Hindawi Publishing Corporation
Journal of Function Spaces and Applications
Volume 2013, Article ID 374268, 5 pages
http://dx.doi.org/10.1155/2013/374268

Research Article

Mixed Equilibrium Problems with Weakly Relaxed α-Monotone Bifunction in Banach Spaces

Wutiphol Sintunavarat

Department of Mathematics and Statistics, Faculty of Science and Technology, Thammasat University Rangsit Center, Pathumthani 12121, Thailand

Correspondence should be addressed to Wutiphol Sintunavarat; poom_teun@hotmail.com

Received 28 May 2013; Accepted 1 August 2013

Academic Editor: Sompong Dhompongsa

We introduce the class of mixed equilibrium problems with the weakly relaxed α-monotone bi-function in Banach spaces. Using the KKM technique, we obtain the existence of solutions for mixed equilibrium problem with weakly relaxed α-monotone bi-function in Banach spaces. The results presented in this paper extend and improve the corresponding results in the existing literature.

1. Introduction

Let K be a nonempty subset of a real reflexive Banach space X. Let $\varphi : K \to \mathbb{R}$ be a real valued function and let $f : K \times K \to \mathbb{R}$ be an equilibrium bi-function; that is, $f(x, x) = 0$, for all $x \in K$. Then the mixed equilibrium problem (for short, (MEP)) is to find $\tilde{x} \in K$ such that

$$f(\tilde{x}, y) + \varphi(y) - \varphi(\tilde{x}) \geq 0, \quad \forall y \in K. \tag{1}$$

In particular, if $\varphi \equiv 0$, this problem reduces to the classical equilibrium problem (for short, (EP)), which is to find $\tilde{x} \in K$ such that

$$f(\tilde{x}, y) \geq 0, \quad \forall y \in K. \tag{2}$$

Equilibrium problems and mixed equilibrium problems play an important role in many fields, such as economics, physics, mechanics, and engineering sciences. Also, the equilibrium problems and mixed equilibrium problems include many mathematical problems as particular cases for example, mathematical programming problems, complementary problems, variational inequality problems, Nash equilibrium problems in noncooperative games, minimax inequality problems, and fixed point problems. Because of their wide applicability, equilibrium problems and mixed equilibrium problems have been generalized in various directions for the past several years.

The monotonicity and generalized monotonicity play an important role in the study of equilibrium problems and mixed equilibrium problems. In recent years, a substantial number of papers on existence results for solving equilibrium problems and mixed equilibrium problems based on different generalization of monotonicity such as pseudomonotonicity, quasimonotonicity, relaxed monotonicity, semimonotonicity, p-monotonicity, and C_x-pseudomonotonicity, see [1–17], appeared.

In 1990, Karamardian and Schaible [14] introduced various kinds of generalized monotone mappings. Afterward, several researcher in [2–4] extended the idea of Karamardian and Schaible [14] for bi-functions to study equilibrium problems.

In 2003, Fang and Huang [10] considered two classes of the variational-like inequalities with the relaxed $\eta - \alpha$ monotone and relaxed $\eta - \alpha$ semimonotone mappings. They obtained the existence solutions of variational-like inequalities with relaxed $\eta - \alpha$ monotone and relaxed $\eta - \alpha$ semimonotone mappings in Banach spaces using the KKM technique. Later, Bai et al. [1] introduced a new concept of relaxed $\eta - \alpha$ pseudomonotone mappings and obtained the solutions for the variational-like inequalities. Afterward, Mahato and Nahak [18] defined the weakly relaxed $\eta - \alpha$ pseudomonotone bi-function to study the equilibrium problems.

Recently, Mahato and Nahak [19] introduce the concept of the relaxed α-monotonicity for bi-functions. They also obtained the existence of solutions for mixed equilibrium

problems with the relaxed α-monotone bi-function in reflexive Banach spaces, by using the KKM technique.

The purpose of this paper is to introduce the class of weakly relaxed α-monotone bi-functions which contain the class of relaxed α-monotone bi-functions. The existence of solutions for mixed equilibrium problems with bi-function in such class is given. Our results in this paper extend and improve the results of Mahato and Nahak [19] and many results in the literature.

2. Preliminaries

In this paper, unless otherwise specified, let K be a nonempty closed convex subset of a real reflexive Banach space X. The following definitions and lemma will be useful in our paper.

Definition 1. A real valued function f defined on a convex subset K of X is said to be hemicontinuous if $\lim_{t\to 0^+} f(tx + (1-t)y) = f(y)$, for each $x, y \in K$.

Definition 2. Let $F : K \to 2^X$ be a set-valued mapping. Then F is said to be KKM mapping if for any finite subset $\{y_1, y_2, \ldots, y_n\}$ of K, we have $\mathrm{co}\{y_1, y_2, \ldots, y_n\} \subset \bigcup_{i=1}^{n} F(y_i)$, where $\mathrm{co}\{y_1, y_2, \ldots, y_n\}$ denotes the convex hull of $\{y_1, y_2, \ldots, y_n\}$.

Remark 3. Let $F, G : K \to 2^X$. If F is KKM mapping and $F(y) \subset G(y)$ for all $y \in K$, then G also is KKM mapping.

Lemma 4 (see [20]). *Let M be a nonempty subset of a Hausdorff topological vector space X and let $F : M \to 2^X$ be a KKM mapping. If $F(y)$ is closed in X for all $y \in M$ and compact for some $y \in M$, then $\bigcap_{y\in M} F(y) \neq \emptyset$.*

Definition 5. Let X be a Banach space. A function $f : X \to \mathbb{R}$ is lower semicontinuous at $x_0 \in X$ if

$$f(x_0) \le \liminf_{n\to\infty} f(x_n) \tag{3}$$

for any sequence $\{x_n\}$ in X such that $\{x_n\}$ converges to x_0.

Definition 6. Let X be a Banach space. A function $f : X \to \mathbb{R}$ is weakly upper semicontinuous at $x_0 \in X$ if

$$f(x_0) \ge \limsup_{n\to\infty} f(x_n) \tag{4}$$

for any sequence $\{x_n\}$ in X such that $\{x_n\}$ converges to x_0 weakly.

Definition 7 (see [19]). A bi-function $f : K\times K \to \mathbb{R}$ is said to be a relaxed α-monotone if there exists a function $\alpha : X \to \mathbb{R}$ with $\alpha(tx) = t^p\alpha(x)$ for all $t > 0$ and $x \in K$ such that

$$f(x, y) + f(y, x) \le \alpha(y - x), \quad \forall x, y \in K, \tag{5}$$

where $p > 1$ is a real constant.

Remark 8. If $\alpha \equiv 0$ then from (5), it follows that f is monotone; that is,

$$f(x, y) + f(y, x) \le 0, \quad \forall x, y \in K. \tag{6}$$

Therefore, the monotonicity implies relaxed α-monotonicity. However, the converse of previous statement is not true in general, which is shown by the following examples.

Example 9. Let $X = \mathbb{R}$, $K = \mathbb{R}$ and let bi-function $f : K \times K \to \mathbb{R}$ be defined by

$$f(x, y) = \sin(x - y)^2 + (x - y)^2, \tag{7}$$

for all $x, y \in K$. Then

$$f(x, y) + f(y, x) = 2\sin(x - y)^2 + 2(x - y)^2 \nleq 0, \tag{8}$$

when $x \neq y$. So f is not monotone.

However, it easy to see that f is a relaxed α-monotone with $\alpha(x) = 5x^2$. In fact,

$$\begin{aligned} f(x, y) &+ f(y, x) \\ &= 2\sin(x - y)^2 + 2(x - y)^2 \le 5(x - y)^2 \\ &= \alpha(y - x). \end{aligned} \tag{9}$$

Example 10. Let $X = \ell^p = \{x = \{x_n\} \subseteq \mathbb{R} : \|x\| = (\sum_{n\ge1} |x_n|^p)^{1/p} < \infty\}$. For $1 < p < \infty$, ℓ^p is a reflexive Banach space. Let $K = \{x \in \ell^p : \|x\| \le 1\}$, which is nonempty, closed, and convex subset of ℓ^p. Define $f : K \times K \to \mathbb{R}$ by

$$f(x, y) = \|x - y\|^2. \tag{10}$$

Then,

$$f(x, y) + f(y, x) = 2\|x - y\|^2 > 0, \tag{11}$$

for $x \neq y$, that is, f is not monotone mapping.

But, if we choose $\alpha : X \to \mathbb{R}$ by $\alpha(x) = 4\|x\|^2$, then f is a relaxed α-monotone.

3. Mixed Equilibrium Problems with Weakly Relaxed α-Monotone Bi-Function

In this section, we introduce the new class of bi-functions. Using KKM technique, we study and prove the existence of solutions for mixed equilibrium problem with bi-function in such class in Banach spaces.

Definition 11. A bi-function $f : K \times K \to \mathbb{R}$ is said to be a weakly relaxed α-monotone if there exists a function $\alpha : X \to \mathbb{R}$ with

$$\lim_{t\to 0^+} \alpha(tx) = 0, \tag{12}$$

$$\lim_{t\to 0^+} \frac{d}{dt}\alpha(tx) = 0, \tag{13}$$

for all $t > 0$ and $x \in K$ such that

$$f(x, y) + f(y, x) \le \alpha(y - x), \quad \forall x, y \in K. \tag{14}$$

Remark 12. We obtain that the relaxed α-monotonicity implies weakly relaxed α-monotonicity. So the class of relaxed α-monotone bi-functions is a subclass of weakly relaxed α-monotone bi-function class.

Next, we discuss the existence solution of the (MEP) (1), using the concept of the weakly relaxed α-monotonicity.

Theorem 13. *Suppose $f : K \times K \to \mathbb{R}$ is a weakly relaxed α-monotone which is hemicontinuous in the first argument, and convex in the second argument let $\varphi : K \to \mathbb{R}$ be a convex function. Then, the (MEP) and the following problem are equivalent:*

$$\begin{aligned} &\text{find } \tilde{x} \in K \text{ such that } f(y,\tilde{x}) + \varphi(\tilde{x}) - \varphi(y) \\ &\quad \le \alpha(y-\tilde{x}), \quad \forall y \in K. \end{aligned} \tag{15}$$

Proof. Suppose that the (MEP) (1) has a solution. So there exists $\tilde{x} \in K$ such that

$$f(\tilde{x}, y) + \varphi(y) - \varphi(\tilde{x}) \ge 0, \quad \forall y \in K. \tag{16}$$

Since f is, weakly relaxed α-monotone, we have

$$f(\tilde{x}, y) + f(y, \tilde{x}) \le \alpha(y-\tilde{x}), \quad \forall y \in K, \tag{17}$$

and then

$$\begin{aligned} &f(y,\tilde{x}) + \varphi(y) - \varphi(\tilde{x}) \\ &\quad \le \alpha(y-\tilde{x}) - f(\tilde{x}, y) + \varphi(y) - \varphi(\tilde{x}) \\ &\quad = \alpha(y-\tilde{x}) - [f(\tilde{x}, y) - \varphi(y) + \varphi(\tilde{x})] \\ &\quad \le \alpha(y-\tilde{x}), \quad \forall y \in K. \end{aligned} \tag{18}$$

Therefore, $\tilde{x} \in K$ is a solution of problem (15).

Conversely, suppose $\tilde{x} \in K$ is a solution of problem (15) and y is any point in K. For $t \in (0,1]$, we let $x_t := ty + (1-t)\tilde{x}$. Since K is convex, we obtain that $x_t \in K$. From (15), we have

$$f(x_t, \tilde{x}) + \varphi(\tilde{x}) - \varphi(x_t) \le \alpha(x_t - \tilde{x}). \tag{19}$$

By the convexity of f in the second argument, we have

$$0 = f(x_t, x_t) \le t f(x_t, y) + (1-t) f(x_t, \tilde{x}); \tag{20}$$

that is,

$$t[f(x_t, \tilde{x}) - f(x_t, y)] \le f(x_t, \tilde{x}). \tag{21}$$

The convexity of φ implies that

$$0 = \varphi(x_t) - \varphi(x_t) \le t\varphi(y) + (1-t)\varphi(\tilde{x}) - \varphi(x_t), \tag{22}$$

and thus

$$t[\varphi(\tilde{x}) - \varphi(y)] \le \varphi(\tilde{x}) - \varphi(x_t). \tag{23}$$

From (19), (21), and (23), we have

$$t[f(x_t, \tilde{x}) - f(x_t, y) + \varphi(\tilde{x}) - \varphi(y)] \le \alpha(x_t - \tilde{x}), \tag{24}$$

and so

$$f(x_t, \tilde{x}) - f(x_t, y) + \varphi(\tilde{x}) - \varphi(y) \le \frac{\alpha(t(y-\tilde{x}))}{t}. \tag{25}$$

Since f is hemicontinuous in the first argument and taking $t \to 0^+$ we get

$$f(\tilde{x}, \tilde{x}) - f(\tilde{x}, y) + \varphi(\tilde{x}) - \varphi(y) \le \lim_{t \to 0^+} \frac{\alpha(t(y-\tilde{x}))}{t}. \tag{26}$$

From (12), we get $\lim_{t \to 0^+}(\alpha(t(y-\tilde{x})/t))$ is indeterminate form. Using L' Hopital's rule, we obtain that

$$\begin{aligned} &f(\tilde{x}, \tilde{x}) - f(\tilde{x}, y) + \varphi(\tilde{x}) - \varphi(y) \\ &\quad \le \lim_{t \to 0^+} \frac{(d/dt)\,\alpha(t(y-\tilde{x}))}{1}. \end{aligned} \tag{27}$$

By property (13) of weakly relaxed α-monotone f, we have

$$f(\tilde{x}, \tilde{x}) - f(\tilde{x}, y) + \varphi(\tilde{x}) - \varphi(y) \le 0, \tag{28}$$

and then

$$f(\tilde{x}, y) - \varphi(\tilde{x}) + \varphi(y) \ge 0, \quad \forall y \in K. \tag{29}$$

Therefore, $\tilde{x}$ is a solution of (MEP). □

Theorem 14. *Let K be a nonempty bounded closed convex subset of a real reflexive Banach space X. Suppose that $f : K \times K \to \mathbb{R}$ is a weakly relaxed α-monotone and hemicontinuous in the first argument; let $\varphi : K \to \mathbb{R}$ be a convex and lower semicontinuous function. Assume that*

(a) *for fixed $z \in K$, the mapping $x \mapsto f(z,x)$ is convex and lower semicontinuous;*

(b) *$\alpha : X \to \mathbb{R}$ is weakly upper semicontinuous.*

Then the problem (MEP) has a solution.

Proof. Consider the set valued mapping $F : K \to 2^X$ defined by

$$F(y) = \{x \in K : f(x,y) + \varphi(y) - \varphi(x) \ge 0\}, \tag{30}$$

for all $y \in K$.

It is easy to see that $\tilde{x} \in K$ solves the problem (MEP) that is

$$f(x,y) + \varphi(y) - \varphi(x) \ge 0, \quad \forall y \in K, \tag{31}$$

if and only if $\tilde{x} \in \bigcap_{y \in K} F(y)$. Thus it is sufficient to prove that $\bigcap_{y \in K} F(y) \ne \emptyset$.

Next, we show that F is a KKM mapping. Assuming this contrary, then there exists $\{x_1, x_2, \ldots, x_m\} \subset K$ such that $\mathrm{co}\{x_1, x_2, \ldots, x_m\} \not\subseteq \bigcup_{i=1}^m F(x_i)$. This implies that there exists $x_0 \in \mathrm{co}\{x_1, x_2, \ldots, x_m\}$ such that $x_0 = \sum_{i=1}^m t_i x_i$, where $t_i \ge 0$, $i = 1, 2, \ldots, m$, and $\sum_{i=1}^m t_i = 1$, but $x_0 \notin \bigcup_{i=1}^m F(x_i)$.

From the construction of F, we have

$$f(x_0, x_i) + \varphi(x_i) - \varphi(x_0) < 0, \quad \text{for } i = 1, 2, \ldots, m. \tag{32}$$

By the convexity of f in the second variable and the convexity of φ and (32), we obtain that

$$\begin{aligned} 0 &= f(x_0, x_0) \\ &= f\left(x_0, \sum_{i=1}^{m} t_i x_i\right) \\ &\leq \sum_{i=1}^{m} t_i f(x_0, x_i) \\ &< \sum_{i=1}^{m} t_i [\varphi(x_0) - \varphi(x_i)] \\ &= \varphi(x_0) - \sum_{i=1}^{m} t_i \varphi(x_i) \\ &\leq \varphi(x_0) - \varphi(x_0) \\ &= 0, \end{aligned} \tag{33}$$

which is a contradiction. Therefore, F is a KKM mapping.

Now we define another set valued mapping $G : K \to 2^X$ such that

$$G(y) = \{x \in K : f(y, x) + \varphi(x) - \varphi(y) \leq \alpha(y - x)\} \tag{34}$$

for all $y \in K$.

Next, we will prove that $F(y) \subset G(y)$ for all $y \in K$. For each $y \in K$, let $x \in F(y)$; then

$$f(x, y) + \varphi(y) - \varphi(x) \geq 0. \tag{35}$$

From the weakly relaxed α-monotonicity of f, we get

$$\begin{aligned} &f(y, x) + \varphi(x) - \varphi(y) \\ &\quad \leq \alpha(y - x) - [f(x, y) + \varphi(y) - \varphi(x)] \\ &\quad \leq \alpha(y - x). \end{aligned} \tag{36}$$

This implies that $x \in G(y)$ and hence $F(y) \subset G(y)$ for all $y \in K$. So G is also a KKM mapping.

By assumption, φ and $x \mapsto f(z, x)$ are convex lower semicontinuous functions, where fixed $z \in K$. Then it is easy to see that they are both weakly lower semicontinuous. From the definition of G and the weakly upper semicontinuity of α, we get $G(y)$ is weakly closed for all $y \in K$.

Since K is closed bounded and convex, it is also weakly compact, and then $G(y)$ is weakly compact in K for each $y \in K$. From Lemma 4 and Theorem 13, we obtain that

$$\bigcap_{y \in K} F(y) = \bigcap_{y \in K} G(y) \neq \emptyset. \tag{37}$$

So there exists $\tilde{x} \in K$, such that

$$f(x, y) + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in K, \tag{38}$$

then the problem (MEP) has a solution. This completes the proof. □

It easy to see that the relaxed α-monotonicity implies the weakly relaxed α-monotonicity. So Theorem 14 can be deduced to the following corollary.

Corollary 15. *Let K be a nonempty bounded closed convex subset of a real reflexive Banach space X. Suppose that $f : K \times K \to \mathbb{R}$ is relaxed α-monotone and hemicontinuous in the first argument; let $\varphi : K \to \mathbb{R}$ be a convex and lower semicontinuous function. Assume that*

(a) *for fixed $z \in K$, the mapping $x \mapsto f(z, x)$ is convex and lower semicontinuous;*

(b) *$\alpha : X \to \mathbb{R}$ is weakly upper semicontinuous.*

Then the problem (MEP) has a solution.

Next, we study and prove result for the case of K is unbounded set.

Theorem 16. *Let K be a nonempty unbounded closed convex subset of a real reflexive Banach space X. Suppose that $f : K \times K \to \mathbb{R}$ is a weakly relaxed α-monotone and hemicontinuous in the first argument; let $\varphi : K \to \mathbb{R}$ be a convex and lower semicontinuous function. Assume that*

(a) *for fixed $z \in K$, the mapping $x \mapsto f(z, x)$ is convex and lower semicontinuous;*

(b) *$\alpha : X \to \mathbb{R}$ is weakly upper semicontinuous.*

(c) *f satisfied the weakly coercivity condition, that is, there exists $x_0 \in K$ such that*

$$f(x, x_0) + \varphi(x_0) - \varphi(x) < 0, \tag{39}$$

whenever $x \in K$ and $\|x\|$ is large enough.

Then the problem (MEP) has a solution.

Proof. For $\epsilon > 0$, define $K_\epsilon := \{y \in K : \|y\| \leq \epsilon\}$.

Consider the problem: find $x_\epsilon \in K_\epsilon$ such that

$$f(x_\epsilon, y) + \varphi(y) - \varphi(x_\epsilon) \geq 0, \quad \forall y \in K_\epsilon. \tag{40}$$

Since K_ϵ is bounded, by Theorem 14, we get that the problem (40) has at least one solution $x_\epsilon \in K_\epsilon$.

For x_0 in the weakly coercivity condition (c), we choose $\epsilon' > \|x_0\|$. From (40), we have

$$f(x_{\epsilon'}, x_0) + \varphi(x_0) - \varphi(x_{\epsilon'}) \geq 0. \tag{41}$$

Since $x_{\epsilon'} \in K_{\epsilon'}$, we have $\|x_{\epsilon'}\| \leq \epsilon'$. If $\|x_{\epsilon'}\| = \epsilon'$, we may choose ϵ' large enough so that by the weakly coercivity condition (c), we get

$$f(x_{\epsilon'}, x_0) + \varphi(x_0) - \varphi(x_{\epsilon'}) < 0, \tag{42}$$

which contradicts (41). Therefore, we must have ϵ' such that $\|x_{\epsilon'}\| < \epsilon'$.

For each $y \in K$, we can choose $0 < t < 1$ small enough such that $ty + (1-t)x_{\epsilon'} \in K_{\epsilon'}$. From (40), for each $y \in K$, we have

$$\begin{aligned} 0 &\le f\left(x_{\epsilon'}, ty + (1-t)x_{\epsilon'}\right) \\ &\quad + \varphi\left(ty + (1-t)x_{\epsilon'}\right) - \varphi\left(x_{\epsilon'}\right) \\ &\le tf\left(x_{\epsilon'}, y\right) + (1-t)f\left(x_{\epsilon'}, x_{\epsilon'}\right) \\ &\quad + t\varphi(y) + (1-t)\varphi\left(x_{\epsilon'}\right) - \varphi\left(x_{\epsilon'}\right) \\ &= t\left[f\left(x_{\epsilon'}, y\right) + \varphi(y) - \varphi\left(x_{\epsilon'}\right)\right]. \end{aligned} \tag{43}$$

This implies that

$$f\left(x_{\epsilon'}, y\right) + \varphi(y) - \varphi\left(x_{\epsilon'}\right) \ge 0 \quad \forall y \in K. \tag{44}$$

Therefore, the problem (MEP) has a solution. This completes the proof. □

Corollary 17. *Let K be a nonempty unbounded closed convex subset of a real reflexive Banach space X. Suppose that $f : K \times K \to \mathbb{R}$ is relaxed α-monotone and hemicontinuous in the first argument; let $\varphi : K \to \mathbb{R}$ be a convex and lower semicontinuous function. Assume that*

(a) *for fixed $z \in K$, the mapping $x \mapsto f(z, x)$ is convex and lower semicontinuous;*

(b) *$\alpha : X \to \mathbb{R}$ is weakly upper semicontinuous;*

(c) *f satisfied the weakly coercivity condition; that is, there exists $x_0 \in K$ such that*

$$f\left(x, x_0\right) + \varphi\left(x_0\right) - \varphi(x) < 0, \tag{45}$$

whenever $x \in K$ and $\|x\|$ large enough.

Then the problem (MEP) has a solution.

Remark 18. Theorems 13, 14, and 16 are improving the results of Fang [10] from the corresponding results of variational-like inequality problems to equilibrium problems. Also, these results are extensions of the main results of Mahato and Nahak [19].

References

[1] M. R. Bai, S. Z. Zhou, and G. Y. Ni, "Variational-like inequalities with relaxed η-α pseudomonotone mappings in Banach spaces," *Applied Mathematics Letters*, vol. 19, no. 6, pp. 547–554, 2006.

[2] M. Bianchi and S. Schaible, "Generalized monotone bifunctions and equilibrium problems," *Journal of Optimization Theory and Applications*, vol. 90, no. 1, pp. 31–43, 1996.

[3] M. Bianchi and S. Schaible, "Equilibrium problems under generalized convexity and generalized monotonicity," *Journal of Global Optimization*, vol. 30, no. 2-3, pp. 121–134, 2004.

[4] M. Bianchi and R. Pini, "A note on equilibrium problems with properly quasimonotone bifunctions," *Journal of Global Optimization*, vol. 20, no. 1, pp. 67–76, 2001.

[5] L. C. Ceng and J. C. Yao, "On generalized variational-like inequalities with generalized monotone multivalued mappings," *Applied Mathematics Letters*, vol. 22, no. 3, pp. 428–434, 2009.

[6] Y. Q. Chen, "On the semi-monotone operator theory and applications," *Journal of Mathematical Analysis and Applications*, vol. 231, no. 1, pp. 177–192, 1999.

[7] R. W. Cottle and J. C. Yao, "Pseudomonotone complementarity problems in Hilbert spaces," *Journal of Optimization Theory and Applications*, vol. 75, no. 2, pp. 281–295, 1992.

[8] X. P. Ding, "Existence and algorithm of solutions for generalized mixed implicit quasi-variational inequalities," *Applied Mathematics and Computation*, vol. 113, no. 1, pp. 67–80, 2000.

[9] K. Fan, "Some properties of convex sets related to fixed point theorems," *Mathematische Annalen*, vol. 266, no. 4, pp. 519–537, 1984.

[10] Y. P. Fang and N. J. Huang, "Variational-like inequalities with generalized monotone mappings in Banach spaces," *Journal of Optimization Theory and Applications*, vol. 118, no. 2, pp. 327–338, 2003.

[11] F. Giannessi, *Vector Variational Inequalities and Vector Equilibria*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 2000.

[12] R. Glowinski, J. L. Lions, and R. Trémolières, *Numerical Analysis of Variational Inequalities*, North-Holland, Amsterdam, The Netherlands, 1981.

[13] D. Goeleven and D. Motreanu, "Eigenvalue and dynamic problems for variational and hemivariational inequalities," *Communications on Applied Nonlinear Analysis*, vol. 3, no. 4, pp. 1–21, 1996.

[14] S. Karamardian and S. Schaible, "Seven kinds of monotone maps," *Journal of Optimization Theory and Applications*, vol. 66, no. 1, pp. 37–46, 1990.

[15] R. U. Verma, "On generalized variational inequalities involving relaxed Lipschitz and relaxed monotone operators," *Journal of Mathematical Analysis and Applications*, vol. 213, no. 1, pp. 387–392, 1997.

[16] R. U. Verma, "On monotone nonlinear variational inequality problems," *Commentationes Mathematicae Universitatis Carolinae*, vol. 39, no. 1, pp. 91–98, 1998.

[17] X. Q. Yang and G. Y. Chen, "A class of nonconvex functions and pre-variational inequalities," *Journal of Mathematical Analysis and Applications*, vol. 169, no. 2, pp. 359–373, 1992.

[18] N. K. Mahato and C. Nahak, "Weakly relaxed α-pseudomonotonicity and equilibrium problem in Banach spaces," *Journal of Applied Mathematics and Computing*, vol. 40, no. 1-2, pp. 499–509, 2012.

[19] N. K. Mahato and C. Nahak, "Mixed equilibrium problems with relaxed α-monotone mapping in Banach spaces," *Rendiconti del Circolo Matematico di Palermo*, 2013.

[20] K. Fan, "A generalization of Tychonoff's fixed point theorem," *Mathematische Annalen*, vol. 142, pp. 305–310, 1961.