Research Article

Relationship between Hardy Spaces Associated with Different Homogeneities and One-Parameter Hardy Spaces

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We prove that the Hardy spaces associated with different homogeneities H_{com}^p , $0 < p < \infty$, are continuously embedded into the intersection of the isotropic Hardy spaces H_{iso}^p and the nonisotropic Hardy spaces H_{non}^p . As a consequence, we obtain that any operator bounded from either H_{iso}^p or H_{non}^p into L^p must be bounded from H_{com}^p to L^p .

1. Introduction

For $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$ and $\delta > 0$, we consider two kinds of homogeneities on $\mathbb{R}^n$:

$$\begin{aligned}\delta \circ_e (x', x_n) &= (\delta x', \delta x_n), \\ \delta \circ_h (x', x_n) &= (\delta x', \delta^2 x_n).\end{aligned}\tag{1}$$

The first are the classical isotropic dilations occurring in the classical Calderón-Zygmund singular integrals, while the second are nonisotropic and related to the heat equations (also the Heisenberg groups). Let $e(\xi)$ and $h(\xi)$ be functions on $\mathbb{R}^n$ homogeneous of degree 0 in the isotropic sense and in the nonisotropic sense, respectively, and both smooth away from the origin. Then, it is well known that the Fourier multipliers T_1 defined by $\widehat{T_1(f)}(\xi) = e(\xi)\widehat{f}(\xi)$ and T_2 given by $\widehat{T_2(f)}(\xi) = h(\xi)\widehat{f}(\xi)$ are both bounded on L^p for $1 < p < \infty$, of weak-type $(1, 1)$, and bounded on the classical isotropic Hardy spaces H_{iso}^p and, nonisotropic Hardy spaces H_{non}^p , respectively. Riviere in [1] asked the question is the composition $T_1 \circ T_2$ still of weak-type $(1, 1)$? Phong and Stein in [2] answered this question and gave a necessary and sufficient condition for which $T_1 \circ T_2$ is of weak-type $(1, 1)$. The operators Phong and Stein studied are in fact a composition of operators with different kinds

of homogeneities which arise naturally in the $\bar{\partial}$ -Neumann problem. Recently, Han et al. [3] developed a theory of the Hardy spaces H_{com}^p , $0 < p < \infty$, associated with the different homogeneities and proved that the composition of the two Calderón-Zygmund convolution operators with different homogeneities is bounded on $H_{\text{com}}^p(\mathbb{R}^n)$. Weighted function spaces associated with different homogeneities and boundedness of composition of operators on them were recently investigated in [4–6].

The Hardy spaces H_{com}^p introduced in [3] have surprising multiparameter structures which reflect the mixed homogeneities arising from the two operators under consideration. A natural question arises: Is there any relationship between H_{com}^p and the two classical one-parameter Hardy spaces H_{iso}^p and H_{non}^p ? The main purpose of this paper is to answer this question. We shall prove that H_{com}^p are continuously embedded into the intersection of H_{iso}^p and H_{non}^p . As an application, we show that any operator boundedness from either H_{iso}^p or H_{non}^p into L^p must be bounded from H_{com}^p into L^p . Our methods are to use the partially discrete Calderón-type formula and the Littlewood-Paley theory in this context, which are appropriately developed.

Before stating the results more precisely, we first recall some notions and notations. For $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$,

we denote $|x|_e = (|x'|^2 + |x_n|^2)^{1/2}$ and $|x|_h = (|x'|^2 + |x_n|)^{1/2}$. Let $\psi^{(1)} \in \mathcal{S}(\mathbb{R}^n)$ satisfy

$$\begin{aligned} \text{supp } \widehat{\psi^{(1)}}(\xi) &\subseteq \left\{ \xi : \frac{1}{2} < |\xi|_e \leq 2 \right\}, \\ \sum_{j \in \mathbb{Z}} \widehat{\psi^{(1)}}(2^j \circ_e \xi) &= 1 \quad \forall \xi \in \mathbb{R}^n \setminus \{0\}. \end{aligned} \quad (2)$$

The isotropic discrete square function $g_{\text{iso}}^d(f)$ is defined by

$$g_{\text{iso}}^d(f)(x) := \left\{ \sum_{j \in \mathbb{Z}} \sum_{Q \in \mathcal{Q}_{\text{iso}}^j} |\psi_j^{(1)} * f(x_Q)|^2 \chi_Q(x) \right\}^{1/2}, \quad (3)$$

where $\psi_j^{(1)}(x) = 2^{-jn} \psi^{(1)}(2^{-j} \circ_e x)$, $\mathcal{Q}_{\text{iso}}^j$ denotes the set of all dyadic cubes with sidelength 2^j , and x_Q denotes the left-lower corner of Q . The isotropic Hardy spaces $H_{\text{iso}}^p(\mathbb{R}^n)$, $0 < p < \infty$, are defined by

$$H_{\text{iso}}^p(\mathbb{R}^n) := \left\{ f \in \frac{\mathcal{S}'}{\mathcal{D}} : \|g_{\text{iso}}^d(f)\|_{L^p(\mathbb{R}^n)} < \infty \right\}. \quad (4)$$

Similarly, let $\psi^{(2)} \in \mathcal{S}(\mathbb{R}^n)$ satisfy

$$\begin{aligned} \text{supp } \widehat{\psi^{(2)}}(\xi) &\subseteq \left\{ \xi : \frac{1}{2} < |\xi|_h \leq 2 \right\}, \\ \sum_{k \in \mathbb{Z}} \widehat{\psi^{(2)}}(2^k \circ_h \xi) &= 1 \quad \forall \xi \in \mathbb{R}^n \setminus \{0\}. \end{aligned} \quad (5)$$

The nonisotropic discrete square function $g_{\text{non}}^d(f)$ is defined by

$$g_{\text{non}}^d(f)(x) := \left\{ \sum_{k \in \mathbb{Z}} \sum_{Q \in \mathcal{Q}_{\text{non}}^k} |\psi_k^{(2)} * f(x_Q)|^2 \chi_Q(x) \right\}^{1/2}, \quad (6)$$

where $\psi_k^{(2)}(x) = 2^{-k(n+1)} \psi^{(2)}(2^{-k} \circ_h x)$, $\mathcal{Q}_{\text{non}}^k$ is the set of dyadic nonisotropic ‘cubes’ $Q = Q_1 \times Q_2 \subset \mathbb{R}^{n-1} \times \mathbb{R}$ with sidelength $\ell(Q_1) = 2^k$ and $|Q_2| = 2^{2k}$, and x_Q is the left-lower corner of Q . The nonisotropic Hardy spaces H_{non}^p , $0 < p < \infty$, are defined by

$$H_{\text{non}}^p(\mathbb{R}^n) := \left\{ f \in \frac{\mathcal{S}'}{\mathcal{D}} : \|g_{\text{non}}^d(f)\|_{L^p(\mathbb{R}^n)} < \infty \right\}. \quad (7)$$

For $j, k \in \mathbb{Z}$, let $\psi_{j,k} = \psi_j^{(1)} * \psi_k^{(2)}$. The discrete square function g_{com}^d associated with different homogeneities is given by

$$g_{\text{com}}^d(f)(x) := \left\{ \sum_{j,k \in \mathbb{Z}} \sum_{R \in \mathcal{R}^{j,k}} |\psi_{j,k} * f(x_R)|^2 \chi_R(x) \right\}^{1/2}, \quad (8)$$

where $\mathcal{R}^{j,k}$ denote the set of dyadic rectangles $R = Q_1 \times Q_2$ in $\mathbb{R}^{n-1} \times \mathbb{R}$ with sidelength $\ell(Q_1) = 2^{j \vee k}$ and $|Q_2| = 2^{j \vee 2k}$ and

x_R is the left-lower corner of R . The Hardy spaces H_{com}^p , $0 < p < \infty$, associated with different homogeneities are defined by

$$H_{\text{com}}^p(\mathbb{R}^n) := \left\{ f \in \frac{\mathcal{S}'}{\mathcal{D}} : \|g_{\text{com}}^d(f)\|_{L^p(\mathbb{R}^n)} < \infty \right\}. \quad (9)$$

The main result of this paper is as follows.

Theorem 1. *Let $0 < p < \infty$. One has*

$$H_{\text{com}}^p(\mathbb{R}^n) \hookrightarrow H_{\text{iso}}^p(\mathbb{R}^n) \cap H_{\text{non}}^p(\mathbb{R}^n). \quad (10)$$

More precisely, there is a constant C depending on p and n such that, for all $f \in H_{\text{com}}^p(\mathbb{R}^n)$,

$$\|f\|_{H_{\text{iso}}^p(\mathbb{R}^n)} + \|f\|_{H_{\text{non}}^p(\mathbb{R}^n)} \leq C \|f\|_{H_{\text{com}}^p(\mathbb{R}^n)}. \quad (11)$$

In [3], it was proved that the composition of the two Calderón-Zygmund convolution operators with different homogeneities is bounded on H_{com}^p (see [3, Theorem 1.9]). This result can be improved by the following.

Corollary 2. *Let $0 < p < \infty$. One has that*

- (1) *any operator T bounded from either $H_{\text{iso}}^p(\mathbb{R}^n)$ or $H_{\text{non}}^p(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$ must be bounded from $H_{\text{com}}^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$;*
- (2) *if T_1 is bounded from $H_{\text{iso}}^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ and T_2 is bounded from $H_{\text{non}}^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$, then $T_1 \circ T_2$ is bounded from $H_{\text{com}}^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$.*

2. New Square Function Characterizations of H_{com}^p

Let $\phi^{(1)} \in \mathcal{S}(\mathbb{R}^n)$ satisfy $\text{supp } \phi^{(1)} \subseteq B(0, 1)$,

$$\begin{aligned} \int_{\mathbb{R}^n} \phi^{(1)}(x) x^\alpha dx &= 0 \quad \forall |\alpha| \leq M_p, \\ \sum_{j \in \mathbb{Z}} \widehat{\phi^{(1)}}(2^j \circ_e \xi) &= 1 \quad \forall \xi \in \mathbb{R}^n \setminus \{0\}, \end{aligned} \quad (12)$$

where M_p is a sufficiently large constant depending on p and n . Similarly, let $\phi^{(2)} \in \mathcal{S}(\mathbb{R}^n)$ satisfy $\text{supp } \phi^{(2)} \subseteq B(0, 1)$,

$$\begin{aligned} \int_{\mathbb{R}^n} \phi^{(2)}(x) x^\beta dx &= 0 \quad \forall |\beta| \leq M_p, \\ \sum_{k \in \mathbb{Z}} \widehat{\phi^{(2)}}(2^k \circ_h \xi) &= 1 \quad \forall \xi \in \mathbb{R}^n \setminus \{0\}. \end{aligned} \quad (13)$$

Let $\phi_j^{(1)}(x) = 2^{-jn} \phi^{(1)}(2^{-j} \circ_e x)$, $\phi_k^{(2)}(x) = 2^{-k(n+1)} \phi^{(2)}(2^{-k} \circ_h x)$, and $\phi_{j,k} = \phi_j^{(1)} * \phi_k^{(2)}$. Then, H_{com}^p can be characterized via the continuous square function g_{com} defined by

$$g_{\text{com}}(f)(x) = \left\{ \sum_{j,k \in \mathbb{Z}} |\phi_{j,k} * f(x)|^2 \right\}^{1/2}. \quad (14)$$

The following Calderón-type identities are well known (cf. [7–9]).

Lemma 3. For $f \in L^2(\mathbb{R}^n)$, there is a sufficiently large integer N such that

$$\begin{aligned} f &= \sum_{j \in \mathbb{Z}} \sum_{Q \in \mathcal{Q}_{\text{iso}}^{j-N}} \phi_j^{(1)}(\cdot - \vartheta_Q) \phi_j^{(1)} * f(\vartheta_Q) \\ &= \sum_{k \in \mathbb{Z}} \sum_{Q \in \mathcal{Q}_{\text{non}}^{k-N}} \phi_k^{(2)}(\cdot - \vartheta_Q) \phi_k^{(2)} * f(\vartheta_Q), \end{aligned} \quad (15)$$

where the two series converge in $L^2(\mathbb{R}^n)$ and ϑ_Q denotes any fixed point of Q .

The Calderón-type identities in Lemma 3 lead to the following square functions:

$$\begin{aligned} g_{\text{com}}^{(1)}(f)(x) &:= \left\{ \sum_{j,k \in \mathbb{Z}} \sum_{Q \in \mathcal{Q}_{\text{iso}}^{j-N}} |\phi_{j,k} * f(x_Q)|^2 \chi_Q(x) \right\}^{1/2}, \\ g_{\text{com}}^{(2)}(f)(x) &:= \left\{ \sum_{j,k \in \mathbb{Z}} \sum_{Q \in \mathcal{Q}_{\text{non}}^{k-N}} |\phi_{j,k} * f(x_Q)|^2 \chi_Q(x) \right\}^{1/2}, \end{aligned} \quad (16)$$

where N is the sufficiently large integer in Lemma 3.

The purpose of this section is to prove the following.

Theorem 4. For $p \in (0, \infty)$, one has that

$$\|f\|_{H_{\text{com}}^p} \sim \|g_{\text{com}}^{(1)}(f)\|_{L^p} \sim \|g_{\text{com}}^{(2)}(f)\|_{L^p}. \quad (17)$$

Proof. We only prove that

$$\|f\|_{H_{\text{com}}^p} \sim \|g_{\text{com}}^{(1)}(f)\|_{L^p}, \quad (18)$$

as $\|f\|_{H_{\text{com}}^p} \sim \|g_{\text{com}}^{(2)}(f)\|_{L^p}$ can be obtained in the same manner. To prove (18), it suffices to show that

$$\begin{aligned} &\left\| \left\{ \sum_{j,k} \sum_{Q \in \mathcal{Q}_{\text{iso}}^{j-N}} \sup_{u \in Q} |\phi_{j,k} * f(u)|^2 \chi_Q \right\}^{1/2} \right\|_{L^p} \\ &\leq \left\| \left\{ \sum_{j',k} \sum_{Q \in \mathcal{Q}_{\text{iso}}^{j'-N}} \inf_{u \in Q} |\phi_{j',k} * f(u)|^2 \chi_Q \right\}^{1/2} \right\|_{L^p}. \end{aligned} \quad (19)$$

To verify (19), we need the following lemma. $\square$

Lemma 5 (see [10]). Let $Q \in \mathcal{Q}_{\text{iso}}^{j-N}$ and $M > 0$. Then, for any $u, \vartheta_Q \in Q, \vartheta_{Q'} \in Q'$, and $\delta \in (n/(n+M), 1]$, one has that

$$\begin{aligned} &\sum_{Q' \in \mathcal{Q}_{\text{iso}}^{j'-N}} \frac{2^{M(j \vee j')}}{(2^{j \vee j'} + |u - \vartheta_{Q'}|)^{n+M}} |g_{j'}(\vartheta_{Q'})| \\ &\leq C_N 2^{n|j-j'|((1/\delta)-1)} \\ &\times \left\{ \mathcal{M} \left[\left(\sum_{Q' \in \mathcal{Q}_{\text{iso}}^{j'-N}} |g_{j'}(\vartheta_{Q'})|^2 \chi_{Q'} \right)^{\delta/2} \right] (\vartheta_Q) \right\}^{1/\delta}, \end{aligned} \quad (20)$$

where $\mathcal{M}$ denotes the classical isotropic Hardy-Littlewood maximal operator and $a \vee b = \max(a, b)$.

We first assume that $f \in L^2 \cap H_{\text{com}}^p(\mathbb{R}^n)$. Set $f_k = \phi_k^{(2)} * f$. Applying the discrete Calderón-type identity in (15), the classical almost orthogonality estimate, and Lemma 5, we deduce that for any $u, \vartheta_Q \in Q, \vartheta_{Q'} \in Q'$, and $\delta \in (n/(n+M), \min(p, 1))$,

$$\begin{aligned} &|\phi_{j,k} * f(u)| \\ &= \left| \sum_{j' \in \mathbb{Z}} \sum_{Q' \in \mathcal{Q}_{\text{iso}}^{j'-N}} \phi_j^{(1)} * \phi_{j'}^{(1)}(u - \vartheta_{Q'}) \phi_{j'}^{(1)} * f_k(\vartheta_{Q'}) \right| \\ &\leq \sum_{j' \in \mathbb{Z}} 2^{-|j-j'|10M} \\ &\times \sum_{Q' \in \mathcal{Q}_{\text{iso}}^{j'-N}} \frac{2^{M(j \vee j')}}{(2^{j \vee j'} + |u - \vartheta_{Q'}|)^{n+M}} |\phi_{j'}^{(1)} * f_k(\vartheta_{Q'})| \\ &\leq \sum_{j' \in \mathbb{Z}} 2^{-|j-j'|M} \\ &\times \left\{ \mathcal{M} \left[\left(\sum_{Q' \in \mathcal{Q}_{\text{iso}}^{j'-N}} |\phi_{j'}^{(1)} * f_k(\vartheta_{Q'})|^2 \right. \right. \right. \\ &\quad \left. \left. \times \chi_{Q'} \right)^{\delta/2} \right] (\vartheta_Q) \right\}^{1/\delta}. \end{aligned} \quad (21)$$

Since u and $\vartheta_{Q'}$ are arbitrary points in Q and Q' , respectively,

$$\begin{aligned} &\sup_{u \in Q} |\phi_{j,k} * f(u)| \\ &\leq \sum_{j' \in \mathbb{Z}} 2^{-|j-j'|M} \end{aligned}$$

$$\times \left\{ \mathcal{M} \left[\left(\sum_{Q' \in \mathcal{Q}_{\text{iso}}^{j'-N}} \inf_{v \in Q'} |\phi_{j'}^{(1)} * f_k(v)|^2 \right)^{\delta/2} \right] (\vartheta_Q) \right\}^{1/\delta}, \quad (22)$$

which, by the Cauchy-Schwarz inequality, implies that

$$\begin{aligned} & \sup_{u \in Q} |\phi_{j,k} * f(u)|^2 \\ & \leq \sum_{j' \in \mathbb{Z}} 2^{-|j-j'|M} \\ & \times \left\{ \mathcal{M} \left[\left(\sum_{Q' \in \mathcal{Q}_{\text{iso}}^{j'-N}} \inf_{v \in Q'} |\phi_{j'}^{(1)} * f_k(v)|^2 \right)^{\delta/2} \right] (\vartheta_Q) \right\}^{2/\delta}, \end{aligned} \quad (23)$$

where we have used $\sum_{j' \in \mathbb{Z}} 2^{-|j-j'|M} \leq C$. Multiplying χ_Q from both sides and summing over $j, k \in \mathbb{Z}, Q \in \mathcal{Q}_{\text{iso}}^{j-N}$ yield that, for any $\vartheta_Q \in Q$,

$$\begin{aligned} & \sum_{j,k \in \mathbb{Z}} \sum_{Q \in \mathcal{Q}_{\text{iso}}^{j-N}} \sup_{u \in Q} |\phi_{j,k} * f(u)|^2 \chi_Q(x) \\ & \leq \sum_{j,k \in \mathbb{Z}} \sum_{j' \in \mathbb{Z}} 2^{-|j-j'|M} \\ & \times \left\{ \mathcal{M} \left[\left(\sum_{Q' \in \mathcal{Q}_{\text{iso}}^{j'-N}} \inf_{v \in Q'} |\phi_{j'}^{(1)} * f_k(v)|^2 \right)^{\delta/2} \right] (\vartheta_Q) \right\}^{2/\delta} \\ & \leq \sum_{j' \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \times \left\{ \mathcal{M} \left[\left(\sum_{Q' \in \mathcal{Q}_{\text{iso}}^{j'-N}} \inf_{v \in Q'} |\phi_{j'}^{(1)} * f_k(v)|^2 \right)^{\delta/2} \right] (\vartheta_Q) \right\}^{2/\delta}. \end{aligned} \quad (24)$$

It follows that

$$\begin{aligned} & \left\{ \sum_{j,k} \sum_{Q \in \mathcal{Q}_{\text{iso}}^{j-N}} \sup_{u \in Q} |\phi_{j,k} * f(u)|^2 \chi_Q(x) \right\}^{1/2} \\ & \leq \left\{ \sum_{j' \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \right. \\ & \times \left\{ \mathcal{M} \left[\left(\sum_{Q' \in \mathcal{Q}_{\text{iso}}^{j'-N}} \inf_{v \in Q'} |\phi_{j'}^{(1)} * f_k(v)|^2 \right)^{\delta/2} \right] (\vartheta_Q) \right\}^{2/\delta} \left. \right\}^{1/2}. \end{aligned} \quad (25)$$

Finally, taking the L^p norm on both sides and applying the Fefferman-Stein vector-valued inequality yield (19). Since $L^2 \cap H_{\text{com}}^p$ is dense in H_{com}^p (see [3]), a limiting argument concludes the proof of Theorem 4.

3. Proofs of Theorem 1 and Corollary 2

We first give the following.

Proof of Theorem 1. For $1 < p < \infty$, Theorem 1 is trivial since $H_{\text{com}}^p = H_{\text{iso}}^p = H_{\text{non}}^p = L^p$. We now assume that $0 < p \leq 1$. We only prove $\|f\|_{H_{\text{non}}^p} \leq \|f\|_{H_{\text{com}}^p}$ as the inequality $\|f\|_{H_{\text{iso}}^p} \leq \|f\|_{H_{\text{com}}^p}$ can be proved similarly. Since $L^2 \cap H_{\text{com}}^p$ is dense in H_{com}^p , we may assume that $f \in L^2 \cap H_{\text{com}}^p$. For $i \in \mathbb{Z}$, set

$$\Omega_i = \{x \in \mathbb{R}^n : g_{\text{com}}^{(1)}(f)(x) > 2^i\}. \quad (26)$$

Denote that

$$\begin{aligned} \mathcal{B}_i &= \left\{ (j, Q) : Q \in \mathcal{Q}_{\text{iso}}^{j-N}, |Q \cap \Omega_i| \right. \\ & \quad \left. > \frac{1}{2|Q|}, |Q \cap \Omega_{i+1}| \leq \frac{1}{2|Q|} \right\}, \end{aligned} \quad (27)$$

where N is the sufficiently large integer in Lemma 3.

Applying the discrete Calderón-type reproducing formula in Lemma 3,

$$f_k(x) = \sum_{i \in \mathbb{Z}} \sum_{(j,Q) \in \mathcal{B}_i} |Q| \phi_j^{(1)} * f_k(x_Q) \phi_j^{(1)}(x - x_Q), \quad (28)$$

$$k \in \mathbb{Z},$$

where $f_k := \phi_k^{(2)} * f$ and the series converges in the $L^2(\mathbb{R}^n)$ norm. We claim that

$$\begin{aligned} & \left\| \left\{ \sum_{k \in \mathbb{Z}} \left| \sum_{(j,Q) \in \mathcal{B}_i} |Q| \phi_j^{(1)} * f_k(x_Q) \phi_j^{(1)}(\cdot - x_Q) \right|^2 \right\}^{1/2} \right\|_p^p \\ & \leq C 2^{ip} |\Omega_i|. \end{aligned} \quad (29)$$

Assume the claim for the moment. Then, by the continuous square function characterization of H_{non}^p ,

$$\begin{aligned}
& \|f\|_{H_{\text{non}}^p}^p \\
& \sim \int_{\mathbb{R}^n} \left\{ \sum_{k \in \mathbb{Z}} \left| \sum_{i \in \mathbb{Z}} \sum_{(j,Q) \in \mathcal{B}_i} |Q| \phi_j^{(1)} * f_k(x_Q) \phi_j^{(1)} \right. \right. \\
& \quad \left. \left. \times |(x - x_Q)|^2 \right\}^{p/2} dx \\
& \leq \int_{\mathbb{R}^n} \left\{ \sum_{i \in \mathbb{Z}} \left[\sum_{k \in \mathbb{Z}} \left| \sum_{(j,Q) \in \mathcal{B}_i} |Q| \phi_j^{(1)} * f_k(x_Q) \phi_j^{(1)} \right. \right. \right. \\
& \quad \left. \left. \times (x - x_Q) \right|^2 \right]^{1/2} \right\}^p dx \\
& \leq \sum_{i \in \mathbb{Z}} \int \left\{ \sum_{k \in \mathbb{Z}} \left| \sum_{(j,Q) \in \mathcal{B}_i} |Q| \phi_j^{(1)} * f_k(x_Q) \phi_j^{(1)} \right. \right. \\
& \quad \left. \left. \times (x - x_Q) \right|^2 \right\}^{p/2} dx \\
& \lesssim \sum_{i \in \mathbb{Z}} 2^{ip} |\Omega_i| \leq \|g_{\text{com}}^{(1)}(f)\|_p^p \sim \|f\|_{H_{\text{com}}^p}^p,
\end{aligned} \tag{30}$$

where in the second inequality, we have used Minkowski's inequality, for ℓ^2 and in the third inequality used the inequality $(\sum_i |a_i|)^p \leq \sum_i |a_i|^p$ for $0 < p \leq 1$.

Thus, to finish the proof of Theorem 1, it suffices to verify claim (29). Since $\phi^{(1)}$ is supported in unit ball of $\mathbb{R}^n$, for $(j, Q) \in \mathcal{B}_i$, $\{\phi_j^{(1)}(\cdot - x_Q)\}$ are supported in

$$\widetilde{\Omega}_i = \left\{ x : \mathcal{M}(\chi_{\Omega_i})(x) > \frac{1}{2} \right\}. \tag{31}$$

Thus, by Hölder's inequality and $|\widetilde{\Omega}_i| \leq C|\Omega_i|$ (which follows from L^2 boundedness of $\mathcal{M}$),

$$\begin{aligned}
& \left\| \left\{ \sum_k \left| \sum_{(j,Q) \in \mathcal{B}_i} |Q| \phi_j^{(1)} * f_k(x_Q) \phi_j^{(1)}(\cdot - x_Q) \right|^2 \right\}^{1/2} \right\|_p^p \\
& = \int_{\widetilde{\Omega}_i} \left\{ \sum_k \left| \sum_{(j,Q) \in \mathcal{B}_i} |Q| \phi_j^{(1)} * f_k(x_Q) \phi_j^{(1)} \right. \right. \\
& \quad \left. \left. \times (x - x_Q) \right|^2 \right\}^{p/2} dx
\end{aligned}$$

$$\begin{aligned}
& \leq |\widetilde{\Omega}_i|^{1-p/2} \int_{\widetilde{\Omega}_i} \sum_k \left| \sum_{(j,Q) \in \mathcal{B}_i} |Q| \phi_j^{(1)} * f_k(x_Q) \phi_j^{(1)} \right. \\
& \quad \left. \times (x - x_Q) \right|^2 dx \\
& \leq |\Omega_i|^{1-p/2} \sum_k \left\| \sum_{(j,Q) \in \mathcal{B}_i} |Q| \phi_j^{(1)} * f_k(x_Q) \phi_j^{(1)} \right. \\
& \quad \left. \times (\cdot - x_Q) \right\|_2^2.
\end{aligned} \tag{32}$$

We now estimate the last L^2 norm by duality argument. For ζ with $\|\zeta\|_2 \leq 1$, applying the Cauchy-Schwarz inequality yields

$$\begin{aligned}
& \left| \left\langle \sum_{(j,Q) \in \mathcal{B}_i} |Q| \phi_j^{(1)} * f_k(x_Q) \phi_j^{(1)}(\cdot - x_Q), \zeta \right\rangle \right| \\
& = \sum_{(j,Q) \in \mathcal{B}_i} |Q| \phi_j^{(1)} * f_k(x_Q) \int \phi_j^{(1)}(x - x_Q) \zeta_k(x) dx \\
& \leq \left(\sum_{(j,Q) \in \mathcal{B}_i} |Q| |\phi_j^{(1)} * f_k(x_Q)|^2 \right)^{1/2} \\
& \quad \times \left(\sum_{(j,Q) \in \mathcal{B}_i} |Q| |\widetilde{\phi_j^{(1)}} * \zeta(x_Q)|^2 \right)^{1/2},
\end{aligned} \tag{33}$$

where $\widetilde{\phi_j^{(1)}}(x) = \overline{\phi_j^{(1)}(-x)}$. Since

$$\begin{aligned}
& \sum_{(j,Q) \in \mathcal{B}_i} |Q| |\widetilde{\phi_j^{(1)}} * \zeta(x_Q)|^2 \\
& = \int_{\mathbb{R}^n} \left\{ \sum_{(j,Q) \in \mathcal{B}_i} |\phi_j^{(1)} * \zeta(x_Q)|^2 \chi_Q(x) \right\}^{(1/2) \cdot 2} dx \\
& \leq \int_{\mathbb{R}^n} |g_{\text{iso}}^d(\zeta)(x)|^2 dx \lesssim \|\zeta\|_{L^2}^2 \leq 1,
\end{aligned} \tag{34}$$

we now have that

$$\begin{aligned}
& \sum_k \left\| \sum_{(j,Q) \in \mathcal{B}_i} |Q| \phi_j^{(1)} * f_k(x_Q) \phi_j^{(1)}(\cdot - x_Q) \right\|_{L^2}^2 \\
& \lesssim \sum_k \sum_{(j,Q) \in \mathcal{B}_i} |Q| |\phi_j^{(1)} * f_k(x_Q)|^2,
\end{aligned} \tag{35}$$

which is, in turn, bounded by

$$\begin{aligned}
 & 2 \sum_k \sum_{(j,Q) \in \mathcal{B}_i} |\phi_j^{(1)} * f_k(x_Q)|^2 |Q \cap (\bar{\Omega}_i - \Omega_{i+1})| \\
 &= 2 \int_{\bar{\Omega}_i \setminus \Omega_{i+1}} \left\{ \sum_k \sum_{(j,Q) \in \mathcal{B}_i} |\phi_{j,k} * f(x_Q)|^2 \chi_Q(x) \right\}^{(1/2) \cdot 2} dx \\
 &\leq 2 \int_{\bar{\Omega}_i \setminus \Omega_{i+1}} [g_{\text{com}}^{(1)}(f)(x)]^2 dx \lesssim 2^{2i} |\Omega_i|,
 \end{aligned} \tag{36}$$

where in the first inequality, we have used $|Q| \leq 2|Q \cap (\bar{\Omega}_i \setminus \Omega_{i+1})|$ when $(j, Q) \in \mathcal{B}_i$. Substituting this estimate back to (32) verifies the claim (29), and hence, Theorem 1 follows. $\square$

Finally, we give the following.

Proof of Corollary 2. (1) follows directly from Theorem 1, while (2) follows from (1). $\square$

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