Research Article

Endpoint Estimates for Generalized Commutators of Hardy Operators on $H^1$ Space

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We study the $H^1$-boundedness of the generalized commutators of Hardy operator with a homogeneous kernel as follows:

$$H^m_{\Omega, A, \beta}f(x) = \frac{1}{|x|^{n-\beta}} \int_{|y|<|x|} (\Omega(x-y)/|x-y|^{m-1}) R_m(A; x, y) f(y) dy,$$

where $m \in \mathbb{Z}^+$, $0 \leq \beta < n$ and $\Omega \in \text{Lip}_1(S^{n-1})$. We prove that, when $m \geq 1$, $H^m_{\Omega, A, \beta}$ is not bounded from $H^1$ to $L^{n/(n-\beta)}$ unless $H^m_{\Omega, A, \beta} \equiv 0$. Finally, we prove that $H^m_{\Omega, A, \beta}$ is bounded from $H^1$ to $L^{n/(n-\beta), \infty}$ with $m \geq 1$.

1. Introduction

Let $f \in L^p(\mathbb{R}^*)$ with $1 < p < \infty$; the classical Hardy operator is defined by

$$Hf(x) = \frac{1}{x} \int_0^x f(t) \, dt, \quad x \neq 0.$$  \hfill (1)

A famous result proved by Hardy [1] can be stated as follows:

$$\|Hf\|_{L^p(\mathbb{R})} \leq \frac{p}{p-1} \|f\|_{L^p(\mathbb{R})}.$$  \hfill (2)

Hardy [1] also pointed out the fact that the constant $p/(p-1)$ in (2) is the best possible. Later, Hardy operator was studied by many mathematicians; please see [2, 3] for more details.

In 1995, Christ and Grafakos [4] studied the following $n$-dimensional Hardy operator:

$$\mathcal{H}_f(x) = \frac{1}{\nu_n |x|^n} \int_{|y|<|x|} f(y) \, dy, \quad x \in \mathbb{R}^n \setminus \{0\},$$  \hfill (3)

where $\nu_n$ is the volume of the unit ball in $\mathbb{R}^n$, and they proved the following inequality:

$$\|\mathcal{H}f\|_{L^p} \leq \frac{p}{p-1} \|f\|_{L^p}, \quad 1 < p < \infty.$$  \hfill (4)

Furthermore, Christ and Grafakos [4] also showed that the constant $p/(p-1)$ in (3) is the best possible.

In 2007, Fu et al. [5] considered the following commutator of fractional Hardy operator:

$$\mathcal{H}_\beta f(x) = b(x) \mathcal{H}_\beta f(x) - \mathcal{H}_\beta (f b)(x),$$  \hfill (5)

where $\mathcal{H}_\beta f(x)$ is defined by

$$\mathcal{H}_\beta f(x) = \frac{1}{|x|^{n-\beta}} \int_{|y|<|x|} f(y) \, dy$$  \hfill (6)

with $-n < \beta < n$. When $\beta = 0$, we simply denote $\mathcal{H}_0$ by $\mathcal{H}_b$ and $\mathcal{H}_0$ is just the $n$-dimensional Hardy operator proposed by Christ and Grafakos in [4] (without considering the constant $\nu_n$).

In 2011, Fu et al. [6] studied the following $n$-dimensional fractional Hardy operator with a homogeneous kernel:

$$\mathcal{H}_{\Omega, \beta} f(x) = \frac{1}{|x|^{n-\beta}} \int_{|y|<|x|} \Omega(x-y) f(y) \, dy,$$  \hfill (7)

where $\Omega \in L^r(S^{n-1})$. Fu et al. [6] proved that $\mathcal{H}_{\Omega, \beta}$ is bounded on Herz type space and $\lambda$-central Morrey space. Here $\mathcal{H}_{\Omega, \beta}$

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is just the commutator of fractional Hardy operator with a homogeneous kernel.

Recently, Zhao et al. [7] gave a counterexample to show that \( \mathcal{H}_b \) is not bounded from \( H^1 \) to \( L^1 \), and they proved that \( \mathcal{H}_b \) is bounded from \( H^1 \) to weak \( L^1 \) space where \( H^1 \) denotes the Hardy space.

On the other hand, in 1982, Cohen [8] studied the following generalized commutator:

\[
T^2_{\alpha} f(x) = \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n+1}} \times (A(x) - A(y) - \nabla A(y) (x-y)) f(y) dy,
\]

(8)

where \( \Omega \in L^1(S^{n-1}) \) is homogeneous of degree zero and satisfies the moment condition

\[
\int_{S^{n-1}} \Omega(x) x^\alpha d\sigma(x) = 0
\]

(9)

for \( |\alpha| = 1 \). Cohen [8] proved that, if \( \Omega \in \text{Lip}_1(S^{n-1}) \) and \( \nabla A \in \text{BMO} \), then \( T^2_{\alpha} \) is bounded on \( L^1(\mathbb{R}^n) \) with \( 1 < p < \infty \).

Later, Cohen and Gosselin [9] considered another type of generalized commutator \( T^m_{\alpha} (f) \) as follows:

\[
T^m_{\alpha} f(x) = \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n+1}} R_m(A; x, y) f(y) dy,
\]

(10)

where \( R_m(A; x, y) \) is defined by

\[
R_m(A; x, y) = A(x) - \sum_{|\alpha|=m-1} \frac{1}{\alpha!} D^\alpha A(y)(x-y)^\alpha,
\]

the remainder of Taylor series of the function \( A \) at \( x \) about \( y \), and \( \Omega \) satisfies the following moment conditions:

\[
\int_{S^{n-1}} \Omega(x) x^\alpha d\sigma(x) = 0
\]

(11)

for \( |\alpha| = m - 1 \). Obviously, if we choose \( m = 1 \), \( T^m_{\alpha} \) becomes \([A,T]\), the commutator of \( T \) generalized by \( A \) and \( T \).

Cohen and Gosselin [9] proved that \( T^m_{\alpha} \) is bounded on \( L^p(\mathbb{R}^n) \) for \( 1 < p < \infty \) if \( \Omega \in \text{Lip}_1(S^{n-1}) \) and the function \( A \) has derivatives of order \( m - 1 \) in \( \text{BMO}(\mathbb{R}^n) \). Later, \( T^m_{\alpha} \) was studied by many mathematicians; for example, see [10, 11] or [12] for more details. Particularly in [11], Lu and Wu studied the endpoint estimates of \( T^m_{\alpha} \) on \( H^1 \) space.

It should be pointed out that the generalized commutators of some operators play an important role in the study of partial differential equation. Recently, by using the \( W^{1,p} \) estimate for the elliptic equation of divergence form with partially BMO coefficients and the \( L^p \) boundedness of the Cohen-Gosselin type generalized commutators proved by Yan in [12], Wang and Zhang [13] gave a new proof of Wu’s theorem in [14]. Here we would like to point out that the method proved in [13] is much simpler than that in [14].

In this paper, we will consider the following generalized commutator of fractional Hardy operator with a homogeneous kernel:

\[
\mathcal{H}^m_{\alpha,\beta} f(x) = \frac{1}{|x|^{n-\beta}} \int_{|y|<|x|} \frac{\Omega(x-y)}{|x-y|^{n+1}} f(y) R_m(A; x, y) dy,
\]

(12)

where \( R_m(A; x, y) = A(x) - \sum_{|\alpha|=m} \frac{1}{\alpha!} D^\alpha A(y)(x-y)^\alpha \), \( 0 \leq \beta < n \), and \( \Omega \in \text{Lip}_1(S^{n-1}) \).

As the Hardy operator is controlled by the Hardy-Littlewood maximal function, we have

\[
\mathcal{H}^m_{\alpha,\beta} f(x) \leq CM_{\alpha,\beta,\Omega} f(x),
\]

(13)

where \( M_{\alpha,\beta,B MO} f(x) \) is defined by

\[
M_{\alpha,\beta,B MO} f(x) = \sup_{r>0} \frac{1}{r^{n-\beta}} \int_{|y|<r} \frac{|\Omega(x-y)|}{|x-y|^{n+1}} |R_m(A; x, y) f(y)| dy.
\]

(14)

By a simple computation or from [15, pp. 221-222], we have

\[
M_{\alpha,\beta,f} f(x) = \sup_{r>0} \frac{1}{r^{n-\beta}} \int_{|y|<r} \frac{|\Omega(x-y)|}{|x-y|^{n+1}} |R_m(A; x, y) f(y)| dy,
\]

(15)

and thus

\[
\mathcal{H}^m_{\alpha,\beta} f(x) \leq CM_{\alpha,\beta} f(x),
\]

(16)

where

\[
M_{\alpha,\beta} f(x) = \sup_{r>0} r^{-n-\beta+m-1} \int_{|y|<r} |\Omega(x-y)| \times |R_m(A; x, y) f(y)| dy.
\]

(17)

So we have

\[
\mathcal{H}^m_{\alpha,\beta} f(x) \leq CM_{\alpha,\beta} f(x),
\]

(18)

From [15, p. 222], we have the following lemma.

**Lemma 1** (see [15]). Let \( 0 < \beta < n \) and \( \Omega \in \text{Lip}_1(S^{n-1}) \). If \( 1 < p \), \( q < \infty \) with \( 1/p - 1/q = \beta/n \), and \( A \) has derivatives of order \( m - 1 \) in \( \text{BMO}(\mathbb{R}^n) \), then

\[
\left\| \mathcal{H}^m_{\alpha,\beta} f \right\|_{L^q} \leq C \sum_{|\alpha|=m-1} \|D^\alpha A\|_{\text{BMO}} \left\| f \right\|_{L^p},
\]

(19)

where the constant \( C \) is independent of \( f \) and \( A \).
By checking [15, p. 222] carefully, we deduce that (19) still holds if we take $\beta = 0$. So we have the following proposition.

**Proposition 2.** Let $0 \leq \beta < n$, $1/p - 1/\alpha = \beta/n$ with $1 < p < \infty$, and $\Omega \in \text{Lip}_1(S^{n-1})$ and $A$ has derivatives of order $m - 1$ in $\text{BMO}(\mathbb{R}^n)$, then

$$
\left\| \mathcal{H}^{m}_{\Omega, A, \beta} f \right\|_{L^p} \leq C \sum_{|\alpha|=m-1} \left\| D^\alpha A \right\|_{\text{BMO}} \left\| f \right\|_{L^\infty},
$$

(20)

where the constant $C$ is independent of $f$ and $A$.

**Definition 3.** (see [16]). One says a function $a(x)$ is an $H^1$ atom if $a$ satisfies the following conditions:

(i) $\text{supp} (a) \subset B(x_0, r)$,

(ii) $\|a\|_{L^\infty} \leq |B(x_0, r)|^{-1}$,

(iii) $\int a(x) \, dx = 0$.

It is well known that, if a function $f$ belongs to $H^1$, then it can be written as $f = \sum_{i=1}^{\infty} \lambda_i a_i$ where each $a_i$ is an $H^1$ atom. Moreover, one has

$$
\left\| f \right\|_{H^1} \sim \inf \left\{ \sum_{i=-\infty}^{\infty} |\lambda_i| \right\},
$$

(22)

where the infimum is taken over all decompositions of $f$.

**Definition 4.** (see [17]). A function $f$ is said to belong to $\text{BMO}(\mathbb{R}^n)$ if the following sharp maximal function is bounded:

$$
\left\{ f \right\}_{\text{BMO}} = \sup_{B} \frac{1}{|B|} \int_{B} \left| f(y) - f_B \right| \, dy < \infty,
$$

(23)

where the supreme is taken over all balls $B \subset \mathbb{R}^n$ and $f_B = (1/|B|) \int_{B} f(x) \, dx$ and $\|f\|_{\text{BMO}} = \|f\|_{L^{\infty}}$.

**Proposition 5.** (see [17]). Let $1 < p < \infty$ and $f \in \text{BMO}(\mathbb{R}^n)$; then one has

(a) $\|f\|_{\text{BMO}} \sim \sup_{B} \left( (1/|B|) \int_{B} |f(x) - f_B|^p \, dx \right)^{1/p}$;

(b) $\|f\|_{\text{BMO}} \sim \sup_{B} \inf_{a \in \mathbb{R}} \left( (1/|B|) \int_{B} |f(x) - a|^p \, dx \right)^{1/p}$.

Obviously, when $m = 1$, $\mathcal{H}^{m}_{\Omega, A, \beta}$ can be written as

$$
\mathcal{H}^{1}_{\Omega, A, \beta} f(x) = A(x) \mathcal{H}^1_{\Omega, \beta} f(x) - \mathcal{H}^1_{\Omega, \beta} (A f)(x),
$$

just the commutator of fractional Hardy operator with a homogeneous kernel.

For the case $m = 2$, $\beta = 0$, and $\Omega \equiv 1$, Lu and Zhao [18] proved that $\mathcal{H}^{2}_{\Omega, A, 0}$ is bounded on Herz type spaces and Morrey-Herz type spaces.

In this paper, we would like to show that $\mathcal{H}^{m}_{\Omega, A, \beta}$ is not bounded from $H^1$ to $L^{n/(n-\beta)}$ for all $m \in \mathbb{Z}^+$. Furthermore, we will prove that $\mathcal{H}^{m}_{\Omega, A, \beta}$ is bounded from $H^1$ to $L^{n/(n-\beta), \infty}$, where $L^{n/(n-\beta), \infty}$ denotes the weak $L^{n/(n-\beta)}$ space. Some ideas of this paper come from Zhao et al. [7].

In this chapter, we would like to show that, if $A \in \text{BMO}(\mathbb{R}^n)$, $\mathcal{H}^{m}_{\Omega, A, \beta} (0 \leq \beta < n)$ is not bounded from $H^1$ to $L^{n/(n-\beta)}$.

To show this, let $A(x) = \chi_{(\infty, \infty)}(x) \in \text{BMO}$, $\Omega \equiv 1$, and $f_0(x) = \chi_{(0, 0)}(x) - \chi_{(-\infty, 0)}(x)$; then for $x > 8$ and $n = 1$, we have

$$
\left\| \mathcal{H}^{1}_{\Omega, A, \beta} f_0(x) \right\| = \left| \frac{1}{x^{1-\beta}} \int_{0}^{4} (1 - 0) \times 1 \, dy \right| = \frac{4}{x^{1-\beta}},
$$

(24)

and then

$$
\int_{\mathbb{R}^n} \left| \mathcal{H}^{1}_{\Omega, A, \beta} f_0(x) \right|^{1/(1-\beta)} \, dx \geq \int_{8}^{\infty} 4^{1/(1-\beta)} \, dx = \infty,
$$

(25)

which indicates that $\mathcal{H}^{m}_{\Omega, A, \beta}$ is not bounded from $H^1$ to $L^{n/(n-\beta)}$.

**2. Endpoint Estimates for $\mathcal{H}^{m}_{\Omega, A, \beta}$ from $H^1$ to $L^{n/(n-\beta)}$.**

In Section 1, we know that, when $m = 1$, $\mathcal{H}^{1}_{\Omega, A, \beta}$ is not bounded from $H^1$ to $L^{n/(n-\beta)}$. In this section, we will prove that, when $m \geq 2$, $\mathcal{H}^{m}_{\Omega, A, \beta}$ is also not bounded from $H^1$ to $L^{n/(n-\beta)}$ unless $\mathcal{H}^{m}_{\Omega, A, \beta} \equiv 0$. We have the following conclusions.

**Theorem 6.** Let $m \geq 2$, $0 \leq \beta < n$, and $\Omega \in \text{Lip}_1(S^{n-1})$. Assume that $A$ has derivatives of order $m - 1$ in $\text{BMO}(\mathbb{R}^n)$; then the following two statements are equivalent;

(i) $\mathcal{H}^{m}_{\Omega, A, \beta}$ maps $H^1(\mathbb{R}^n)$ continuously into $L^{n/(n-\beta)}$;

(ii) for any $H^1$ atom supported on certain ball $B$ and $u \in 3B \setminus 2B$, there is

$$
\int_{(4B)^c} \left[ \sum_{|\alpha|=m-1} \frac{1}{\alpha!} K_\alpha(x, u) \frac{1}{|x|^{\beta}} \int_{B} D^\alpha A(y) a(y) \, dy \right] \, dx \leq C,
$$

(26)

where $K_\alpha(x, u) = \Omega(x-u)(x-u)^\alpha / |x-u|^{m-1}$ with $|\alpha| = m-1$.

In order to prove Theorem 6, we need the following lemma.

**Lemma 7.** (see [9]). Let $b$ be a function on $\mathbb{R}^n$ with mth order derivatives in $L^q_{\text{loc}}(\mathbb{R}^n)$ for some $q > n$; then

$$
|\mathcal{R}_m(b; x, y)| \leq C_{m, q} |x - y|^m
$$

$$
\times \sum_{|\alpha|=m} \left( \frac{1}{Q(x, y)} \int_{Q(x, y)} |D^\alpha b(z)|^q \, dz \right)^{1/q},
$$

(27)

where $Q(x, y)$ is the cube centered at $x$ and having diameter $5\sqrt{n}|x - y|$. 

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Proof of Theorem 6. Suppose that \( a(x) \) is an \( H^1 \) atom supported on \( B(x_0, r) \) and satisfies (21). Now we take \( \tilde{a}(x) = a(x + x_0) \); then \( \tilde{a} \) is also an \( H^1 \) atom and satisfies

\[
\begin{align*}
(i') & \quad \text{supp} (\tilde{a}) \subset B(0, r), \\
(ii') & \quad \|\tilde{a}\|_{L^\infty} \leq |B(0, r)|^{-1}, \\
(iii') & \quad \int_{\mathbb{R}^n} \tilde{a}(x) \, dx = 0.
\end{align*}
\]

(28)

Thus by the main results in [19] and the atomic decomposition of the space \( H^1(\mathbb{R}^n) \), it suffices to show that, for any \( H^1 \) atom \( \tilde{a} \), we have \( \|\mathcal{H}^m_{\Omega, \Delta, \beta} \tilde{a}(x)\|_{L^{(n-\beta)/p}} \leq C \).

Let \( B = B(0, r) \) and \( \tilde{A}(x) = A(x) - \sum_{|\alpha| = m-1} \frac{1}{\alpha!} m_\beta (D^\alpha A) x^\alpha \); then \( R_m (A; x, y) = R_m (\tilde{A}; x, y) \). For each \( H^1 \) atom \( \tilde{a} \), we split each \( \mathcal{H}^m_{\Omega, \Delta, \beta} \tilde{a}(x) \) as

\[
\mathcal{H}^m_{\Omega, \Delta, \beta} \tilde{a}(x) = \chi_{4B} (x) \mathcal{H}^m_{\Omega, \Delta, \beta} \tilde{a}(x) + \chi_{(4B)^c} (x) \mathcal{H}^m_{\Omega, \Delta, \beta} \tilde{a}(x) \quad \text{(29)}
\]

\[
:= \mu_1 (x) + \mu_2 (x).
\]

For \( \mu_1 (x) \), taking \( n/(n-\beta) < q < \infty \) and \( p \) so that \( 1/p - 1/q = \beta/n \), it follows from Proposition 2 that

\[
\mu_1 (x) \leq C \sum_{|\alpha| = m-1} \|D^\alpha A\|_{\text{BMO}} \frac{1}{|x|^n} \|\tilde{a}\|_{L^\infty} \quad \text{(30)}
\]

\[
\leq C \sum_{|\alpha| = m-1} \|D^\alpha A\|_{\text{BMO}} |x|^{-n\beta/p}.
\]

For \( \mu_2 (x) \), as \( x \in (4B)^c \) and \( |y| < |x| \), we can deduce \( \{y : |y| < |x|\} \cap \{y : y \in B(0, r)\} = \{y : y \in B(0, r)\} \); thus we have

\[
\mu_2 (x) = \chi_{(4B)^c} (x) \frac{1}{|x|^\beta} \times \int_{B(0, r)} \frac{\Omega (x-y)}{|x-y|^{m-1}} R_m (\tilde{A}; x, y) \tilde{a}(y) \, dy.
\]

(31)

Next we denote \( B(0, r) = B \) and \( B(0, kr) = kB \); then by the vanishing condition of \( \tilde{a} \), we decompose \( \mu_2 \) as follows:

\[
\mu_2 (x) = \chi_{(4B)^c} (x) \frac{1}{|x|^\beta} \times \int_{B} \left( \frac{\Omega (x-y)}{|x-y|^{m-1}} R_m (\tilde{A}; x, y) - \frac{\Omega (x-u)}{|x-u|^{m-1}} R_m (\tilde{A}; x, u) \right) \tilde{a}(y) \, dy.
\]

For the term I, by Lemma 7, we have

\[
I \leq C \frac{|\Omega (x-y)|}{|x-y|^{m}} |y-u| R_m (\tilde{A}; x, y)
\]

\[
\leq C |y-u| |x-y|^{m-1} k \sum_{|\alpha| = m-1} R_m (\tilde{A}; x, u) \quad \text{(35)}
\]

\[
\leq C k^{2}\sum_{|\alpha| = m-1} R_m (\tilde{A}; x, u) \quad \text{(36)}
\]

For the term II, by the similar estimates of I, we obtain

\[
II \leq C k^{2}\sum_{|\alpha| = m-1} R_m (\tilde{A}; x, u) \quad \text{(37)}
\]

For the term III, by the following formula (see [20]):

\[
R_m (A; x, y) = R_m (A; x, z) + \sum_{|\alpha| = m} \frac{1}{\alpha!} R_m (D^\alpha A; z, y) (x-z)^\alpha,
\]

where \( u \in 3B \setminus 2B. \)

For \( \mu_{23} (x, u) \), as \( u \in 3B \setminus 2B, y \in B, \) and \( x \in (4B)^c \), we have

\[
|y-u| \sim |x-u| \sim |x|.
\]

Thus we have

\[
\left| \Omega (x-y) |x-y|^{m-1} R_m (\tilde{A}; x, y) - \Omega (x-u) |x-u|^{m-1} R_m (\tilde{A}; x, u) \right| \leq \Omega (x-y) |x-y|^{m-1} R_m (\tilde{A}; x, y) - \Omega (x-u) |x-u|^{m-1} R_m (\tilde{A}; x, u)
\]

\[
\leq |x-y|^{m-1} R_m (\tilde{A}; x, u) \quad \text{(33)}
\]

\[
\left| \Omega (x-y) |x-y|^{m-1} R_m (\tilde{A}; x, y) - \Omega (x-u) |x-u|^{m-1} R_m (\tilde{A}; x, u) \right| \leq \Omega (x-y) |x-y|^{m-1} R_m (\tilde{A}; x, u)
\]

\[
\leq |x-u|^{m-1} R_m (\tilde{A}; x, u) \quad \text{(34)}
\]

\[
\left| \Omega (x-y) |x-y|^{m-1} R_m (\tilde{A}; x, y) - \Omega (x-u) |x-u|^{m-1} R_m (\tilde{A}; x, u) \right| \leq \Omega (x-y) |x-y|^{m-1} R_m (\tilde{A}; x, u)
\]

\[
\leq |x-u|^{m-1} R_m (\tilde{A}; x, u) \quad \text{(35)}
\]

\[
\left| \Omega (x-y) |x-y|^{m-1} R_m (\tilde{A}; x, y) - \Omega (x-u) |x-u|^{m-1} R_m (\tilde{A}; x, u) \right| \leq \Omega (x-y) |x-y|^{m-1} R_m (\tilde{A}; x, u)
\]

\[
\leq |x-u|^{m-1} R_m (\tilde{A}; x, u) \quad \text{(36)}
\]

\[
\left| \Omega (x-y) |x-y|^{m-1} R_m (\tilde{A}; x, y) - \Omega (x-u) |x-u|^{m-1} R_m (\tilde{A}; x, u) \right| \leq \Omega (x-y) |x-y|^{m-1} R_m (\tilde{A}; x, u)
\]

\[
\leq |x-u|^{m-1} R_m (\tilde{A}; x, u) \quad \text{(37)}
\]
and then together with Lemma 7, we have

\[
III \leq C \sum_{|\alpha|=m-1} \left\| D^\alpha A \right\|_{BMO} \sum_{i=0}^{m-2} |x-y|^{-(m-1)+i} |y-u|^{m-1-i} \\
\leq C \sum_{|\alpha|=m-1} \left\| D^\alpha A \right\|_{BMO} k^2. 
\]

Thus for \( \mu_2(x, u) \), by the size condition of \( \tilde{a} \), we get the following estimates:

\[
\left\| \mu_2 (\cdot, u) \right\|_{L^{n/(n-\beta)}} \leq C \sum_{|\alpha|=m-1} \left\| D^\alpha A \right\|_{BMO} \sum_{k=2}^{\infty} k^2 \left( \int_{2^{k+1}B} \Omega(\tilde{a}) \right)^{n/(n-\beta)} \\
\leq C \sum_{|\alpha|=m-1} \left\| D^\alpha A \right\|_{BMO} \sum_{k=2}^{\infty} k^2 \leq C \sum_{|\alpha|=m-1} \left\| D^\alpha A \right\|_{BMO}.
\]

(38)

For \( \mu_2(x, u) \), since \( \Omega \in \text{Lip}_1(S^{n-1}) \), we have the following estimates of \( |K_\alpha(x-y) - K_\alpha(x-u)| \):

\[
|K_\alpha(x-y) - K_\alpha(x-u)| \\
\leq \frac{\Omega(x-y)(x-y)^\alpha}{|x-y|^{m-1}} - \frac{\Omega(x-y)(x-u)^\alpha}{|x-y|^{m-1}} \\
\leq \frac{\Omega(x-y)(x-u)}{|x-y|^{m-1}} - \frac{\Omega(x-u)(x-u)^\alpha}{|x-u|^{m-1}} \\
\leq C \frac{|x-u|}{|x-y|}. 
\]

Thus by the size condition of \( \tilde{a} \) and Lemma 7, we have

\[
\left\| \mu_{22} (\cdot, u) \right\|_{L^{n/(n-\beta)}} \leq C \sum_{|\alpha|=m-1} \left\| D^\alpha A \right\|_{BMO} \sum_{k=2}^{\infty} k^2 \left( \int_{2^{k+1}B} \Omega(\tilde{a}) \right)^{n/(n-\beta)} \\
\times \left\{ \int_{B} \frac{|y-u|}{|x-y| |x|^{n-\beta}} \left| \tilde{a}(y) \right| \left| D^\alpha \left( \tilde{a}(y) \right) \right| dx \right\}^{n/(n-\beta)/n} \\
\leq C \sum_{|\alpha|=m-1} \left\| D^\alpha A \right\|_{BMO} \sum_{k=2}^{\infty} k^2 \left( \int_{B} \Omega(\tilde{a}) \right)^{n/(n-\beta)} \\
\times \sum_{k=2}^{\infty} 2^k B^{(\beta-\beta)/n} |B| 2^k B^{(\beta-\beta)/2} \\
\leq C \sum_{|\alpha|=m-1} \left\| D^\alpha A \right\|_{BMO}.
\]

(41)

Now we can deduce that \( \left\| \mathcal{H}_{\Omega, A, \beta}^{m} \tilde{a} \right\|_{L^{n/(n-\beta)}} \leq C \) is equivalent to \( \left\| \mu_2 (\cdot, u) \right\|_{L^{n/(n-\beta)}} \leq C \). By the vanishing condition of \( \tilde{a} \), we can easily get

\[
\left( \int_{B} \left| \Omega(\tilde{a}) \right| \right)^{n/(n-\beta)} \left( \int_{B} \left| \tilde{a}(y) \right| \left| D^\alpha \left( \tilde{a}(y) \right) \right| dx \right)^{n/(n-\beta)/n} \leq C. 
\]

(42)

Consequently, we have finished the proof of Theorem 6.
\( \square \)

Next we would like to show that \( \mathcal{H}_{\Omega, A, \beta}^{m} \) is not bounded from \( H^1 \) to \( L^{n/(n-\beta)} \) unless \( \mathcal{H}_{\Omega, A, \beta}^{m} \equiv 0 \). We have the following theorem.

**Theorem 8.** Let \( m \geq 2 \), \( 0 \leq \beta < n \), and \( \Omega \in \text{Lip}_1(S^{n-1}) \), and assume that \( A \) has derivatives of order \( m-1 \) in \( \text{BMO}(\mathbb{R}^n) \). Then the following two statements are equivalent:

(i) \( \mathcal{H}_{\Omega, A, \beta}^{m} \) maps \( H^1 \) continuously into \( L^{n/(n-\beta)} \),

(ii) \( A \) is a polynomial of degree no more than \( m-1 \) or \( \Omega \equiv 0 \).

**Remark 9.** From Theorem 8, we can draw the conclusion that, when \( m \geq 2 \), \( \mathcal{H}_{\Omega, A, \beta}^{m} \) is not bounded from \( H^1 \) to \( L^{n/(n-\beta)} \) unless \( \mathcal{H}_{\Omega, A, \beta}^{m} \equiv 0 \).

**Proof of Theorem 8.** It is clear that (ii) \( \Rightarrow \) (i) is obvious. We only need to prove (i) \( \Rightarrow \) (ii).

Let \( \tilde{a} \) be an \( H^1 \) atom supported on the ball \( B = B(0, r) \), and denote \( C_a = (1/\alpha!) \int_B D^\alpha A(y) \tilde{a}(y) dy \) with \( |\alpha| = m-1 \).

By Theorem 6, for any \( u \in 3B \setminus 2B \) and \( N > 8 \) with \( N \in \mathbb{Z}^+ \), we have

\[
C \geq \int_{(4B)^c} \left\{ \frac{1}{|x|^{n-\beta}} \sum_{|\alpha|=m-1} C_a \Omega(x-u)(x-u)^\alpha \left| \tilde{a}(y) \right| \left| D^\alpha \left( \tilde{a}(y) \right) \right| dx \right\}^{n/(n-\beta)} \\
\geq C_1 \int_{(4B)^c} \left\{ \frac{1}{|x-u|^{n-\beta}} \sum_{|\alpha|=m-1} C_a \Omega(x-u)(x-u)^\alpha \left| \tilde{a}(y) \right| \left| D^\alpha \left( \tilde{a}(y) \right) \right| dx \right\}^{n/(n-\beta)}
\]
\[ \geq C_1 \int_{|x| \leq Nr} \frac{1}{|x|^{n-\beta}} \sum_{|\alpha|=m-1} C_\alpha \Omega(x) \left| x^{\alpha} \right| |x|^{n/(n-\beta)} \, dx \]

\[ = C_1 \log \left( \frac{N}{8} \right) \int_{|x|=m-1} \sum_{|\alpha|=m-1} C_\alpha \Omega(x') \left( x' \right)^{n/(n-\beta)} \, d\sigma(x'). \]

(43)

Let \( N \to +\infty \), we know log(N/8) \( \to +\infty \). Thus we have

\[ \int_{|x|=m-1} \sum_{|\alpha|=m-1} C_\alpha \Omega(x') \left( x' \right)^{n/(n-\beta)} \, d\sigma(x') = 0. \]

(44)

From (44) we can deduce

\[ \sum_{|\alpha|=m-1} C_\alpha \Omega(x') \left( x' \right)^{\alpha} = 0. \]

(45)

If \( \Omega \equiv 0 \), (45) is obviously true. Otherwise, we can easily obtain

\[ C_\alpha = \frac{1}{\alpha!} \int_B D^\alpha A(y) \, dy = 0. \]

(46)

Since \( \alpha \) is arbitrary, \( D^\alpha A \) must be a constant for each \( \alpha \) with \( |\alpha| = m-1 \). So we can deduce that \( A \) is a polynomial with degree no more than \( m-1 \).

Consequently, we have finished the proof of Theorem 8.

\[ \square \]

3. Boundedness of \( H_{m,\Omega,\alpha,\beta} \) from \( H^1 \) to \( L^{n/(n-\beta)} \)

In Section 2, we prove that, when \( m \geq 2 \), \( H_{m,\Omega,\alpha,\beta} \) is not bounded from \( H^1 \) to \( L^{n/(n-\beta)} \) unless \( H_{m,\Omega,\alpha,\beta} = 0 \). In this section, we will prove that \( H_{m,\Omega,\alpha,\beta} \) is bounded from \( H^1 \) to \( L^{n/(n-\beta)} \) with \( m \geq 1 \). Here \( L^{n/(n-\beta)} \) is defined by

\[ \|f\|_{L^{n/(n-\beta)}} = \sup_{\lambda > 0} \lambda \| |x| \in \mathbb{R}^n : |f(x)| > \lambda \|^{|n-\beta|/n} < \infty. \]

(47)

Our results can be stated as follows.

**Theorem 10.** Suppose that \( m \geq 2 \), \( 0 \leq \beta < n \), and \( \Omega \in \text{Lip}_1(S^{n-1}) \). If \( A \) has derivatives of order \( m-1 \) in BMO(\( \mathbb{R}^n \)), then there exists a constant \( C \) independent of \( f \) and \( A \), such that

\[ \|x \in \mathbb{R}^n : H_{m,\Omega,\alpha,\beta} f(x) > \lambda \|^{|n-\beta|/n} \leq C \sum_{|\alpha|=m-1} \| D^\alpha A \|_{\text{BMO}} \|f\|_{L^1} / \lambda \]

(48)

for any \( \lambda > 0 \).

For the case \( m = 1 \), we have the following.

**Theorem 11.** Let \( 0 < \beta < n \), \( A \in \text{BMO}(\mathbb{R}^n) \), and \( \Omega \in \text{Lip}_1(S^{n-1}) \); then there exists a constant \( C \) independent of \( f \) and \( A \), such that

\[ \|x \in \mathbb{R}^n : H_{1,\Omega,\alpha,\beta} f(x) > \lambda \|^{|n-\beta|/n} \leq C \sum_{|\alpha|=m-1} \| D^\alpha A \|_{\text{BMO}} \|f\|_{L^1} / \lambda \]

(49)

for any \( \lambda > 0 \).

Before the proof of Theorems 10 and 11, we need the following lemma.

**Lemma 12.** Let \( \hat{H}_{\beta} \) be defined by

\[ \hat{H}_{\beta} f(x) = \frac{1}{|x|^{n-\beta}} \int_{|y|<|x|} h(y) \left( \frac{x-y}{|x-y|} \right)^{\beta} \, dy, \]

(50)

where \( 0 \leq \beta < n \), \( \Omega \in \text{Lip}_1(S^{n-1}) \), and \( |\alpha| = m-1 \); then \( \hat{H}_{\beta} \) is bounded from \( L^1 \) to \( L^{n/(n-\beta)} \).

**Proof.** For any \( \lambda > 0 \), we have

\[ \lambda \|x \in \mathbb{R}^n : \hat{H}_{\beta} f(x) > \lambda \|^{|n-\beta|/n} \]

\[ = \lambda \left\{ x \in \mathbb{R}^n : \frac{1}{|x|^{n-\beta}} \int_{|y|<|x|} h(y) \left( \frac{x-y}{|x-y|} \right)^{\beta} \, dy > \lambda \right\} \]

\[ \leq \lambda \left\{ x \in \mathbb{R}^n : \frac{1}{|x|^{n-\beta}} \int_{|y|<|x|} |h(y)| \, dy > \lambda \right\} \]

\[ \leq \lambda \left\{ x \in \mathbb{R}^n : \frac{1}{|x|^{n-\beta}} \int_{|y|<|x|} |f(x')| \, dy > \lambda \right\} \]

\[ \leq \lambda \left( \int_{\mathbb{R}^{n-1}} \left( \frac{|h(x')|}{\lambda^{1/(n-\beta)}} \right)^{|n-\beta|/n} r^{-n+1} \, d\sigma(x') \right) \]

\[ \leq C \lambda \left( \frac{|h(x')|}{\lambda^{1/(n-\beta)}} \right)^{|n-\beta|/n} = C \|h\|_{L^1}. \]

(51)

**Proposition 13.** By the proof of Lemma 12 with minor changes, one can draw the conclusion that \( H_{1,\Omega,\alpha,\beta} \) is bounded from \( L^1 \) to \( L^{n/(n-\beta)} \) with \( \Omega \in \text{Lip}_1(S^{n-1}) \).
Proof of Theorem 10. Before giving the proof of Theorem 10, we introduce some notations that are very useful in this section.

For multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, \ldots, \beta_n)$, we denote $\alpha - \beta = (\alpha_1 - \beta_1, \ldots, \alpha_n - \beta_n)$. Furthermore, $\beta < \alpha$ means that for each $i$, we have $\beta_i < \alpha_i$. Finally, we denote $C_\alpha^\beta = \prod_{j=1}^m C_{\alpha_j}^{\beta_j}$.

From [19] and by the atomic decomposition of $H^1$, it suffices to show
\[
\left| \left\{ x \in \mathbb{R}^n : \mathcal{H}^{m}_{\Omega, A, \beta} \tilde{a} (x) > \lambda \right\} \right|^{(n-\beta)/n} \leq C \sum_{|\alpha| = m-1} \| D^\alpha A \|_{BMO} \Omega (x) |
\]
for any $H^1$ atom $\tilde{a}$, where $\tilde{a}$ is defined in Section 2. First we have the following decomposition:
\[
\left| \left\{ x \in \mathbb{R}^n : \mathcal{H}^{m}_{\Omega, A, \beta} \tilde{a} (x) > \lambda \right\} \right|^{(n-\beta)/n} \leq \left| \left\{ x \in B(0, 2r) : \mathcal{H}^{m}_{\Omega, A, \beta} \tilde{a} (x) > \lambda \right\} \right|^{(n-\beta)/n} + \left| \left\{ x \in \mathbb{R}^n \setminus B(0, 2r) : \mathcal{H}^{m}_{\Omega, A, \beta} \tilde{a} (x) > \lambda \right\} \right|^{(n-\beta)/n} := L_1 + L_2.
\]

For the term $L_1$, choosing $B(0, r) = B$ and $B(0, kr) = kB$ with $k \in \mathbb{Z}^+$, then by Proposition 2, Hölder inequality, and the size condition of $\tilde{a}$, we have
\[
\lambda \left| \left\{ x \in B(0, 2r) : \mathcal{H}^{m}_{\Omega, A, \beta} \tilde{a} (x) > \lambda \right\} \right|^{(n-\beta)/n} \leq C \left( \int_{B(0, 2r)} \left| \mathcal{H}^{m}_{\Omega, A, \beta} \tilde{a} (x) \right|^n dx \right)^{1/n} \leq C 2B^{(n-\beta)/n-1/q} \left( \int_{B(0, 2r)} \left| \mathcal{H}^{m}_{\Omega, A, \beta} \tilde{a} (x) \right|^q dx \right)^{1/q} \leq C 2B^{(n-\beta)/n-1/q} \sum_{|\alpha| = m-1} \left\| D^\alpha A \right\|_{BMO} \| \tilde{a} \|_{L^p} \]
\[
\leq C \sum_{|\alpha| = m-1} \left\| D^\alpha A \right\|_{BMO},
\]
where $1/p - 1/q = \beta/n$ and $p > 1$.

For the term $L_2$, as $x \in \mathbb{R}^n \setminus B(0, 2r)$, we have
\[
\left\{ y : |y| < |x| \cap \left\{ y : |y| < r \right\} = \left\{ y : |y| < r \right\}. \quad (55)
\]

Then for a fixed $B = B(0, r)$, we set $A_B(x) = A(x) - \sum_{|\alpha| = m-1} (1/\alpha!) (D^\alpha A)_B x^\alpha$. It is easy to check $R_m (A; x, y) = R_m (A_B; x, y)$.

Thus by the vanishing condition of $\tilde{a}$ and the fact $\text{supp}(\tilde{a}) \subset B(0, r)$, we can decompose $\mathcal{H}^{m}_{\Omega, A, \beta} \tilde{a} (x)$ as follows:
\[
\mathcal{H}^{m}_{\Omega, A, \beta} \tilde{a} (x) = \frac{1}{|x|^\beta} \int_{B(0, r)} \Omega (x - y) \frac{\tilde{a} (x) - \sum_{|\alpha| = m-1} \frac{1}{\alpha!} D^\alpha A_B (x) x^\alpha}{|x|^{m-1}} \]
\[
\times \left[ \frac{\Omega (x) - \Omega (x - y)}{|x|^{m-1} - |x - y|^{m-1}} \right] \tilde{a} (y) dy
\]
\[
= \frac{1}{|x|^{\beta-1}} \int_{B(0, r)} \Omega (x - y) \frac{\tilde{a} (x) - \sum_{|\alpha| = m-1} \frac{1}{\alpha!} D^\alpha A_B (x) x^\alpha}{|x|^{m-1}} \]
\[
\times \left[ \frac{\Omega (x) - \Omega (x - y)}{|x|^{m-1} - |x - y|^{m-1}} \right] \tilde{a} (y) dy + \frac{1}{|x|^\beta}
\]
\[
\times \int_{B(0, r)} \sum_{|\alpha| = m-1} \frac{1}{\alpha!} D^\alpha A_B (y) x^\alpha \left[ \Omega (x - y) \frac{\tilde{a} (y)}{|x - y|^{m-1}} \right] dy + \frac{1}{|x|^\beta}
\]
\[
\times \int_{B(0, r)} \sum_{|\alpha| = m-1} \frac{1}{\alpha!} D^\alpha A_B (y) \left[ \Omega (x - y) \frac{\tilde{a} (y)}{|x - y|^{m-1}} \right] dy
\]
\[
:= I_1 + I_2 + I_3 + I_4. \quad (56)
\]

Here we can simply denote each $I_j$ by
\[
I_j = \frac{1}{|x|^\beta} \int_{B(0, r)} \Omega (x - y) \frac{\tilde{a} (y)}{|x - y|^{m-1}} \] (57)

For $K_1$, by the fact that $\Omega \in \text{Lip}_1(S^{m-1})$ and $|x - y| \sim |x|$, we have
\[
\frac{\Omega (x) - \Omega (x - y)}{|x|^{m-1} - |x - y|^{m-1}} \leq \frac{\Omega (x) - \Omega (x - y)}{|x|^{m-1} - |x - y|^{m-1}}
\]
\[
+ \frac{\Omega (x - y)}{|x|^{m-1} - |x - y|^{m-1}}
\]
\[
\int_{B(0, r)} \frac{\tilde{a} (y)}{|x|^{m-1} - |x - y|^{m-1}} \mathcal{H}^{m}_{\Omega, A, \beta} \tilde{a} (x) dx.
\]


As $|x| > r$ for $x \in \mathbb{R}^n \setminus 2B$, Lemma 7 in this paper and Lemma 2.2 in [8] tell us that

$$K_1(A_B; x, y) \leq C \left| \sum_{|\alpha| = m - 1} |D^\alpha A(x) - (D^\alpha A)_B| \right| \frac{|y|}{|x|^m} + \sum_{|\alpha| = m - 1} |D^\alpha A(x) - (D^\alpha A)_B| \left| \frac{y}{|x|^m} \right|,$$

(58)

where $C_1$ is a constant only depending on $n$, and $Q$ is a cube centered at $x$ and having diameter $5 \sqrt{n}|x|$.

Thus we obtain

$$|K_1(A_B; x, y)| \leq C |x|^{m-1} \left[ \sum_{|\alpha| = m - 1} \left( \frac{1}{|Q|} \int_{|Q|} |D^\alpha A(z) - (D^\alpha A)_B|^q dz \right)^{\frac{q}{q}} + \sum_{|\alpha| = m - 1} |D^\alpha A(x) - (D^\alpha A)_B| \right] \left| \frac{y}{|x|^m} \right|,$$

(59)

Next we will give the estimates of $I_2$. First by a cumbersome but straightforward computation, we have

$$|I_2| \leq \frac{m-2}{m-1} \sum_{k=0}^{m-1} \sum_{|\alpha| = k} R_{m-k-1} (D^\alpha A_B; 0, y) \left| x \right|^{-m+k+1}.$$

(60)

Also noting the fact that

$$R_{m-k-1} (D^\alpha A_B; 0, y) \leq C |y|^{m-k-1} \left( \sum_{|\alpha| = m - 1} \left( \frac{1}{|Q_0|} \int_{|Q_0|} |D^\alpha A(z) - (D^\alpha A)_B|^q dz \right)^{\frac{1}{q}} \right)^{\frac{q}{n}} + \sum_{|\alpha| = m - 1} |D^\alpha A(x) - (D^\alpha A)_B| \left| \frac{y}{|x|^m} \right|.$$

(61)

where $q > n$, and $Q_0$ is a cube centered at 0 and having diameter $5 \sqrt{n}|y|$.

Thus we obtain

$$|K_2(A_B; x, y)| \leq C \left( \sum_{|\alpha| = m - 1} |D^\alpha A(x) - (D^\alpha A)_B| \right) \left| \frac{y}{|x|^m} \right| \leq C \left( \sum_{|\alpha| = m - 1} |D^\alpha A(x) - (D^\alpha A)_B| \right) \left| \frac{y}{|x|^m} \right|.$$
So we have the following estimates of $I_2$:

\[
I_2 \leq C \frac{1}{|x|^{n-\beta}} \int_{\mathbb{R}^n} \sum_{k=0}^{m-2} r^{m-k-1} |x|^{-m+k+1} |\tilde{a}(y)| dy \\
= C \sum_{k=0}^{m-2} r^{m-k-1} |x|^{-m+k+1+n(\beta-1)} \int_B |\tilde{a}(y)| dy \\
\leq C \sum_{k=0}^{m-2} r^{m-k-1} |x|^{-m+k+1+n\beta}.
\]

Now we get

\[
\lambda |x| \in \mathbb{R}^n \setminus B (0, 2r) : |I_2| > \lambda \]^{(n-\beta)/n} \\
\leq C \sum_{|\alpha|=m-1} \|D^\alpha a\|_{BMO} \left( \int_{\mathbb{R}^n \setminus B (0,2r)} |I_2|^{(n-\beta)/n} \right) \\
\leq C \sum_{|\alpha|=m-1} \|D^\alpha a\|_{BMO} \\
\times \left( \sum_{k=0}^{m-2} \left( \frac{r^{m-k-1} |x|^{-m+k+1+n\beta}}{n^2} \right)^{(n-\beta)/n} \right) \\
\times \left( \sum_{j=1}^{\infty} \int_{2^j B / 2} |x|^{n((1+k-m)/(n-\beta)-1)} dx \right)^{(n-\beta)/n} \\
\leq C \sum_{|\alpha|=m-1} \|D^\alpha a\|_{BMO} \\
\times \left( \sum_{k=0}^{m-2} |B|^{(m-k-1)/(n-\beta)} \times \sum_{j=1}^{\infty} |2^j B|^{((1+k-m)/(n-\beta)-1) + 1} \right) \\
\leq C \sum_{|\alpha|=m-1} \|D^\alpha a\|_{BMO} \\
\times \left( \sum_{k=0}^{m-2} |B|^{(m-k-1)/(n-\beta)} \times \sum_{j=1}^{\infty} 2^{jn((1+k-m)/(n-\beta)-1) + 1} \right)^{(n-\beta)/n}.
\]

For $I_3$, by the vanishing condition of $\tilde{a}(y)$, we can split $I_3$ as follows:

\[
I_3 = \frac{1}{|x|^{n-\beta}} \int_{\mathbb{R}^n} \sum_{|\alpha|=m-1} \frac{1}{\alpha!} D^\alpha A_B (y) x^\alpha \\
- \sum_{|\alpha|=m-1} \frac{1}{\alpha!} D^\alpha A_B (x) x^\alpha \\
\times \Omega (x - y) |x - y|^{m-1-\tilde{a}(y)} dy \\
\leq \frac{1}{|x|^{n-\beta}} \sum_{|\alpha|=m-1} \frac{1}{\alpha!} \int_{\mathbb{R}^n} \left[ D^\alpha A (y) - (D^\alpha A) B \right] \\
\times (x - y)^{\alpha} \Omega (x - y) |x - y|^{m-1-\tilde{a}(y)} dy \\
+ \frac{1}{|x|^{n-\beta}} \sum_{|\alpha|=m-1} \frac{1}{\alpha!} \int_{\mathbb{R}^n} D^\alpha A_B (y) \\
\times \sum_{|y|<\alpha} C^i (x - y)^{1/y} y^{-\gamma} \\
\times \Omega (x - y) |x - y|^{m-1-\tilde{a}(y)} dy \\
+ \frac{1}{|x|^{n-\beta}} \sum_{|\alpha|=m-1} \frac{1}{\alpha!} \int_{\mathbb{R}^n} x^\alpha \left[ D^\alpha A (x) - (D^\alpha A) B \right] \\
\times \Omega (x - y) |x - y|^{m-1-\tilde{a}(y)} dy \\
= I_{31} + I_{32} + I_{33}.
\]

For the term $I_{31}$, by Lemma 12 and the size condition of $\tilde{a}$, we obtain

\[
\lambda |x| \in \mathbb{R}^n \setminus B (0, 2r) : |I_{31}| > \lambda \]^{(n-\beta)/n} \\
\leq \sum_{|\alpha|=m-1} \| (D^\alpha A - (D^\alpha A)_B ) \tilde{a} \|_{L^1} \\
\leq \sum_{|\alpha|=m-1} \|D^\alpha a\|_{BMO}.
\]

For the term $I_{32}$, by the vanishing condition of $\tilde{a}$ and the fact $\Omega \in \text{Lip}_1 (S^{n-1})$, we have

\[
I_{32} = \frac{1}{|x|^{n-\beta}} \sum_{|\alpha|=m-1} \frac{1}{\alpha!} \int_{\mathbb{R}^n} \left[ D^\alpha A_B (y) - (D^\alpha A)_B \right] \\
\times \sum_{|y|<\alpha} C^i (x - y)^{1/y} y^{-\gamma} \\
\times \Omega (x - y) |x - y|^{m-1-\tilde{a}(y)} dy.
\]
\[ \leq C \frac{1}{|x|^{n-\beta}} \sum_{|\alpha|=m-1} \frac{1}{\alpha!} \int_{\mathbb{R}^n} |D^\alpha A_B (y) - (D^\alpha A)_B| \times |x-y|^{1-m+|\gamma|} dy \]
\[ \times \|\tilde{a}(y)\|_{BMO}. \]
\[
\leq C \sum_{|\alpha|=m-1} \|D^\alpha A\|_{\text{BMO}}.
\]

(71)

Finally, we will give the estimates of \(I_4\). By a similar argument as in the estimates of \(I_{32}\), we can easily get

\[
\lambda \left| \{ x \in \mathbb{R}^n \setminus B(0,2r) : |I_4| > \lambda \} \right| \leq C \sum_{|\alpha|=m-1} \|D^\alpha A\|_{\text{BMO}}.
\]

(72)

Combining the estimates of \(I_1, I_2, I_3, \) and \(I_4\), we finish the proof of Theorem 10.

\[\square\]

**Proof of Theorem 11.** Theorem 11 was proved in [7] in the case \(\beta = 0\) and \(\Omega \equiv 1\). For the case \(0 \leq \beta < n\) and \(\Omega \equiv 1\), we can easily prove Theorem 11 by the proof of Theorem 5.3 in [7] with minor changes. Then for the case \(0 \leq \beta < n\) and \(\Omega \in \text{Lip}_1(S^{n-1})\), by the main idea used in the proof of Theorem 10, we can prove Theorem 11 easily and we omit the details here.

\[\square\]

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