Research Article

On Generalized Statistical Convergence of Order α of Difference Sequences

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We introduce the concept of statistical convergence of order α of difference sequences, and we give some relations between the set of statistical convergence of order α of difference sequences and strong $w_\lambda^\alpha(\Delta^m, p)$ -summability of order α . Furthermore some relations between the spaces $w_\lambda^\alpha(\Delta^m, (p), M)$ and $S_\lambda^\alpha(\Delta^m)$ are examined.

1. Introduction

The idea of statistical convergence was given by Zygmund [1] in the first edition of his monograph published in Warsaw in 1935. The concept of statistical convergence was introduced by Steinhaus [2] and Fast [3] and later reintroduced by Schoenberg [4] independently. Over the years and under different names statistical convergence has been discussed in the theory of Fourier analysis, ergodic theory, number theory, measure theory, trigonometric series, turnpike theory, and Banach spaces. Later on it was further investigated from the sequence space point of view and linked with summability theory by Connor [5], Fridy [6], Khan and Orhan [7], Miller and Orhan [8], Mursaleen [9], Rath and Tripathy [10], Šalát [11], Savaş [12], and many others. In recent years, generalizations of statistical convergence have appeared in the study of strong integral summability and the structure of ideals of bounded continuous functions on locally compact spaces. Statistical convergence and its generalizations are also connected with subsets of the Stone-Čech compactification of the natural numbers. Moreover, statistical convergence is closely related to the concept of convergence in probability.

Let w be the set of all sequences of real or complex numbers, and let ℓ_∞ , c , and c_0 be, respectively, the Banach spaces of bounded, convergent, and null sequences $x = (x_k)$ with the usual norm $\|x\| = \sup |x_k|$, where $k \in \mathbb{N} = \{1, 2, \dots\}$, the set of positive integers. Also by bs , cs , ℓ_1 , and ℓ_p , we

denote the spaces of all bounded, convergent, absolutely, and p -absolutely convergent series, respectively.

Let $\lambda = (\lambda_n)$ be a nondecreasing sequence of positive numbers tending to ∞ such that $\lambda_{n+1} \leq \lambda_n + 1$, $\lambda_1 = 1$. The generalized de la Vallée-Poussin mean is defined by $t_n(x) = (1/\lambda_n) \sum_{k \in I_n} x_k$, where $I_n = [n - \lambda_n + 1, n]$ for $n = 1, 2, \dots$. A sequence $x = (x_k)$ is said to be (V, λ) -summable to a number ℓ if $t_n(x) \rightarrow \ell$ as $n \rightarrow \infty$ [13]. If $\lambda_n = n$, then (V, λ) -summability is reduced to the Cesàro summability.

The notion of difference sequence spaces was introduced by Kızmaz [14], and it was generalized by Et and Çolak [15] such as

$$\Delta^m(X) = \{x = (x_k) : (\Delta^m x_k) \in X\} \quad (1)$$

for $X = \ell_\infty$, c , or c_0 , where $m \in \mathbb{N}$, $\Delta^0 x = (x_k)$, $\Delta^m x = (\Delta^{m-1} x_k - \Delta^{m-1} x_{k+1})$, and so $\Delta^m x_k = \sum_{i=0}^m (-1)^i \binom{m}{i} x_{k+i}$. The sequence spaces $\Delta^m(X)$ are the Banach spaces normed by

$$\|x\|_\Delta = \sum_{i=1}^m |x_i| + \|\Delta^m x_k\|_\infty \quad (2)$$

for $X = \ell_\infty$, c or c_0 . Recently the difference sequence spaces have been studied in [16–21].

The order of statistical convergence of a sequence of numbers was given by Gadjiev and Orhan [22] and after that statistical convergence of order α and strongly p -Cesàro summability of order α were studied by Çolak [23] and generalized by Çolak and Bektaş [24].

Let $0 < \alpha \leq 1$ be given. The sequence (x_k) is said to be statistically convergent of order α if there is a real number L such that

$$\lim_{n \rightarrow \infty} \frac{1}{\lambda_n^\alpha} |\{k \leq n : |x_k - L| \geq \varepsilon\}| = 0 \quad (3)$$

for every $\varepsilon > 0$. In this case we write $S^\alpha - \lim x_k = L$. The set of all statistically convergent sequences of order α will be denoted by S^α .

Let X be a sequence space. Then X is called

- (i) solid (or normal) if $(\alpha_k x_k) \in X$ whenever $(x_k) \in X$ for all sequences (α_k) of scalars with $|\alpha_k| \leq 1$,
- (ii) symmetric if $(x_k) \in X$ implies $(x_{\pi(k)}) \in X$, where $\pi(k)$ is a permutation of $\mathbb{N}$,
- (iii) sequence algebra if X is closed under multiplication.

2. Main Results

In this section we give the main results of the paper. In Theorem 5 we give the inclusion relations between the sets of $S_\lambda^\alpha(\Delta^m)$ -statistically convergent sequences of order α for different α 's. In Theorem 9 we give the relationship between the strong $w_\lambda^\alpha(\Delta^m, p)$ -summability of order α and the strong $w_\lambda^\beta(\Delta^m, p)$ -summability of order β . In Theorem 12 we give the relationship between the strong $w_\lambda^\alpha(\Delta^m, p)$ -summability of order α and the $S_\lambda^\beta(\Delta^m)$ -statistical convergence of order β .

Definition 1. Let the sequence $\lambda = (\lambda_n)$ of real numbers be defined as above, and let $0 < \alpha \leq 1$ be given. The sequence $x = (x_k) \in w$ is said to be $S_\lambda^\alpha(\Delta^m)$ -statistically convergent of order α if there is a real number L such that

$$\lim_{n \rightarrow \infty} \frac{1}{\lambda_n^\alpha} |\{k \in I_n : |\Delta^m x_k - L| \geq \varepsilon\}| = 0, \quad (4)$$

where $I_n = [n - \lambda_n + 1, n]$ and λ_n^α denote the α th power $(\lambda_n)^\alpha$ of λ_n ; that is $\lambda^\alpha = (\lambda_n^\alpha) = (\lambda_1^\alpha, \lambda_2^\alpha, \dots, \lambda_n^\alpha, \dots)$. In this case we write $S_\lambda^\alpha(\Delta^m) - \lim x_k = L$. The set of all $S_\lambda^\alpha(\Delta^m)$ -statistically convergent sequences of order α will be denoted by $S_\lambda^\alpha(\Delta^m)$. For $\lambda_n = n$ for all $n \in \mathbb{N}$, we will write $S^\alpha(\Delta^m)$ instead of $S_\lambda^\alpha(\Delta^m)$ and in the special case $\alpha = 1$, we will write $S_\lambda(\Delta^m)$ instead of $S_\lambda^\alpha(\Delta^m)$, and also in the special case $\alpha = 1$ and $\lambda_n = n$ for all $n \in \mathbb{N}$ we will write $S(\Delta^m)$ instead of $S_\lambda^\alpha(\Delta^m)$, which were defined and studied by Et and Nuray [25].

The $S_\lambda^\alpha(\Delta^m)$ -statistical convergence of order α is well defined for $0 < \alpha \leq 1$, but it is not well defined for $\alpha > 1$ in general. For this let $\lambda_n = n$ for all $n \in \mathbb{N}$ and $x = (x_k)$ be defined as follows:

$$x_k = \begin{cases} -1, & \text{if } k \text{ is odd} \\ 1, & \text{if } k \text{ is even,} \end{cases} \quad (5)$$

and so

$$\Delta^m x_k = \begin{cases} -2^m, & \text{if } k \text{ is odd} \\ 2^m, & \text{if } k \text{ is even;} \end{cases} \quad (6)$$

then both

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{\lambda_n^\alpha} |\{k \in I_n : |\Delta^m x_k - (-2^m)| \geq \varepsilon\}| &= \lim_{n \rightarrow \infty} \frac{\lambda_n}{2\lambda_n^\alpha} = 0, \\ \lim_{n \rightarrow \infty} \frac{1}{\lambda_n^\alpha} |\{k \in I_n : |\Delta^m x_k - 2^m| \geq \varepsilon\}| &= \lim_{n \rightarrow \infty} \frac{\lambda_n}{2\lambda_n^\alpha} = 0 \end{aligned} \quad (7)$$

for $\alpha > 1$, and so $S_\lambda^\alpha(\Delta^m)$ -statistically converges of order α , both to -2^m and 2^m ; that is, $S_\lambda^\alpha(\Delta^m) - \lim x_k = 2^m$ and $S_\lambda^\alpha(\Delta^m) - \lim x_k = -2^m$. But this is impossible.

Theorem 2. Let $0 < \alpha \leq 1$ and $x = (x_k), y = (y_k)$ be sequences of real numbers; then

- (i) if $S_\lambda^\alpha(\Delta^m) - \lim x_k = x_0$ and $c \in \mathbb{R}$, then $S_\lambda^\alpha(\Delta^m) - \lim(cx_k) = cx_0$,
- (ii) if $S_\lambda^\alpha(\Delta^m) - \lim x_k = x_0$ and $S_\lambda^\alpha(\Delta^m) - \lim y_k = y_0$, then $S_\lambda^\alpha(\Delta^m) - \lim(x_k + y_k) = x_0 + y_0$.

Proof. Omitted. $\square$

Definition 3. Let the sequence $\lambda = (\lambda_n)$ be as above, $\alpha \in (0, 1]$ any real number, and p a positive real number. A sequence x is said to be strongly $w_\lambda^\alpha(\Delta^m, p)$ -summable of order α if there is a real number L such that

$$\lim_{n \rightarrow \infty} \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} |\Delta^m x_k - L|^p = 0, \quad (8)$$

where $I_n = [n - \lambda_n + 1, n]$. In this case we write $w_\lambda^\alpha(\Delta^m, p) - \lim x_k = L$. The strong $w_\lambda^\alpha(\Delta^m, p)$ -summability of order α reduces to the strong $w_\lambda(\Delta^m, p)$ -summability for $\alpha = 1$. The set of all strongly $w_\lambda^\alpha(\Delta^m, p)$ -summable sequences of order α will be denoted by $w_\lambda^\alpha(\Delta^m, p)$.

Theorem 4. Let the sequence $\lambda = (\lambda_n)$ be as above, $\alpha_o \in (0, 1]$, and p_o a positive real number. The sequence space $w_\lambda^{\alpha_o}(\Delta^m, p_o)$ is a Banach space for $1 \leq p_o < \infty$ normed by

$$\|x\|_\Delta = \sum_{i=1}^m |x_i| + \sup_n \left(\frac{1}{\lambda_n^{\alpha_o}} \sum_{k \in I_n} |\Delta^m x_k|^{p_o} \right)^{1/p_o} \quad (9)$$

and a complete p -normed space for $0 < p_o < 1$ by

$$\|x\|_{\Delta'} = \sum_{i=1}^m |x_i|^{p_o} + \sup_n \frac{1}{\lambda_n^{\alpha_o}} \sum_{k \in I_n} |\Delta^m x_k|^{p_o}. \quad (10)$$

Proof. It is a routine verification that $w_\lambda^{\alpha_o}(\Delta^m, p_o)$ is a normed space normed by (9). Let (x^s) be a Cauchy sequence such that $x^s = (x_k^s)_k = (x_1^s, x_2^s, \dots) \in w_\lambda^{\alpha_o}(\Delta^m, p_o)$ for each $s \in \mathbb{N}$. Then we have

$$\begin{aligned} \|x^s - x^t\|_\Delta &= \sum_{i=1}^m |x_i^s - x_i^t| \\ &+ \sup_n \left(\frac{1}{\lambda_n^{\alpha_o}} \sum_{k \in I_n} |\Delta^m x_k^s - \Delta^m x_k^t|^{p_o} \right)^{1/p_o} \rightarrow 0 \end{aligned} \quad (11)$$

as $s, t \rightarrow \infty$ and so that

$$\sum_{i=1}^m |x_i^s - x_i^t| \rightarrow 0, \quad (12)$$

$$\left(\frac{1}{\lambda_n^{\alpha_o}} \sum_{k \in I_n} |\Delta^m x_k^s - \Delta^m x_k^t|^{p_o} \right)^{1/p_o} \rightarrow 0, \quad \forall n \in \mathbb{N}$$

as $s, t \rightarrow \infty$. Now for $k \leq m$ we get $|x_i^s - x_i^t| \rightarrow 0$ as $s, t \rightarrow \infty$, and for all $k \in \mathbb{N}$ we get $|\Delta^m x_k^s - \Delta^m x_k^t| \rightarrow 0$ as $s, t \rightarrow \infty$. Since the inequality

$$\begin{aligned} |\Delta^m x_{k+m}^s - \Delta^m x_{k+m}^t| &\leq |\Delta^m x_k^s - \Delta^m x_k^t| \\ &+ \binom{m}{0} |x_k^s - x_k^t| + \cdots + \binom{m}{m-1} \\ &\times |\Delta^m x_{k+m-1}^s - \Delta^m x_{k+m-1}^t| \end{aligned} \quad (13)$$

holds for all $k \in \mathbb{N}$, we get $|x_i^s - x_i^t| \rightarrow 0$ as $s, t \rightarrow \infty$ and for each $k \in \mathbb{N}$. This implies that the sequence $(x_k^s)_k = (x_k^1, x_k^2, \dots)$ is a Cauchy in $\mathbb{R}$ for each $k \in \mathbb{N}$. Also, it is convergent, because $\mathbb{R}$ is complete. Assume that $\lim_s x_k^s = x_k$ for each $k \in \mathbb{N}$. Since (x^s) is a Cauchy sequence, then for every $\varepsilon > 0$ there exists a number $n_0 = n_0(\varepsilon)$ such that

$$\|x^s - x^t\|_{\Delta} < \varepsilon \quad (14)$$

for all $s, t \geq n_0$. Hence for all $n \in \mathbb{N}$ and all $s, t \geq n_0$ we get

$$\begin{aligned} \sum_{i=1}^m |x_i^s - x_i^t| &< \varepsilon, \\ \frac{1}{\lambda_n^{\alpha_o}} \sum_{k \in I_n} |\Delta^m x_k^s - \Delta^m x_k^t|^{p_o} &< \varepsilon^{p_o}, \quad \forall n \in \mathbb{N}. \end{aligned} \quad (15)$$

Taking limit as $t \rightarrow \infty$ in the last inequalities, we get

$$\begin{aligned} \lim_t \sum_{i=1}^m |x_i^s - x_i^t| &= \sum_{i=1}^m |x_i^s - x_i| < \varepsilon, \\ \lim_t \frac{1}{\lambda_n^{\alpha_o}} \sum_{k \in I_n} |\Delta^m x_k^s - \Delta^m x_k^t|^{p_o} &= \frac{1}{\lambda_n^{\alpha_o}} \sum_{k \in I_n} |\Delta^m x_k^s - \Delta^m x_k|^{p_o} < \varepsilon^{p_o} \end{aligned} \quad (16)$$

for $s \geq n_0$ and all $n \in \mathbb{N}$. This implies that

$$\begin{aligned} \|x^s - x\|_{\Delta} &= \sum_{i=1}^m |x_i^s - x_i| \\ &+ \sup_n \left(\frac{1}{\lambda_n^{\alpha_o}} \sum_{k \in I_n} |\Delta^m x_k^s - \Delta^m x_k|^{p_o} \right)^{1/p_o} \rightarrow 0, \end{aligned} \quad (17)$$

and hence $x^s \rightarrow x$ as $s \rightarrow \infty$. Since

$$\begin{aligned} \frac{1}{\lambda_n^{\alpha_o}} \sum_{k \in I_n} |\Delta^m x_k - L|^{p_o} \\ \leq 2^{p_o} \left\{ \frac{1}{\lambda_n^{\alpha_o}} \sum_{k \in I_n} |\Delta^m x_k^{n_0} - L|^{p_o} + \frac{1}{\lambda_n^{\alpha_o}} \sum_{k \in I_n} |\Delta^m x_k^{n_0} - \Delta^m x_k|^{p_o} \right\}, \end{aligned} \quad (18)$$

we get $x \in w_{\lambda}^{\alpha_o}(\Delta^m, p_o)$.

It can be shown that $w_{\lambda}^{\alpha_o}(\Delta^m, p_o)$ is a complete p -normed space for $0 < p_o < 1$ by (10). $\square$

Theorem 5. If $0 < \alpha \leq \beta \leq 1$, then $S_{\lambda}^{\alpha}(\Delta^m) \subseteq S_{\lambda}^{\beta}(\Delta^m)$ and the inclusion is strict.

Proof. Proof follows from the inequality

$$\begin{aligned} \frac{1}{\lambda_n^{\beta}} |\{k \in I_n : |\Delta^m x_k - L| \geq \varepsilon\}| \\ \leq \frac{1}{\lambda_n^{\alpha}} |\{k \in I_n : |\Delta^m x_k - L| \geq \varepsilon\}|. \end{aligned} \quad (19)$$

To show that the inclusion is strict, define $x = (x_k)$ such that

$$\Delta^m x_k = \begin{cases} k, & \text{if } k \text{ is square} \\ 0, & \text{otherwise.} \end{cases} \quad (20)$$

Then $x \in S_{\lambda}^{\beta}(\Delta^m)$ for $1/2 < \beta \leq 1$, but $x \notin S_{\lambda}^{\alpha}(\Delta^m)$ for $0 < \alpha \leq 1/2$. $\square$

From Theorem 5 we have the following.

Corollary 6. If a sequence is $S_{\lambda}^{\alpha}(\Delta^m)$ -statistically convergent of order α , to L , then it is $S_{\lambda}(\Delta^m)$ -statistically convergent to L .

Theorem 7. $S^{\alpha}(\Delta^m) \subseteq S_{\lambda}^{\alpha}(\Delta^m)$ if

$$\liminf_{n \rightarrow \infty} \frac{\lambda_n^{\alpha}}{n^{\alpha}} > 0. \quad (21)$$

Proof. For given $\varepsilon > 0$ we have

$$\{k \leq n : |\Delta^m x_k - L| \geq \varepsilon\} \supset \{k \in I_n : |\Delta^m x_k - L| \geq \varepsilon\} \quad (22)$$

and so

$$\begin{aligned} \frac{1}{n^{\alpha}} |\{k \leq n : |\Delta^m x_k - L| \geq \varepsilon\}| \\ \geq \frac{\lambda_n^{\alpha}}{n^{\alpha} \lambda_n^{\alpha}} |\{k \in I_n : |\Delta^m x_k - L| \geq \varepsilon\}|. \end{aligned} \quad (23)$$

Taking the limit as $n \rightarrow \infty$ and using (21), we get $S_{\lambda}^{\alpha}(\Delta^m) = \lim x_k = L$. $\square$

Theorem 8. $S(\Delta^m) = S_{\lambda}^{\alpha}(\Delta^m)$ if and only if

$$\liminf_{n \rightarrow \infty} \frac{\lambda_n^{\alpha}}{n} > 0. \quad (24)$$

Proof. From Theorem 7 we have $S(\Delta^m) \subseteq S_\lambda^\alpha(\Delta^m)$.

Conversely, suppose that $\lim_{n \rightarrow \infty} \inf \lambda_n^\alpha/n = 0$. We can choose a subsequence $(n(j))_{j=1}^\infty$ such that $\lambda_n^\alpha(j)/n(j) < 1/j$. If we consider a sequence $x = (x_k)$ defined by

$$\Delta^m x_i = \begin{cases} 1, & \text{if } i \in I_{n(j)} \text{ for some } j = 1, 2, 3, \dots \\ 0, & \text{otherwise,} \end{cases} \quad (25)$$

then $x \in S(\Delta^m)$ but $x \notin S_\lambda^\alpha(\Delta^m)$. From Corollary 6 we have $x \notin S_\lambda^\alpha(\Delta^m)$. This completes the proof. $\square$

Theorem 9. Let $0 < \alpha \leq \beta \leq 1$ and p a positive real number; then $w_\lambda^\alpha(\Delta^m, p) \subseteq w_\lambda^\beta(\Delta^m, p)$ and the inclusion is strict.

Proof. Let $x = (x_k) \in w_\lambda^\alpha(\Delta^m, p)$. Then given α and β such that $0 < \alpha \leq \beta \leq 1$ and a positive real number p , we may write

$$\frac{1}{\lambda_n^\beta} \sum_{k \in I_n} |\Delta^m x_k - L|^p \leq \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} |\Delta^m x_k - L|^p \quad (26)$$

and this gives that $w_\lambda^\alpha(\Delta^m, p) \subseteq w_\lambda^\beta(\Delta^m, p)$.

To show that the inclusion is strict define $x = (x_k)$ such that

$$\Delta^m x_k = \begin{cases} 1, & \text{if } k \text{ is square} \\ 0, & \text{otherwise.} \end{cases} \quad (27)$$

Then $x \in w_\lambda^\beta(\Delta^m, p)$ for $1/2 < \beta \leq 1$ but $x \notin w_\lambda^\alpha(\Delta^m, p)$ for $0 < \alpha \leq 1/2$, for $\lambda = (n)$. $\square$

The following result is a consequence of Theorem 9.

Corollary 10. Let $0 < \alpha \leq \beta \leq 1$ and p a positive real number. Then

- (i) if $\alpha = \beta$, then $w_\lambda^\alpha(\Delta^m, p) = w_\lambda^\beta(\Delta^m, p)$,
- (ii) $w_\lambda^\alpha(\Delta^m, p) \subseteq w_\lambda^\beta(\Delta^m, p)$ for each $\alpha \in (0, 1]$ and $0 < p < \infty$.

Theorem 11. Let $0 < \alpha \leq 1$ and $0 < p < q < \infty$. Then $w_\lambda^\alpha(\Delta^m, q) \subset w_\lambda^\alpha(\Delta^m, p)$.

Proof is seen from the Hölder inequality.

Theorem 12. Let α and β be fixed real numbers such that $0 < \alpha \leq \beta \leq 1$ and $0 < p < \infty$. If a sequence is strongly $w_\lambda^\alpha(\Delta^m, p)$ -summable of order α , to L , then it is $S_\lambda^\beta(\Delta^m)$ -statistically convergent of order β , to L .

Proof. For any sequence $x = (x_k)$ and $\varepsilon > 0$, we have

$$\begin{aligned} \sum_{k \in I_n} |\Delta^m x_k - L|^p &= \sum_{\substack{k \in I_n \\ |\Delta^m x_k - L| \geq \varepsilon}} |\Delta^m x_k - L|^p \\ &\quad + \sum_{\substack{k \in I_n \\ |\Delta^m x_k - L| < \varepsilon}} |\Delta^m x_k - L|^p \\ &\geq \sum_{\substack{k \in I_n \\ |\Delta^m x_k - L| \geq \varepsilon}} |\Delta^m x_k - L|^p \\ &\geq |\{k \in I_n : |\Delta^m x_k - L| \geq \varepsilon\}| \varepsilon^p \end{aligned} \quad (28)$$

and so

$$\begin{aligned} &\frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} |\Delta^m x_k - L|^p \\ &\geq \frac{1}{\lambda_n^\alpha} |\{k \in I_n : |\Delta^m x_k - L| \geq \varepsilon\}| \varepsilon^p \\ &\geq \frac{1}{\lambda_n^\beta} |\{k \in I_n : |\Delta^m x_k - L| \geq \varepsilon\}| \varepsilon^p. \end{aligned} \quad (29)$$

From this it follows that if $x = (x_k)$ is strongly $w_\lambda^\alpha(\Delta^m, p)$ -summable of order α , to L , then it is $S_\lambda^\beta(\Delta^m)$ -statistically convergent of order β , to L . $\square$

Corollary 13. Let α be a fixed real number such that $0 < \alpha \leq 1$ and $0 < p < \infty$.

- (i) If a sequence is strongly $w_\lambda^\alpha(\Delta^m, p)$ -summable of order α , to L , then it is $S_\lambda^\alpha(\Delta^m)$ -statistically convergent of order α , to L .
- (ii) If a sequence is strongly $w_\lambda^\alpha(\Delta^m, p)$ -summable of order α , to L , then it is $S_\lambda(\Delta^m)$ -statistically convergent to L .

3. Results Related to the Orlicz Function

In this section we give the inclusion relations between the sets of $S_\lambda^\alpha(\Delta^m)$ -statistically convergent sequences of order α and strongly $w_\lambda^\alpha(\Delta^m, (p), M)$ -summable sequences of order α with respect to the Orlicz function M .

An Orlicz function is a function $M : [0, \infty) \rightarrow [0, \infty)$, which is continuous, nondecreasing, and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \rightarrow \infty$ as $x \rightarrow \infty$. An Orlicz function M can always be represented in the following integral form:

$$M(x) = \int_0^x q(t) dt, \quad (30)$$

where q known as the kernel of M is right differentiable for $t \geq 0$, $q(0) = 0$, $q(t) > 0$ for $t > 0$, q is nondecreasing, and $q(t) \rightarrow \infty$ as $t \rightarrow \infty$.

Let (Ω, Σ, μ) be a finite measure space. We denote by $E(\mu)$ the space of all (equivalence classes of) Σ -measurable functions x from Ω into $[0, \infty)$. Given an Orlicz function M , we define on $E(\mu)$ a convex functional I_M by

$$I_M(x) = \int_\Omega M(x(t)) d\mu, \quad (31)$$

and an Orlicz space $L^M(\mu)$ by $L^M(\mu) = \{x \in E(\mu) : I_M(\lambda x) < +\infty \text{ for some } \lambda > 0\}$ (for detail see [26]).

The study of Orlicz sequence spaces was initiated with a certain specific purpose in the Banach space theory. Indeed, Lindberg [27] got interested in Orlicz spaces in connection with finding the Banach spaces with symmetric Schauder bases having complementary subspaces isomorphic to c_0 or

ℓ_p ($1 \leq p < \infty$). Subsequently, Lindenstrauss and Tzafriri [28] used the idea of the Orlicz function to construct the sequence space

$$\ell_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty \text{ for some } \rho > 0 \right\}. \quad (32)$$

The space ℓ_M with the norm

$$\|x\| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\} \quad (33)$$

becomes a Banach space, called an Orlicz sequence space. The space ℓ_M is closely related to the space ℓ_p which is an Orlicz sequence space with $M(x) = |x|^p$ for $1 \leq p < \infty$. Lindenstrauss and Tzafriri [28] proved that every Orlicz sequence space ℓ_M contains a subspace isomorphic to ℓ_p ($1 \leq p < \infty$). The Orlicz sequence spaces are the special cases of Orlicz spaces studied in [26].

It is well known that if M is a convex function and $M(0) = 0$, then $M(\lambda x) \leq \lambda M(x)$ for all λ with $0 < \lambda < 1$.

Recently, the Orlicz sequence spaces have been studied by Altin et al. [29], Bhardwaj and Singh [30], Et et al. [31, 32], Mursaleen et al. [33], Savaş and Rhoades [34], and many others.

Definition 14. Let M an Orlicz function, $p = (p_k)$ a sequence of strictly positive real numbers, and $\alpha \in (0, 1]$ any real number. Now we define

$$w_{\lambda}^{\alpha}(\Delta^m, (p), M) = \left\{ x = (x_k) : \lim_{n \rightarrow \infty} \frac{1}{\lambda_n^{\alpha}} \times \sum_{k \in I_n} \left[M\left(\frac{|\Delta^m x_k - L|}{\rho}\right) \right]^{p_k} = 0, \right. \\ \left. \text{for some } L \text{ and } \rho > 0 \right\}. \quad (34)$$

If $x \in w_{\lambda}^{\alpha}(\Delta^m, (p), M)$, then we say that x is strongly $w_{\lambda}^{\alpha}(\Delta^m, (p), M)$ -summable of order α with respect to the Orlicz function M . For $p_k = p$ for all $k \in \mathbb{N}$ and $M(x) = x$ we will write $w_{\lambda}^{\alpha}(\Delta^m, p)$ instead of $w_{\lambda}^{\alpha}(\Delta^m, (p), M)$ and in the special case $M(x) = x$ we will write $w_{\lambda}^{\alpha}(\Delta^m, (p))$ instead of $w_{\lambda}^{\alpha}(\Delta^m, (p), M)$.

In the following two theorems we will assume that the sequence $p = (p_k)$ is bounded and $0 < h = \inf_k p_k \leq p_k \leq \sup_k p_k = H < \infty$.

Theorem 15. Let $\alpha, \beta \in (0, 1]$ be real numbers such that $\alpha \leq \beta$ and M an Orlicz function; then $w_{\lambda}^{\alpha}(\Delta^m, (p), M) \subset S_{\lambda}^{\beta}(\Delta^m)$.

Proof. Let $x \in w_{\lambda}^{\alpha}(\Delta^m, (p), M)$, $\varepsilon > 0$ be given, and let $\sum_1$ and $\sum_2$ denote the sums over $k \in I_n$, $|\Delta^m x_k - L| \geq \varepsilon$ and $k \in I_n$,

$|\Delta^m x_k - L| < \varepsilon$, respectively. Since $\lambda_n^{\alpha} \leq \lambda_n^{\beta}$ for each n we may write

$$\begin{aligned} & \frac{1}{\lambda_n^{\alpha}} \sum_{k \in I_n} \left[M\left(\frac{|\Delta^m x_k - L|}{\rho}\right) \right]^{p_k} \\ &= \frac{1}{\lambda_n^{\alpha}} \left[\sum_1 \left[M\left(\frac{|\Delta^m x_k - L|}{\rho}\right) \right]^{p_k} \right. \\ & \quad \left. + \sum_2 \left[M\left(\frac{|\Delta^m x_k - L|}{\rho}\right) \right]^{p_k} \right] \\ &\geq \frac{1}{\lambda_n^{\beta}} \left[\sum_1 \left[M\left(\frac{|\Delta^m x_k - L|}{\rho}\right) \right]^{p_k} \right. \\ & \quad \left. + \sum_2 \left[M\left(\frac{|\Delta^m x_k - L|}{\rho}\right) \right]^{p_k} \right] \quad (35) \\ &\geq \frac{1}{\lambda_n^{\beta}} \sum_1 [f(\varepsilon)]^{p_k} \\ &\geq \frac{1}{\lambda_n^{\beta}} \sum_1 \min \left([M(\varepsilon_1)]^h, [M(\varepsilon_1)]^H \right), \quad \varepsilon_1 = \frac{\varepsilon}{\rho} \\ &\geq \frac{1}{\lambda_n^{\beta}} |\{k \in I_n : |\Delta^m x_k - L| \geq \varepsilon\}| \\ & \quad \times \min \left([f(\varepsilon_1)]^h, [f(\varepsilon_1)]^H \right). \end{aligned}$$

Since $x \in w_{\lambda}^{\alpha}(\Delta^m, (p), M)$, the left-hand side of the above inequality tends to zero as $n \rightarrow \infty$. Therefore the right-hand side tends to zero as $n \rightarrow \infty$ and hence $x \in S_{\lambda}^{\beta}(\Delta^m)$, because $\min([f(\varepsilon_1)]^h, [f(\varepsilon_1)]^H) > 0$. $\square$

Corollary 16. Let $0 < \alpha \leq 1$ and M an Orlicz function; then $w_{\lambda}^{\alpha}(\Delta^m, (p), M) \subset S_{\lambda}^{\alpha}(\Delta^m)$.

Theorem 17. Let M be an Orlicz function and $x = (x_k)$ Δ^m -bounded sequence; then $S_{\lambda}^{\alpha}(\Delta^m) \subset w_{\lambda}^{\alpha}(\Delta^m, (p), M)$.

Proof. Suppose that $x = (x_k)$ a Δ^m -bounded sequence; that is, $x \in \ell_{\infty}(\Delta^m)$ and $S_{\lambda}^{\alpha}(\Delta^m) - \lim x_k = L$. Since $x \in \ell_{\infty}(\Delta^m)$, then there is a constant $T > 0$ such that $|\Delta^m x_k| \leq T$. Given $\varepsilon > 0$ we have

$$\begin{aligned} & \frac{1}{\lambda_n^{\alpha}} \sum_{k \in I_n} \left[M\left(\frac{|\Delta^m x_k - L|}{\rho}\right) \right]^{p_k} \\ &= \frac{1}{\lambda_n^{\alpha}} \sum_1 \left[M\left(\frac{|\Delta^m x_k - L|}{\rho}\right) \right]^{p_k} \\ & \quad + \frac{1}{\lambda_n^{\alpha}} \sum_2 \left[M\left(\frac{|\Delta^m x_k - L|}{\rho}\right) \right]^{p_k} \\ &\leq \frac{1}{\lambda_n^{\alpha}} \sum_1 \max \left\{ \left[M\left(\frac{T}{\rho}\right) \right]^h, \left[M\left(\frac{T}{\rho}\right) \right]^H \right\} \\ & \quad + \frac{1}{\lambda_n^{\alpha}} \sum_2 \left[M\left(\frac{\varepsilon}{\rho}\right) \right]^{p_k} \end{aligned}$$

$$\begin{aligned}
&\leq \max \{ [M(K)]^h, [M(K)]^H \} \\
&\quad \times \frac{1}{\lambda_n^\alpha} |\{k \in I_n : |\Delta^m x_k - L| \geq \varepsilon\}| \\
&\quad + \max \{ [M(\varepsilon_1)]^h, [M(\varepsilon_1)]^H \}, \\
&\quad \frac{T}{\rho} = K, \quad \frac{\varepsilon}{\rho} = \varepsilon_1.
\end{aligned} \tag{36}$$

Hence $x \in w_\lambda^\alpha(\Delta^m, (p), M)$. $\square$

Theorem 18. *The sequence spaces $w_\lambda^\alpha(\Delta^m, p)$, $S_\lambda^\alpha(\Delta^m)$, and $w_\lambda^\alpha(\Delta^m, (p), M)$ are neither solid nor symmetric, nor sequence algebras for $m \geq 1$.*

Proof. Let $\lambda_k = k$ for all $k \in \mathbb{N}$; then $(x_k) = (k^{m-1}) \in w_\lambda^\alpha(\Delta^m, p)$ but $(\alpha_k x_k) \notin w_\lambda^\alpha(\Delta^m, p)$ when $\alpha_k = (-1)^k$ for all $k \in \mathbb{N}$. Hence $w_\lambda^\alpha(\Delta^m, p)$ is not solid. The other cases can be proved on considering similar examples. $\square$

Theorem 19. *If $\lim p_k > 0$ and $x = (x_k)$ is strongly $w_\lambda^\alpha(\Delta^m, (p), M)$ -summable of order α , to L with respect to the Orlicz function M , then $w_\lambda^\alpha(\Delta^m, (p), M) - \lim x_k = L$ uniquely.*

Proof. Let $\lim p_k = s > 0$. Suppose that $w_\lambda^\alpha(\Delta^m, (p), M) - \lim x_k = L$ and $w_\lambda^\alpha(\Delta^m, (p), M) - \lim x_k = L_1$. Then

$$\begin{aligned}
\lim_{n \rightarrow \infty} \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} \left[M \left(\frac{|\Delta^m x_k - L|}{\rho_1} \right) \right]^{p_k} &= 0, \quad \text{for some } \rho_1 > 0, \\
\lim_{n \rightarrow \infty} \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} \left[M \left(\frac{|\Delta^m x_k - L_1|}{\rho_2} \right) \right]^{p_k} &= 0, \quad \text{for some } \rho_2 > 0.
\end{aligned} \tag{37}$$

Define $\rho = \max(2\rho_1, 2\rho_2)$. Since M is nondecreasing and convex, we have

$$\begin{aligned}
&\frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} \left[M \left(\frac{|L - L_1|}{\rho} \right) \right]^{p_k} \\
&\leq \frac{D}{\lambda_n^\alpha} \sum_{k \in I_n} \frac{1}{2^{p_k}} \left(\left[M \left(\frac{|\Delta^m x_k - L|}{\rho_1} \right) \right]^{p_k} \right. \\
&\quad \left. + \left[M \left(\frac{|\Delta^m x_k - L_1|}{\rho_2} \right) \right]^{p_k} \right) \\
&\leq \frac{D}{\lambda_n^\alpha} \sum_{k \in I_n} \left[M \left(\frac{|\Delta^m x_k - L|}{\rho_1} \right) \right]^{p_k} \\
&\quad + \frac{D}{\lambda_n^\alpha} \sum_{k \in I_n} \left[M \left(\frac{|\Delta^m x_k - L_1|}{\rho_2} \right) \right]^{p_k} \rightarrow 0, \quad (n \rightarrow \infty),
\end{aligned} \tag{38}$$

where $\sup_k p_k = H$ and $D = \max(1, 2^{H-1})$. Hence

$$\lim_{n \rightarrow \infty} \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} \left[M \left(\frac{|L - L_1|}{\rho} \right) \right]^{p_k} = 0. \tag{39}$$

Since $\lim_{k \rightarrow \infty} p_k = s$ we have

$$\lim_{k \rightarrow \infty} \left[M \left(\frac{|L - L_1|}{\rho} \right) \right]^{p_k} = \left[M \left(\frac{|L - L_1|}{\rho} \right) \right]^s \tag{40}$$

and so $L = L_1$. Thus the limit is unique. $\square$

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