mathcal{H}^p f_0(x) dx \\
&= \frac{1}{|B_\gamma|_H} \int_{B_\gamma} \frac{1 - p^{-n}}{1 - p^{-n(1+\lambda)}} f_0(x) dx \\
&= \frac{1 - p^{-n}}{1 - p^{-n(1+\lambda)}} (f_0)_{B_\gamma}. \tag{44}
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \left(\frac{1}{|B_\gamma|_H^{1+\lambda q}} \int_{B_\gamma} |\mathcal{H}^p f_0(x) - (\mathcal{H}^p f_0)_{B_\gamma}|^q dx \right)^{1/q} \\
&= \left(\frac{1}{|B_\gamma|_H^{1+\lambda q}} \int_{B_\gamma} \left(\frac{1 - p^{-n}}{1 - p^{-n(1+\lambda)}} \right)^q |f_0 - (f_0)_{B_\gamma}|^q dx \right)^{1/q} \\
&= \frac{1 - p^{-n}}{1 - p^{-n(1+\lambda)}} \left(\frac{1}{|B_\gamma|_H^{1+\lambda q}} \int_{B_\gamma} |f_0 - (f_0)_{B_\gamma}|^q dx \right)^{1/q}. \tag{45}
\end{aligned}$$

We arrive at

$$\|\mathcal{H}^p f_0\|_{\text{CBMO}^{q,\lambda}(\mathbb{Q}_p^n)} = \frac{1 - p^{-n}}{1 - p^{-n(1+\lambda)}} \|f_0\|_{\text{CBMO}^{q,\lambda}(\mathbb{Q}_p^n)}. \tag{46}$$

As a result,

$$\|\mathcal{H}^p\|_{\text{CBMO}^{q,\lambda}(\mathbb{Q}_p^n) \rightarrow \text{CBMO}^{q,\lambda}(\mathbb{Q}_p^n)} \geq \frac{1 - p^{-n}}{1 - p^{-n(1+\lambda)}}. \tag{47}$$

Then Theorem 10 follows from (43) and (47). $\square$

Proof of Theorem 12. Let $f \in \mathcal{L}^{q,\lambda}(\mathbb{Q}_p^n)$. Then $f(x) = f(|x|_p^{-1})$. Using Minkowski's inequality, we have

$$\begin{aligned}
& \left(\frac{1}{|B_\gamma(a)|_H^{1+\lambda q}} \int_{B_\gamma(a)} |\mathcal{H}^p f(x)|^q dx \right)^{1/q} \\
&= \left(\frac{1}{|B_\gamma(a)|_H^{1+\lambda q}} \int_{B_\gamma(a)} \left| \int_{B_0} f(|x|_p^{-1} y) dy \right|^q dx \right)^{1/q} \\
&\leq \left[\frac{1}{|B_\gamma(a)|_H^{1+\lambda q}} \int_{B_\gamma(a)} \left(\int_{B_0} |f(|y|_p^{-1} x)| dy \right)^q dx \right]^{1/q} \\
&\leq \int_{B_0} \left[\frac{1}{|B_\gamma(a)|_H^{1+\lambda q}} \int_{B_\gamma(a)} |f(|y|_p^{-1} x)|^q dx \right]^{1/q} dy \\
&\leq \int_{B_0} \left[\frac{1}{|B_\gamma(a)|_H^{1+\lambda q}} \int_{B(a|y|_p^{-1}, p^\gamma|y|_p)} |f(z)|^q |y|_p^{-n} dz \right]^{1/q} dy
\end{aligned}$$

$$\begin{aligned}
&\leq \int_{B_0} \left[\frac{|y|_p^{n\lambda q}}{|B(a|y|_p^{-1}, p^\gamma|y|_p)|_H^{1+\lambda q}} \right. \\
&\quad \left. \times \int_{B(a|y|_p^{-1}, p^\gamma|y|_p)} |f(z)|^q dz \right]^{1/q} dy \\
&\leq \|f\|_{L^{q,\lambda}(\mathbb{Q}_p^n)} \int_{B_0} |y|_p^{n\lambda} dy = \frac{1-p^{-n}}{1-p^{-n(1+\lambda)}} \|f\|_{L^{q,\lambda}(\mathbb{Q}_p^n)}. \quad (48)
\end{aligned}$$

On the other hand, as in the proof of Theorem 9, we take $f_0(x) = |x|_p^{n\lambda}$, and we only need to show that $f_0 \in L^{q,\lambda}(\mathbb{Q}_p^n)$. Consider the following.

(I) If $|a|_p > p^\gamma$ and $x \in B_\gamma(a)$, then $|x|_p = \max\{|x - a|_p, |a|_p\} > p^\gamma$. Since $-1/q \leq \lambda < 0$, we have

$$\begin{aligned}
&\frac{1}{|B_\gamma(a)|_H^{1+\lambda q}} \int_{B_\gamma(a)} |x|_p^{n\lambda q} dx \\
&< \frac{1}{|B_\gamma(a)|_H^{1+\lambda q}} \int_{B_\gamma(a)} p^{n\lambda q} dx = 1. \quad (49)
\end{aligned}$$

(II) If $|a|_p \leq p^\gamma$ and $x \in B_\gamma(a)$, then $|x|_p \leq \max\{|x - a|_p, |a|_p\} \leq p^\gamma$; therefore, $x \in B_\gamma$. Recall that two balls in $\mathbb{Q}_p^n$ are either disjoint or one is contained in the other (cf. page 21 in [39]). So we have $B_\gamma(a) = B_\gamma$; thus,

$$\begin{aligned}
&\frac{1}{|B_\gamma(a)|_H^{1+\lambda q}} \int_{B_\gamma(a)} |x|_p^{n\lambda q} dx \\
&= \frac{1}{|B_\gamma|_H^{1+\lambda q}} \int_{B_\gamma} |x|_p^{n\lambda q} dx \quad (50) \\
&= p^{-\gamma n(1+\lambda q)} \sum_{k=-\infty}^{\gamma} \int_{S_k} p^{kn\lambda q} dx = \frac{1-p^{-n}}{1-p^{-n(1+\lambda q)}}.
\end{aligned}$$

From the previous discussion, we can see that $f_0 \in L^{q,\lambda}(\mathbb{Q}_p^n)$. Then by (33),

$$\|\mathcal{H}^p f_0\|_{L^{q,\lambda}(\mathbb{Q}_p^n)} = \frac{1-p^{-n}}{1-p^{-n(1+\lambda)}} \|f_0\|_{L^{q,\lambda}(\mathbb{Q}_p^n)}. \quad (51)$$

This completes the proof. $\square$

3. Boundedness for Commutators of p -Adic Hardy Operators on p -Adic Central Morrey Spaces

The boundedness of commutators is an active topic in harmonic analysis due to its important applications. For example, it can be applied to characterizing some function spaces [40]. In this section, we consider the boundedness for commutators generated by $\mathcal{H}^p$ and λ -central BMO functions on p -adic central Morrey spaces.

Definition 13. Let $b \in \text{CBMO}^{q,\lambda}(\mathbb{Q}_p^n)$. The commutator of $\mathcal{H}^p$ is defined by

$$\mathcal{H}_b^p f := b\mathcal{H}^p f - \mathcal{H}^p(bf) \quad (52)$$

for some suitable functions f .

Theorem 14. Let $1 < q_1 < \infty$, $q'_1 < q_2 < \infty$, $n(1/q_2 - 1/q'_1) < n/q_1$, $1/q = 1/q_1 + 1/q_2$, $\lambda_1 > -1/q$, $0 \leq \lambda_2 < 1/n$, and $\lambda = \lambda_1 + \lambda_2$. If $b \in \text{CBMO}^{q_2,\lambda_2}(\mathbb{Q}_p^n)$; then $\mathcal{H}_b^p$ is bounded from $\dot{B}^{q_1,\lambda_1}(\mathbb{Q}_p^n)$ to $\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)$ and satisfies

$$\|\mathcal{H}_b^p f\|_{\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)} \leq C \|b\|_{\text{CBMO}^{q_2,\lambda_2}(\mathbb{Q}_p^n)} \|f\|_{\dot{B}^{q_1,\lambda_1}(\mathbb{Q}_p^n)}. \quad (53)$$

Before the proof of this theorem, we need the following calculations.

Lemma 15. Suppose that $b \in \text{CBMO}^{q,\lambda}(\mathbb{Q}_p^n)$ and $j, k \in \mathbb{Z}$, $\lambda \geq 0$. Then,

$$|b_{B_j} - b_{B_k}| \leq p^n |j - k| \|b\|_{\text{CBMO}^{q,\lambda}(\mathbb{Q}_p^n)} \max\{|B_j|_H^\lambda, |B_k|_H^\lambda\}. \quad (54)$$

Proof. Without loss of generality, we may assume that $k > j$. Recall that $b_{B_i} = (1/|B_i|_H) \int_{B_i} b(x) dx$. By Hölder's inequality, we have

$$\begin{aligned}
&|b_{B_{i+1}} - b_{B_i}| \\
&\leq \frac{1}{|B_i|_H} \int_{B_i} |b(x) - b_{B_{i+1}}| dx \leq \frac{1}{|B_i|_H} \int_{B_{i+1}} |b(x) - b_{B_{i+1}}| dx \\
&\leq \frac{1}{|B_i|_H} \left(\int_{B_{i+1}} |b(x) - b_{B_{i+1}}|^q dx \right)^{1/q} |B_{i+1}|_H^{1-1/q} \\
&\leq \frac{|B_{i+1}|_H^{1+\lambda}}{|B_i|_H} \|b\|_{\text{CBMO}^{q,\lambda}(\mathbb{Q}_p^n)} = p^n |B_{i+1}|_H^\lambda \|b\|_{\text{CBMO}^{q,\lambda}(\mathbb{Q}_p^n)}. \quad (55)
\end{aligned}$$

Therefore,

$$\begin{aligned}
&|b_{B_j} - b_{B_k}| \\
&\leq \sum_{i=j}^{k-1} |b_{B_{i+1}} - b_{B_i}| \leq p^n \|b\|_{\text{CBMO}^{q,\lambda}(\mathbb{Q}_p^n)} \sum_{i=j}^{k-1} |B_{i+1}|_H^\lambda \quad (56) \\
&\leq (k - j) p^n \|b\|_{\text{CBMO}^{q,\lambda}(\mathbb{Q}_p^n)} |B_k|_H^\lambda.
\end{aligned}$$

$\square$

Proof of Theorem 14. Assume that $f \in \dot{B}^{q_1, \lambda_1}(\mathbb{Q}_p^n)$. Fix $\gamma \in \mathbb{Z}$, and by Minkowski's inequality, we have

$$\begin{aligned}
 & \left(\frac{1}{|B_\gamma|_H^{1+\lambda q}} \int_{B_\gamma} |\mathcal{H}_b^p f(x)|^q dx \right)^{1/q} \\
 &= \left(\frac{1}{|B_\gamma|_H^{1+\lambda q}} \right. \\
 & \quad \times \left. \int_{B_\gamma} \left| \frac{1}{|x|_p^n} \int_{B(0, |x|_p)} f(y) (b(x) - b(y)) dy \right|^q dx \right)^{1/q} \\
 &\leq \left(\frac{1}{|B_\gamma|_H^{1+\lambda q}} \right. \\
 & \quad \times \left. \int_{B_\gamma} \left| \frac{1}{|x|_p^n} \int_{B(0, |x|_p)} f(y) (b(x) - b_{B_\gamma}) dy \right|^q dx \right)^{1/q} \\
 & \quad + \left(\frac{1}{|B_\gamma|_H^{1+\lambda q}} \right. \\
 & \quad \times \left. \int_{B_\gamma} \left| \frac{1}{|x|_p^n} \int_{B(0, |x|_p)} f(y) (b(y) - b_{B_\gamma}) dy \right|^q dx \right)^{1/q} \\
 &:= I + J.
 \end{aligned} \tag{57}$$

In the following, we will estimate I and J , respectively. For I , since $\mathcal{H}^p$ is bounded from $L^r(\mathbb{Q}_p^n)$ to $L^r(\mathbb{Q}_p^n)$, $1 < r < \infty$ [32], then, by Hölder's inequality ($q/q_1 + q/q_2 = 1$), we get

$$\begin{aligned}
 I &\leq |B_\gamma|_H^{-1/q-\lambda} \left(\int_{B_\gamma} |b(x) - b_{B_\gamma}|^{q_2} dx \right)^{1/q_2} \\
 & \quad \times \left(\int_{B_\gamma} |\mathcal{H}^p f(x)|^{q_1} dx \right)^{1/q_1} \\
 &\leq |B_\gamma|_H^{-1/q_1-\lambda_1} \|b\|_{\text{CBMO}^{q_2, \lambda_2}(\mathbb{Q}_p^n)} \\
 & \quad \times \left(\int_{\mathbb{Q}_p^n} |\mathcal{H}^p(f\chi_{B_\gamma})(x)|^{q_1} dx \right)^{1/q_1} \\
 &\leq C |B_\gamma|_H^{-1/q_1-\lambda_1} \|b\|_{\text{CBMO}^{q_2, \lambda_2}(\mathbb{Q}_p^n)} \left(\int_{B_\gamma} |f(x)|^{q_1} dx \right)^{1/q_1} \\
 &\leq C \|b\|_{\text{CBMO}^{q_2, \lambda_2}(\mathbb{Q}_p^n)} \|f\|_{\dot{B}^{q_1, \lambda_1}(\mathbb{Q}_p^n)}.
 \end{aligned} \tag{58}$$

Next, let us estimate J as follows:

$$\begin{aligned}
 J^q &\leq \frac{1}{|B_\gamma|_H^{1+\lambda q}} \int_{B_\gamma} \left| \frac{1}{|x|_p^n} \int_{B(0, |x|_p)} f(y) (b(y) - b_{B_\gamma}) dy \right|^q dx \\
 &\leq \frac{1}{|B_\gamma|_H^{1+\lambda q}} \\
 & \quad \times \sum_{k=-\infty}^{\gamma} \int_{S_k} p^{-knq} \left(\int_{B_k} |f(y) (b(y) - b_{B_\gamma})| dy \right)^q dx \\
 &= \frac{1-p^{-n}}{|B_\gamma|_H^{1+\lambda q}} \\
 & \quad \times \sum_{k=-\infty}^{\gamma} p^{kn(1-q)} \left(\sum_{j=-\infty}^k \int_{S_j} |f(y) (b(y) - b_{B_\gamma})| dy \right)^q \\
 &= \frac{1-p^{-n}}{|B_\gamma|_H^{1+\lambda q}} \\
 & \quad \times \sum_{k=-\infty}^{\gamma} p^{kn(1-q)} \left(\sum_{j=-\infty}^k \int_{S_j} |f(y) (b(y) - b_{B_j})| dy \right)^q \\
 & \quad + \frac{1-p^{-n}}{|B_\gamma|_H^{1+\lambda q}} \\
 & \quad \times \sum_{k=-\infty}^{\gamma} p^{kn(1-q)} \left(\sum_{j=-\infty}^k \int_{S_j} |f(y) (b_{B_j} - b_{B_\gamma})| dy \right)^q \\
 &:= J_1 + J_2.
 \end{aligned} \tag{59}$$

For J_1 , by Hölder's inequality ($1/q_1 + 1/q_2 + 1 - 1/q = 1$) and the fact that $\lambda_1 > -1/q_1$, $\lambda_2 > -1/q_2$, we have

$$\begin{aligned}
 J_1 &\leq \frac{1-p^{-n}}{|B_\gamma|_H^{1+\lambda q}} \sum_{k=-\infty}^{\gamma} p^{kn(1-q)} \\
 & \quad \times \left[\sum_{j=-\infty}^k |B_j|_H^{1-1/q} \left(\int_{B_j} |f(y)|^{q_1} dy \right)^{1/q_1} \right. \\
 & \quad \times \left. \left(\int_{B_j} |b(y) - b_{B_j}|^{q_2} dy \right)^{1/q_2} \right]^q \\
 &\leq \frac{1-p^{-n}}{|B_\gamma|_H^{1+\lambda q}} \sum_{k=-\infty}^{\gamma} p^{kn(1-q)} \\
 & \quad \times \left[\sum_{j=-\infty}^k |B_j|_H^{1+\lambda} \|b\|_{\text{CBMO}^{q_2, \lambda_2}(\mathbb{Q}_p^n)} \|f\|_{\dot{B}^{q_1, \lambda_1}(\mathbb{Q}_p^n)} \right]^q
 \end{aligned}$$

$$\begin{aligned}
&\leq \frac{1-p^{-n}}{|B_\gamma|_H^{1+\lambda q}} \|b\|_{\text{CBMO}^{q_2, \lambda_2}}^q \|f\|_{\dot{B}^{q_1, \lambda_1}(\mathbb{Q}_p^n)}^q \\
&\quad \times \sum_{k=-\infty}^{\gamma} p^{kn(1-q)} \left(\sum_{j=-\infty}^k p^{jn(1+\lambda)} \right)^q \\
&= \frac{C}{|B_\gamma|_H^{1+\lambda q}} \|b\|_{\text{CBMO}^{q_2, \lambda_2}(\mathbb{Q}_p^n)}^q \\
&\quad \times \|f\|_{\dot{B}^{q_1, \lambda_1}(\mathbb{Q}_p^n)}^q \sum_{k=-\infty}^{\gamma} p^{kn(1+\lambda q)} \\
&= C \|b\|_{\text{CBMO}^{q_2, \lambda_2}(\mathbb{Q}_p^n)}^q \|f\|_{\dot{B}^{q_1, \lambda_1}(\mathbb{Q}_p^n)}^q.
\end{aligned} \tag{60}$$

For J_2 , by Lemma 15 and Hölder's inequality, we obtain

$$\begin{aligned}
J_2 &= \frac{1-p^{-n}}{|B_\gamma|_H^{1+\lambda q}} \sum_{k=-\infty}^{\gamma} p^{kn(1-q)} \left(\sum_{j=-\infty}^k \int_{S_j} |f(y)(b_{B_j} - b_{B_\gamma})| dy \right)^q \\
&\leq \frac{C}{|B_\gamma|_H^{1+\lambda q}} \sum_{k=-\infty}^{\gamma} p^{kn(1-q)} \\
&\quad \times \left[\sum_{j=-\infty}^k (\gamma - j) \|b\|_{\text{CBMO}^{q_2, \lambda_2}} |B_\gamma|_H^{\lambda_2} \int_{B_j} |f(y)| dy \right]^q \\
&\leq \frac{C}{|B_\gamma|_H^{1+\lambda_1 q}} \|b\|_{\text{CBMO}^{q_2, \lambda_2}(\mathbb{Q}_p^n)}^q \sum_{k=-\infty}^{\gamma} p^{kn(1-q)} \\
&\quad \times \left[\sum_{j=-\infty}^k (\gamma - j) |B_j|_H^{1-1/q_1} \left(\int_{B_j} |f(y)|^{q_1} dy \right)^{1/q_1} \right]^q \\
&\leq \frac{C}{|B_\gamma|_H^{1+\lambda_1 q}} \|b\|_{\text{CBMO}^{q_2, \lambda_2}(\mathbb{Q}_p^n)}^q \|f\|_{\dot{B}^{q_1, \lambda_1}(\mathbb{Q}_p^n)}^q \\
&\quad \times \sum_{k=-\infty}^{\gamma} p^{kn(1-q)} \left[\sum_{j=-\infty}^k (\gamma - j) |B_j|_H^{1+\lambda_1} \right]^q \\
&= \frac{C}{|B_\gamma|_H^{1+\lambda_1 q}} \|b\|_{\text{CBMO}^{q_2, \lambda_2}(\mathbb{Q}_p^n)}^q \|f\|_{\dot{B}^{q_1, \lambda_1}(\mathbb{Q}_p^n)}^q \\
&\quad \times \sum_{k=-\infty}^{\gamma} p^{kn(1-q)} (\gamma - k)^q |B_k|_H^{(1+\lambda_1)q} \\
&= C \|b\|_{\text{CBMO}^{q_2, \lambda_2}(\mathbb{Q}_p^n)}^q \|f\|_{\dot{B}^{q_1, \lambda_1}(\mathbb{Q}_p^n)}^q.
\end{aligned} \tag{61}$$

Notice that since $\lambda_1 > -1/q_1$, then $1+\lambda_1 q_1 > 0$ and $1+\lambda_1 > 0$.

The above estimates imply that

$$\begin{aligned}
&\left(\frac{1}{|B_\gamma|_H^{1+\lambda q}} \int_{B_\gamma} |\mathcal{H}_b^p f(x)|^q dx \right)^{1/q} \\
&\leq C \|b\|_{\text{CBMO}^{q_2, \lambda_2}(\mathbb{Q}_p^n)} \|f\|_{\dot{B}^{q_1, \lambda_1}(\mathbb{Q}_p^n)}.
\end{aligned} \tag{62}$$

Consequently,

$$\|\mathcal{H}_b^p f\|_{\dot{B}^{q_1, \lambda_1}(\mathbb{Q}_p^n)} \leq C \|b\|_{\text{CBMO}^{q_2, \lambda_2}(\mathbb{Q}_p^n)} \|f\|_{\dot{B}^{q_1, \lambda_1}(\mathbb{Q}_p^n)}. \tag{63}$$

Theorem 14 is proved. $\square$

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