Research Article

Estimates of the Fundamental Solution for Higher Order Schrödinger Type Operators and Their Applications

Satoko Sugano

Kobe City College of Technology, 8-3 Gakuen-higashimachi, Nishi-ku, Kobe 651-2194, Japan

Correspondence should be addressed to Satoko Sugano; sugano@kobe-kosen.ac.jp

Received 3 October 2013; Accepted 21 November 2013

Academic Editor: Hendra Gunawan

Copyright © 2013 Satoko Sugano. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We consider higher order Schrödinger type operators with nonnegative potentials. We assume that the potential belongs to the reverse Hölder class which includes nonnegative polynomials. We establish estimates of the fundamental solution and show L^p boundedness of some Schrödinger type operators.

1. Introduction

Let $V(x)$ be a nonnegative potential and consider the Schrödinger type operators $H_k = (-\Delta)^k + V^k$ on $\mathbf{R}^n$, where k is a positive integer and $n \geq 2k + 1$. When V is a nonnegative polynomial, Zhong proved estimates of the fundamental solution for H_1 and H_2 and showed some estimates for H_1 and H_2 (see [1]). More precisely, he showed the L^p boundedness of the operators VH_1^{-1} , $V^{1/2}\nabla H_1^{-1}$, and $V^{2-j/2}\nabla^j H_2^{-1}$, where $j = 0, 1, 2, 3, 4$. He also proved that the operators $\nabla^2 H_1^{-1}$ and $\nabla^4 H_2^{-1}$ are Calderón-Zygmund operators. Recently, in [2], the authors showed the L^p boundedness of the operators $V^k H_k^{-1}$, $\nabla^{2k} H_k^{-1}$, and $\nabla^k H_k^{-1/2}$ for nonnegative polynomial potential V .

For the potential V which belongs to the reverse Hölder class, which includes nonnegative polynomials, Shen generalized Zhong's results on H_1 (see [3]). Actually, he established estimates of the fundamental solution for H_1 and showed the L^p estimates for the operators VH_1^{-1} , $V^{1/2}\nabla H_1^{-1}$, $\nabla^2 H_1^{-1}$, and so on. For the operator H_1 with reverse Hölder class potentials, further results have been investigated by many researchers. See [4–7], for example. For the operator H_2 with reverse Hölder class potentials, in [8], the author established estimates of the fundamental solution for H_2 and showed the L^p boundedness of the operators $V^{2-j/2}\nabla^j H_2^{-1}$, where $j =$

$0, 1, 2, 3, 4$. In [8], the author also showed that the operator $\nabla^4 H_2^{-1}$ is a Calderón-Zygmund operator. Recently, in [9], the authors showed L^p boundedness of the operator $\nabla^2 H_2^{-1/2}$ and that the operator $\nabla^2 H_2^{-1/2}$ is a Calderón-Zygmund operator.

In this paper, we study $H_{2^m} = (-\Delta)^{2^m} + V^{2^m}$ with reverse Hölder class potential V on $\mathbf{R}^n$, where m is a nonnegative integer and $n \geq 2^{m+1} + 1$. We establish estimates of the fundamental solution for H_{2^m} and show the L^p boundedness of the operators $V^{2^m-j/2}\nabla^j H_{2^m}^{-1}$, where j is an integer satisfying $0 \leq j \leq 2^{m+1}$.

As mentioned above, in [2], the authors proved some results on $H_k = (-\Delta)^k + V^k$, where V is a nonnegative polynomial and k is an integer, $k \geq 3$. They proved their results by making use of [2, Lemma 3.3 and Corollary 3.1] which have been proved only for nonnegative polynomial V . In this paper, our strategy is different from the one in [2], since the question whether the above two results can be proved for reverse Hölder class potentials V is yet to be settled. The purpose of this paper is to show some results on $H_k = (-\Delta)^k + V^k$ with potential V which belongs to the reverse Hölder class, which includes nonnegative polynomials. However, our results are only for $k = 2^m$, where m is an integer, $m \geq 2$.

We recall the definitions of the reverse Hölder class (e.g., [3]). We denote by $B_r(x)$ the ball centered at x with radius r .

Definition 1 (reverse Hölder class). Let $V \geq 0$.

(1) For $1 < p < \infty$ one says that $V \in (RH)_p$ if $V \in L^p_{\text{loc}}(\mathbf{R}^n)$ and there exists a positive constant C such that

$$\left(\frac{1}{|B_r(x)|} \int_{B_r(x)} V(y)^p dy \right)^{1/p} \leq \frac{C}{|B_r(x)|} \int_{B_r(x)} V(y) dy \quad (1)$$

holds for every $x \in \mathbf{R}^n$ and $0 < r < \infty$.

(2) One says that $V \in (RH)_{\infty}$ if $V \in L^{\infty}_{\text{loc}}(\mathbf{R}^n)$ and there exists a positive constant C such that

$$\|V\|_{L^{\infty}(B_r(x))} \leq \frac{C}{|B_r(x)|} \int_{B_r(x)} V(y) dy \quad (2)$$

holds for every $x \in \mathbf{R}^n$ and $0 < r < \infty$.

Remark 2. (1) If $P(x)$ is a polynomial and $\alpha > 0$, then $V(x) = |P(x)|^{\alpha}$ belongs to $(RH)_{\infty}$ (see [10, page 146]).

(2) For $1 < p < \infty$, it is easy to see $(RH)_{\infty} \subset (RH)_p$.

Definition 3 (see [3, Definition 1.3]). (1) Let $V \in (RH)_{n/2}$ and $V \neq 0$. Then it is well known that there exists a positive ϵ such that $V \in (RH)_{n/2+\epsilon}$ (see [11, Lemma 2]). Then the function $\rho(x, V)$ is well defined by

$$\frac{1}{\rho(x, V)} = \sup \left\{ r > 0 : \frac{r^2}{|B_r(x)|} \int_{B_r(x)} V(y) dy \leq 1 \right\} \quad (3)$$

and satisfies $0 < \rho(x, V) < \infty$ for every $x \in \mathbf{R}^n$.

One recalls that

$$\frac{r^2}{|B_r(x)|} \int_{B_r(x)} V(y) dy = 1 \quad \text{for } r = \frac{1}{\rho(x, V)} \quad (4)$$

(see [3, page 519]).

Remark 4. (1) If $V \in (RH)_p$, $p \geq n/2$, then there exists a positive constant C such that

$$\left(\frac{1}{|B_r(x)|} \int_{B_r(x)} V(y)^p dy \right)^{1/p} \leq C \rho(x, V)^2 \quad (5)$$

(cf. [3, Lemma 1.8] and [5, Lemma 2.2 (a)]).

(2) If $V \in (RH)_{\infty}$ then there exists a positive constant C such that $V(x) \leq C \rho(x, V)^2$ (see [3, Remark 2.9]).

Let $\alpha = (\alpha_1, \dots, \alpha_n)$ denote the multi-index with $\alpha_i \in \mathbf{N}$, $i \in \mathbf{N}$, and $1 \leq i \leq n$. Define $x^{\alpha} = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $\nabla^j = \partial^{\alpha} = \partial^{|\alpha|} / (\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n})$ for $j = |\alpha| = \alpha_1 + \cdots + \alpha_n$. For any positive integer j and a function $u \in C^j(\mathbf{R}^n)$, denote that $\nabla^j u(x) = (\partial^{\alpha} u(x) : |\alpha| = j)$ and $|\nabla^j u(x)|^2 = \sum_{|\alpha|=j} |\partial^{\alpha} u(x)|^2$. We denote by $\Gamma_{H_k}(x, y)$ the fundamental solution for H_k . The operator H_k^{-1} is the integral operator with $\Gamma_{H_k}(x, y)$ as its kernel.

Now we state our theorems.

Theorem 5. Let j, m , and n be integers, $m \geq 0$, $n \geq 2^{m+1} + 1$, and $0 \leq j \leq 2^{m+1} - 1$. Suppose that $V \in (RH)_{q_0}$ for some

$q_0 \geq 2^m n / (2^{m+1} - j)$. Then there exist positive constants C_j such that

$$\|V^{2^m-j/2} \nabla^j H_{2^m}^{-1} f\|_{L^p(\mathbf{R}^n)} \leq C_j \|f\|_{L^p(\mathbf{R}^n)}, \quad (6)$$

where $1 \leq p \leq 2q_0 / (2^{m+1} - j)$.

In Theorem 5, the case $m = 0$ was shown in [3, Theorems 2.7 and 4.13].

Remark 6. In Theorem 5, for the case $m = j = 0$, inequality (6) was shown in [3, Remark 2.9] under the assumption $V \in (RH)_{\infty}$. If we take the limit $q_0 \rightarrow \infty$, then class $(RH)_{q_0}$ becomes $(RH)_{\infty}$ and $V \in (RH)_{\infty}$ implies “ $V \in (RH)_{n/2}$ and $V(x) \leq C \rho(x, V)^2$.” See [8, Remark 3] and also [8, Theorem 1].

Remark 7. We assume “ $V \in (RH)_{n/2}$ and $V(x) \leq C \rho(x, V)^2$.” Then inequality (6) is true for $1 \leq p \leq \infty$. See Section 7 for the details. We also remark that the inequality (6) is true for $j = 2^{m+1}$ and $1 < p < \infty$ under the assumption “ $V \in (RH)_{n/2}$ and $V(x) \leq C \rho(x, V)^2$.” See [3, Remark 2.9] and also [8, Theorem 1 (2)].

Theorem 8. Let j, m , and n be integers, $m \geq 0$, $n \geq 2^{m+1} + 1$, and $1 \leq j \leq 2^{m+1} - 1$. Suppose that $V \in (RH)_{q_0}$ for some q_0 , $n/2 \leq q_0 < 2^m n / (2^{m+1} - j)$. Then there exist positive constants C_j such that (6) holds, where $1 \leq p \leq p_0$ and $1/p_0 = (2^{m+2} - j) / (2q_0 - (2^{m+1} - j)/n)$.

To prove Theorems 5 and 8 we need the estimates of the fundamental solution (Theorems 9 and 10). The following Theorem 9 generalizes the results in [2, Theorem 3.1] to the operator H_{2^m} with potential V which belongs to the reverse Hölder class.

Theorem 9. Let m and n be integers, $m \geq 0$, and $n \geq 2^{m+1} + 1$. Suppose that $V \in (RH)_{n/2}$. Then for any positive integer N there exists a positive constant C_N such that

$$(0 \leq) \Gamma_{H_{2^m}}(x, y) \leq \frac{C_N}{\{1 + \rho(x, V)|x - y|\}^N} \cdot \frac{1}{|x - y|^{n-2^{m+1}}}. \quad (7)$$

In Theorem 9, the cases $m = 0$ and $m = 1$ were shown in [3, Theorem 2.7] and [8, Theorem 2], respectively. We prove Theorem 9 by induction; that is, we assume that Theorem 9 is true for $m = l$ and show the case $m = l + 1$. We also prove the following theorem which states derivative estimates of the fundamental solution.

Theorem 10. Let j, m , and n be integers, $m \geq 0$, $n \geq 2^{m+1} + 1$, and $1 \leq j \leq 2^{m+1} - 1$. Suppose that $V \in (RH)_{2^m n / (2^{m+1} - j)}$. Then for any positive integer N there exists a positive constant C_N such that

$$|\nabla_x^j \Gamma_{H_{2^m}}(x, y)| \leq \frac{C_N}{\{1 + \rho(x, V)|x - y|\}^N} \cdot \frac{1}{|x - y|^{n-2^{m+1}+j}}. \quad (8)$$

In Theorem 10, the cases $m = 0$ and $m = 1$ were shown in [3, page 537] and [8, Theorem 6], respectively.

The plan of this paper is as follows. In Section 2, we describe some lemmas needed later. In Section 3, we assume that Theorem 9 is true for $m = l$ and show some estimates for H_{2^l} which are needed to prove the case $m = l + 1$ in Theorem 9. In Section 4, we prove the case $m = l + 1$ in Theorem 9. Section 5 is devoted to proof of Theorem 10. In Section 6, we prove Theorems 5 and 8. Finally, in Section 7, we state some remarks. Throughout this paper the letter C stands for a constant not necessarily the same at each occurrence.

2. Preliminaries

In this section, we describe some lemmas needed later. First, we remark that $V \in (RH)_q$, $q > 1$, implies that $V(x)dx$ is a doubling measure; that is, there exists a positive constant C_0 such that

$$\int_{B_{2r}(x)} V(y) dy \leq C_0 \int_{B_r(x)} V(y) dy \quad (9)$$

(see [3, page 518] and also [5, Remark 1.2 (1)]).

Lemma 11 (see [3, Lemma 1.2]). Assume that $V \in (RH)_q$ for some $q > n/2$. Then there exists a positive constant C such that, for $0 < r < R < \infty$,

$$\frac{r^2}{|B_r(x)|} \int_{B_r(x)} V(y) dy \leq C \left(\frac{R}{r} \right)^{n/q-2} \frac{R^2}{|B_R(x)|} \int_{B_R(x)} V(y) dy. \quad (10)$$

Lemma 12 (see [3, Lemma 1.4 (b), (c)]). (1) Suppose that $V \in (RH)_{n/2}$. Then there exist positive constants C and k_0 such that, for $x, y \in \mathbf{R}^n$,

$$\rho(y, V) \leq C \{1 + \rho(x, V) |x - y|\}^{k_0} \rho(x, V). \quad (11)$$

(2) Suppose $V \in (RH)_{n/2}$. Then there exist positive constants C' and k_0 such that, for $x, y \in \mathbf{R}^n$,

$$\rho(y, V) \geq \frac{C' \rho(x, V)}{\{1 + \rho(x, V) |x - y|\}^{k_0/(k_0+1)}}. \quad (12)$$

Lemma 13 (Caccioppoli type inequality). Let a, j , and k be integers, $k \geq 2$, $1 \leq a \leq k/2$, and $1 \leq j \leq 2a$. Assume that $(-\Delta)^k u + V^k u = 0$ in $B_R(x_0)$ for some $x_0 \in \mathbf{R}^n$. Then there exists a positive constant C such that

$$\begin{aligned} & \int_{B_{R/2}(x_0)} |\Delta^{k-a} u(x)|^2 dx \\ & + \int_{B_{R/2}(x_0)} V(x)^k |u(x)| |\Delta^{k-2a} u(x)| dx \\ & \leq C \sum_{j=1}^{2a} \frac{1}{R^{2j}} \int_{B_R(x_0)} |\nabla^{2a-j} (\Delta^{k-2a} u)(x)|^2 dx. \end{aligned} \quad (13)$$

For readers' convenience, we give the proof of Lemma 13 at the end of this section. If k is an even number, then letting $a = k/2$ in Lemma 13, we have the following.

Corollary 14 (see [2, Lemma 3.5]). Let j be an integer, k an even number, $k \geq 2$, and $0 \leq j \leq k - 1$. Assume that $(-\Delta)^k u + V^k u = 0$ in $B_R(x_0)$ for some $x_0 \in \mathbf{R}^n$. Then there exists a positive constant C such that

$$\begin{aligned} & \int_{B_{R/2}(x_0)} |\Delta^{k/2} u(x)|^2 dx + \int_{B_{R/2}(x_0)} V(x)^k |u(x)|^2 dx \\ & \leq C \sum_{j=0}^{k-1} \frac{1}{R^{2(k-j)}} \int_{B_R(x_0)} |\nabla^j u(x)|^2 dx. \end{aligned} \quad (14)$$

Lemma 15 (Caccioppoli type inequality). Let a, j , and k be integers, $k \geq 1$, $1 \leq a \leq (k+1)/2$, and $1 \leq j \leq 2a - 1$. Assume that $(-\Delta)^k u + V^k u = 0$ in $B_R(x_0)$ for some $x_0 \in \mathbf{R}^n$. Then there exists a positive constant C such that

$$\begin{aligned} & \int_{B_{R/2}(x_0)} |\nabla (\Delta^{k-a} u)(x)|^2 dx \\ & + \int_{B_{R/2}(x_0)} V(x)^k |u(x)| |\Delta^{k-2a+1} u(x)| dx \\ & \leq C \sum_{j=1}^{2a-1} \frac{1}{R^{2j}} \int_{B_R(x_0)} |\nabla^{2a-j-1} (\Delta^{k-2a+1} u)(x)|^2 dx. \end{aligned} \quad (15)$$

If k is an odd number, then letting $a = (k+1)/2$ in Lemma 15, we have the following.

Corollary 16 (see [2, Lemma 3.5]). Let j be an integer, k an odd number, $k \geq 1$, and $0 \leq j \leq k - 1$. Assume that $(-\Delta)^k u + V^k u = 0$ in $B_R(x_0)$ for some $x_0 \in \mathbf{R}^n$. Then there exists a positive constant C such that

$$\begin{aligned} & \int_{B_{R/2}(x_0)} |\nabla (\Delta^{(k-1)/2} u)(x)|^2 dx + \int_{B_{R/2}(x_0)} V(x)^k |u(x)|^2 dx \\ & \leq C \sum_{j=0}^{k-1} \frac{1}{R^{2(k-j)}} \int_{B_R(x_0)} |\nabla^j u(x)|^2 dx. \end{aligned} \quad (16)$$

Lemma 17. Let j, k , and n be integers, $k \geq 1$, $n \geq 2k + 1$, and $0 \leq j \leq k - 1$. Assume that $(-\Delta)^k u + V^k u = 0$, $u \geq 0$, in $B_R(x_0)$ for some $x_0 \in \mathbf{R}^n$. Then there exists a positive constant C such that

$$\begin{aligned} & \sup_{y \in B_{R/2}(x_0)} |u(y)| \\ & \leq C \sum_{j=0}^{k-1} R^j \left(\frac{1}{|B_R(x_0)|} \int_{B_R(x_0)} |\nabla^j u(x)|^2 dx \right)^{1/2}. \end{aligned} \quad (17)$$

In Lemma 17, the cases $k = 1$ and $k = 2$ have been proved (see [3, page 523] and [1, Corollary 5.6]).

Proof of Lemma 17. We proceed following the proof of [1, Corollary 5.6]. We prove Lemma 17 for u which satisfies $(-\Delta)^k u + V^k u = 0$, $u \geq 0$, in $B_R(x_0)$. Let a and b be nonnegative integers and choose $\eta \in C_0^\infty(B_{R/4}(x_0))$ such that $\eta \equiv 1$

on $B_{R/8}(x_0)$ and $|\nabla^b(\Delta^a \eta)| \leq C/R^{2a+b}$. Let $\Gamma_{H_{k-1},0}(x, y) = C|x-y|^{2(k-1)-n}$ be the fundamental solution for $(-\Delta)^{k-1}$. Note that

$$\begin{aligned} & \Delta^{k-1}(u(y)\eta(y)) \\ &= \sum_{a,b} \binom{2(k-1)}{2a+b} \nabla^b(\Delta^{k-1-a-b}u)(y) \cdot \nabla^b(\Delta^a \eta)(y), \end{aligned} \quad (18)$$

where

$$\begin{aligned} & \nabla^b(\Delta^{k-1-a-b}u)(y) \cdot \nabla^b(\Delta^a \eta)(y) \\ &= \sum_{i_1, i_2, \dots, i_{k-1}=1}^n \frac{\partial^b}{\partial y_{i_1} \partial y_{i_2} \cdots \partial y_{i_b}} \\ & \quad \times \left(\frac{\partial^2}{\partial y_{i_1}^2} \left(\frac{\partial^2}{\partial y_{i_2}^2} \left(\cdots \left(\frac{\partial^2}{\partial y_{i_{k-1}-a-b}^2} u \right) \cdots \right) \right) \right) \\ & \quad \cdot \frac{\partial^b}{\partial y_{i_1} \partial y_{i_2} \cdots \partial y_{i_b}} \left(\frac{\partial^2}{\partial y_{i_1}^2} \left(\frac{\partial^2}{\partial y_{i_2}^2} \left(\cdots \left(\frac{\partial^2}{\partial y_{i_a}^2} \eta \right) \cdots \right) \right) \right) \end{aligned} \quad (19)$$

and the summation is taken over all integers a and b satisfying $a \geq 0, b \geq 0$, and $0 \leq a+b \leq k-1$. Then, integrating by parts, for $x \in B_{R/4}(x_0)$ we have

$$\begin{aligned} & u(x)\eta(x) \\ &= \int_{\mathbb{R}^n} \Gamma_{H_{k-1},0}(x, y) (-\Delta)^{k-1}(u\eta)(y) dy \\ &= \int_{\mathbb{R}^n} \Gamma_{H_{k-1},0}(x, y) \\ & \quad \times \sum_{a,b} \left\{ (-1)^{k-1} \binom{2(k-1)}{2a+b} \nabla^b(\Delta^{k-1-a-b}u)(y) \right. \\ & \quad \left. \cdot \nabla^b(\Delta^a \eta)(y) \right\} dy \\ &= \int_{\mathbb{R}^n} \Gamma_{H_{k-1},0}(x, y) (-\Delta)^{k-1}u(y)\eta(y) dy \\ & \quad + C \int_{\mathbb{R}^n} \sum_{(a,b) \neq (0,0)} \left\{ \nabla^b(\Delta^{k-1-a-b}\Gamma_{H_{k-1},0}(x, y)) \right. \\ & \quad \left. \cdot \nabla^b(\Delta^a \eta(y))u(y) \right\} dy. \end{aligned} \quad (20)$$

Let

$$\begin{aligned} |u(x_0)| &\leq \int_{B_{R/4}(x_0)} \frac{|(-\Delta)^{k-1}u(y)|}{|x_0 - y|^{n-2(k-1)}} dy \\ & \quad + \frac{C}{R^{2a+b}} \sum_{(a,b) \neq (0,0)} \int_{B_{R/4}(x_0) \setminus B_{R/8}(x_0)} \frac{|u(y)|}{|x_0 - y|^{n-2a-b}} dy \\ &= I_1 + I_2. \end{aligned} \quad (21)$$

Then we have

$$\begin{aligned} I_2 &\leq \frac{C}{R^n} \int_{B_{R/4}(x_0)} |u(y)| dy \\ &\leq C \left(\frac{1}{|B_R(x_0)|} \int_{B_R(x_0)} |u(y)|^2 dy \right)^{1/2}. \end{aligned} \quad (22)$$

It remains to estimate I_1 . Since $u \geq 0$, it follows that $\Delta((-\Delta)^{k-1}u) = -(-\Delta)^k u = V^k u \geq 0$. Hence $(-\Delta)^{k-1}u$ is subharmonic. Then for all $x \in B_{R/4}(x_0)$,

$$\begin{aligned} (-\Delta)^{k-1}u(x) &\leq \frac{1}{|B_{R/4}(x)|} \int_{B_{R/4}(x)} (-\Delta)^{k-1}u(y) dy \\ &\leq C \left(\frac{1}{|B_{R/2}(x_0)|} \int_{B_{R/2}(x_0)} |(-\Delta)^{k-1}u(y)|^2 dy \right)^{1/2}. \end{aligned} \quad (23)$$

Then we have

$$\begin{aligned} I_1 &\leq C \sup_{y \in B_{R/4}(x_0)} |(-\Delta)^{k-1}u(y)| \int_{B_{R/4}(x_0)} \frac{dy}{|x_0 - y|^{n-2(k-1)}} \\ &\leq CR^{2(k-1)} \sup_{y \in B_{R/4}(x_0)} |(-\Delta)^{k-1}u(y)| \\ &\leq CR^{2k-2} \left(\frac{1}{|B_{R/2}(x_0)|} \int_{B_{R/2}(x_0)} |(-\Delta)^{k-1}u(y)|^2 dy \right)^{1/2}. \end{aligned} \quad (24)$$

Then using Lemmas 13 and 15 repeatedly, we have

$$\begin{aligned} I_1 &\leq CR^{2k-2-n/2} \left(\frac{1}{R^2} \int_{B_R(x_0)} |\nabla(\Delta^{k-2}u)(x)|^2 dx \right. \\ & \quad \left. + \frac{1}{R^4} \int_{B_R(x_0)} |\Delta^{k-2}u(x)|^2 dx \right)^{1/2} \\ &\leq \cdots \leq C \sum_{j=0}^{k-1} R^j \left(\frac{1}{R^n} \int_{B_R(x_0)} |\nabla^j u(x)|^2 dx \right)^{1/2}. \end{aligned} \quad (25)$$

It follows that

$$|u(x_0)| \leq C \sum_{j=0}^{k-1} R^j \left(\frac{1}{|B_R(x_0)|} \int_{B_R(x_0)} |\nabla^j u(x)|^2 dx \right)^{1/2}. \quad (26)$$

From (26) we have, for all $y \in B_{R/2}(x_0)$,

$$|u(y)| \leq C \sum_{j=0}^{k-1} R^j \left(\frac{1}{|B_{R/4}(y)|} \int_{B_{R/4}(y)} |\nabla^j u(x)|^2 dx \right)^{1/2}. \quad (27)$$

Then we arrive at the desired estimate. $\square$

Lemma 18. Let j, k , and n be integers, $k \geq 1, n \geq 2k+1$, and $1 \leq j \leq 2k-1$. Suppose that $V \in (RH)_{q_0}$ for some $q_0, n/2 \leq q_0 < kn/(2k-j)$. Assume also that $(-\Delta)^k u + V^k u = 0$ in

$B_R(x_0)$ for some $x_0 \in \mathbf{R}^n$. Then there exists a positive constant C such that

$$\left(\int_{B_{R/2}(x_0)} |\nabla^j u(x)|^t dx \right)^{1/t} \leq CR^{kn/q_0-2k} \{1 + R\rho(x_0, V)\}^{2k} \sup_{y \in B_R(x_0)} |u(y)|, \quad (28)$$

where $1/t = k/q_0 - (2k - j)/n$.

In Lemma 18, the cases $k = 1$ and $k = 2$ were shown in [3, Lemma 4.6] and [8, Lemma 7], respectively.

Proof of Lemma 18. We show Lemma 18 by a method similar to the one used in the proof of [3, Lemma 4.6]. Let $\eta \in C_0^\infty(B_R(x_0))$ such that $\eta \equiv 1$ on $B_{3R/4}(x_0)$ and $|\nabla^b(\Delta^a \eta)| \leq C/R^{2a+b}$, where a and b are nonnegative integers. We denote by $\Gamma_{H_k,0}(x, y)$ the fundamental solution for $(-\Delta)^k$. Note that

$$\begin{aligned} u(x) \eta(x) &= \int_{\mathbf{R}^n} \Gamma_{H_k,0}(x, y) \left\{ -V(y)^k u(y) \eta(y) + (-1)^k \right. \\ &\quad \times \sum_{a,b} \binom{2k}{2a+b} \nabla^b (\Delta^{k-a-b} u)(y) \\ &\quad \left. \cdot \nabla^b (\Delta^a \eta)(y) \right\} dy, \end{aligned} \quad (29)$$

where the summation is taken over all integers a and b satisfying $a \geq 0$, $b \geq 0$, and $1 \leq a + b \leq k$. Then integrating by parts, for $x \in B_{R/2}(x_0)$, we have

$$\begin{aligned} |\nabla^j u(x)| &\leq C \int_{B_R(x_0)} \frac{V(y)^k |u(y)| |\eta(y)|}{|x - y|^{n-2k+j}} dy \\ &\quad + \frac{C}{R^{n-2k+j+b+2(k-a-b)+2a+b}} \int_{B_R(x_0)} |u(y)| dy \\ &\leq C \sup_{y \in B_R(x_0)} |u(y)| \cdot \int_{B_R(x_0)} \frac{V(y)^k}{|x - y|^{n-2k+j}} dy \\ &\quad + \frac{C}{R^{n+j}} \int_{B_R(x_0)} |u(y)| dy. \end{aligned} \quad (30)$$

It then follows from the well-known theorem on fractional integrals that

$$\begin{aligned} \left(\int_{B_{R/2}(x_0)} |\nabla^j u(x)|^t dx \right)^{1/t} &\leq C \sup_{y \in B_R(x_0)} |u(y)| \left(\int_{B_R(x_0)} V(x)^{q_0} dx \right)^{k/q_0} \end{aligned}$$

$$\begin{aligned} &+ CR^{kn/q_0-2k} \sup_{y \in B_R(x_0)} |u(y)| \\ &\leq CR^{kn/q_0-2k} \{1 + R\rho(x_0, V)\}^{2k} \sup_{y \in B_R(x_0)} |u(y)|, \end{aligned} \quad (31)$$

where $1/t = k/q_0 - (2k - j)/n$ and we have used Remark 4 (1). $\square$

At the end of this section, we give the following.

Proof of Lemma 13. Let l and m be nonnegative integers and choose $\eta \in C_0^\infty(B_R(x_0))$ such that $\eta \equiv 1$ on $B_{R/2}(x_0)$ and $|\nabla^m(\Delta^l \eta)| \leq C/R^{2l+m}$. Multiplying $(-\Delta)^k u + V^k u = 0$ by $((-\Delta)^{k-2a} u) \eta^{k+1}$ and integrating over $\mathbf{R}^n$ by integrating by parts, we have

$$\begin{aligned} &\int_{B_{R/2}(x_0)} |\Delta^{k-a} u(x)|^2 dx \\ &\quad + \int_{B_{R/2}(x_0)} V(x)^k |u(x)| |\Delta^{k-2a} u(x)| dx \\ &\leq C \int_{B_R(x_0)} \sum_{l,m} |\Delta^{k-a} u(x)| |\nabla^m (\Delta^{k-a-l-m} u)(x) \\ &\quad \cdot \nabla^m (\Delta^l \eta^{k+1})(x)| dx, \end{aligned} \quad (32)$$

where the summation is taken over all integers l and m satisfying $l \geq 0$, $m \geq 0$, and $1 \leq l + m \leq a$. Let ϵ be a positive real number which will be determined later. Then the right-hand side of (32) is bounded by

$$\begin{aligned} &C \int_{B_R(x_0)} \sum_{l,m} \sqrt{\epsilon} |\Delta^{k-a} u(x)| \cdot \frac{1}{\sqrt{\epsilon}} \\ &\quad \cdot \frac{1}{R^{2l+m}} |\nabla^m (\Delta^{k-a-l-m} u)(x)| dx \\ &\leq C \int_{B_R(x_0)} \sum_{l,m} \left(\epsilon |\Delta^{k-a} u(x)|^2 + \frac{1}{\epsilon} \cdot \frac{1}{R^{2(2l+m)}} \right. \\ &\quad \left. \times |\nabla^m (\Delta^{k-a-l-m} u)(x)|^2 \right) dx. \end{aligned} \quad (33)$$

Then choosing ϵ such that $C(a-1)(a+2)\epsilon = 1$, we arrive at the desired estimate. $\square$

3. Estimates for H_{2^l}

In this section, we assume that Theorem 9 is true for $m = l$ and show estimates for H_{2^l} which is needed to prove the case $m = l + 1$ in Theorem 9.

Lemma 19. Let j, l , and n be integers, $l \geq 0$, and $n \geq 2^{l+1} + 1$. Suppose that $V \in (RH)_{n/2}$. Assume also that Theorem 9 is true for $m = l$. Then there exists a positive constant C such that

$$\|\rho(\cdot, V)^{2^{l+1}} H_{2^l}^{-1} f\|_{L^p(\mathbf{R}^n)} \leq C \|f\|_{L^p(\mathbf{R}^n)}, \quad (34)$$

where $1 \leq p \leq \infty$.

Proof. We can prove Lemma 19 by the same way as in the proof of [3, Corollary 2.8]. It follows from the inductive assumption that

$$\begin{aligned} & \int_{\mathbf{R}^n} |\Gamma_{H_{2^l}}(x, y)| dy \\ & \leq C \int_{\mathbf{R}^n} \frac{dy}{\{1 + \rho(x, V) |x - y|\}^{2^{l+1}+1} |x - y|^{n-2^{l+1}}} \quad (35) \\ & = \frac{C}{\rho(x, V)^{2^{l+1}}}. \end{aligned}$$

If $f \in L^p(\mathbf{R}^n)$, then

$$u(x) = H_{2^l}^{-1} f(x) = \int_{\mathbf{R}^n} \Gamma_{H_{2^l}}(x, y) f(y) dy. \quad (36)$$

Let $p' = p/(p-1)$. Then using Hölder's inequality and (35) we have

$$\begin{aligned} |u(x)| & \leq \left(\int_{\mathbf{R}^n} |\Gamma_{H_{2^l}}(x, y)| dy \right)^{1/p'} \\ & \quad \times \left(\int_{\mathbf{R}^n} |\Gamma_{H_{2^l}}(x, y)| |f(y)|^p dy \right)^{1/p} \\ & \leq \frac{C}{\rho(x, V)^{2^{l+1}/p'}} \left(\int_{\mathbf{R}^n} |\Gamma_{H_{2^l}}(x, y)| |f(y)|^p dy \right)^{1/p}. \quad (37) \end{aligned}$$

It follows that

$$\begin{aligned} & \int_{\mathbf{R}^n} |\rho(x, V)^{2^{l+1}} u(x)|^p dx \\ & \leq C \int_{\mathbf{R}^n} |f(y)|^p \left(\int_{\mathbf{R}^n} \rho(x, V)^{2^{l+1}} |\Gamma_{H_{2^l}}(x, y)| dx \right) dy. \quad (38) \end{aligned}$$

Finally, note that, by the inductive assumption and Lemma 12 (1),

$$\begin{aligned} & \int_{\mathbf{R}^n} \rho(x, V)^{2^{l+1}} |\Gamma_{H_{2^l}}(x, y)| dx \\ & \leq CC_N \int_{\mathbf{R}^n} \frac{\rho(y, V)^{2^{l+1}}}{\{1 + \rho(y, V) |x - y|\}^{N-2^{l+1}k_0} |x - y|^{n-2^{l+1}}} dx \\ & \leq C \quad (39) \end{aligned}$$

if we choose $N = 2^{l+1}k_0 + 2^{l+1} + 1$. Then Lemma 19 follows. $\square$

We also need the following lemma.

Lemma 20. Let j, l , and n be integers, $l \geq 0$, $n \geq 2^{l+1} + 1$, and $1 \leq j \leq 2^{l+1} - 1$. Suppose that $V \in (RH)_{q_0}$ for some q_0 , $n/2 \leq q_0 < 2^l n / (2^{l+1} - j)$. Assume also that Theorem 9 is true

for $m = l$. Then for $1 \leq p \leq p_0$ there exists a positive constant C such that

$$\|\rho(\cdot, V)^{2^{l+1}-j} \nabla^j H_{2^l}^{-1} f\|_{L^p(\mathbf{R}^n)} \leq C \|f\|_{L^p(\mathbf{R}^n)}, \quad (40)$$

where $1/p_0 = 2^l/q_0 - (2^{l+1} - j)/n$.

Proof. We show Lemma 20 by a method similar to the one used in the proof of [3, Theorem 4.13]. Suppose that $V \in (RH)_{q_0}$ for some $q_0 \geq n/2$. Then $V \in (RH)_{q_1}$ for some q_1 , $2^l n / (2^{l+1} - j) > q_1 > q_0$. Let

$$T_{2^l} f(x) = \rho(x, V)^{2^{l+1}-j} \int_{\mathbf{R}^n} \nabla_y^j \Gamma_{H_{2^l}}(x, y) f(y) dy. \quad (41)$$

The adjoint of T_{2^l} is given by

$$T_{2^l}^* f(x) = \int_{\mathbf{R}^n} \nabla_y^j \Gamma_{H_{2^l}}(y, x) \rho(y, V)^{2^{l+1}-j} f(y) dy. \quad (42)$$

By duality, it suffices to show that

$$\|T_{2^l}^* f\|_{L^p(\mathbf{R}^n)} \leq C \|f\|_{L^p(\mathbf{R}^n)} \quad \text{for } p'_0 \leq p \leq \infty, \quad (43)$$

where $1/p_0 + 1/p'_0 = 1$. Let $r = 1/\rho(x, V)$. We choose t and p_1 such that $1/t = 2^l/q_1 - (2^{l+1} - j)/n$, $1/p_1 = 1 - 2^l/q_1 + (2^{l+1} - j)/n$. Thus $1/t + 1/p_1 = 1$. By Hölder's inequality we have

$$\begin{aligned} & |T_{2^l}^* f(x)| \\ & \leq \sum_{i=-\infty}^{\infty} \int_{2^{i-1}r < |y-x| \leq 2^i r} |\nabla_y^j \Gamma_{H_{2^l}}(y, x)| \\ & \quad \times \rho(y, V)^{2^{l+1}-j} |f(y)| dy \\ & \leq \sum_{i=-\infty}^{\infty} \left(\int_{2^{i-1}r < |y-x| \leq 2^i r} \left\{ |\nabla_y^j \Gamma_{H_{2^l}}(y, x)| \right. \right. \\ & \quad \times \rho(y, V)^{2^{l+1}-j} \Big\}^t dy \Big)^{1/t} \\ & \quad \cdot \left(\int_{|y-x| \leq 2^i r} |f(y)|^{p_1} dy \right)^{1/p_1}. \quad (44) \end{aligned}$$

Let a be a positive integer and $y_a \in \mathbf{R}^n$. It follows from Lemma 18, Lemma 12 (1), and the inductive assumption that

$$\begin{aligned}
& \left(\int_{2^{i-1}r < |y-x| \leq 2^i r} \left\{ \left| \nabla_y^j \Gamma_{H_{2^i}}(y, x) \right| \rho(y, V)^{2^{i+1}-j} \right\}^t dy \right)^{1/t} \\
& \leq \sum_{a=1}^A \left(\int_{\substack{|y_a-x|=3 \cdot 2^{i-2}r \\ |z-y_a| \leq 2^{i-1}r}} \left\{ \left| \nabla_z^j \Gamma_{H_{2^i}}(z, x) \right| \rho(z, V)^{2^{i+1}-j} \right\}^t dz \right)^{1/t} \\
& \leq C(2^{i-1}r)^{2^{i-1}/q_1-2^{i+1}} \left\{ 1 + 2^i r \rho(x, V) \right\}^{2^{i+1}+(2^{i+1}-j)k_0} \\
& \quad \times \rho(x, V)^{2^{i+1}-j} \cdot \sum_{a=1}^A \sup_{z \in B_{5 \cdot 2^{i-3}r}(y_a)} \left| \Gamma_{H_{2^i}}(z, x) \right| \\
& \leq C(2^i r)^{2^{i-1}/q_1-2^{i+1}} (1+2^i)^{2^{i+1}+(2^{i+1}-j)k_0} \cdot \frac{1}{r^{2^{i+1}-j}} \\
& \quad \cdot \frac{C_N}{\{1 + \rho(x, V)|z-x|\}^N} \cdot \sum_{a=1}^A \sup_{z \in B_{5 \cdot 2^{i-3}r}(y_a)} \frac{1}{|z-x|^{n-2^{i+1}}} \\
& \leq CC_N \frac{(2^i r)^{2^{i-1}/q_1-2^{i+1}} (1+2^i)^{2^{i+1}+(2^{i+1}-j)k_0}}{r^{2^{i+1}-j} (1+2^{i-3})^N (2^{i-3}r)^{n-2^{i+1}}} \\
& \leq CC_N \frac{(2^i r)^{2^{i-1}/q_1-n}}{r^{2^{i+1}-j} (1+2^i)^{N-2^{i+1}-(2^{i+1}-j)k_0}}, \tag{45}
\end{aligned}$$

where A is a finite integer not depending on i and r . Thus

$$\begin{aligned}
|T_{2^i}^* f(x)| & \leq CC_N \sum_{i=-\infty}^{\infty} \frac{(2^i)^{2^{i+1}-j}}{(1+2^i)^{N-2^{i+1}-(2^{i+1}-j)k_0}} \\
& \quad \times \left\{ \frac{1}{(2^i r)^n} \int_{B_{2^i r}(x)} |f(y)|^{p_1} dy \right\}^{1/p_1} \\
& \leq C \{M(|f|^{p_1})(x)\}^{1/p_1}, \tag{46}
\end{aligned}$$

where we choose $N \geq 2^{l+2} + (2^{l+1} - j)k_0 - j + 1$ and M is the Hardy-Littlewood maximal operator. Hence it follows that

$$\|T_{2^i}^* f\|_{L^p(\mathbf{R}^n)} \leq C \|f\|_{L^p(\mathbf{R}^n)} \quad \text{for } p_1 < p \leq \infty. \tag{47}$$

Then (43) follows since $p'_0 > p_1$. $\square$

For the case $q_0 = n/2$ in Lemma 20, if $n \geq 2^{l+2} + 1$, letting $p = 2$, we have the following.

Corollary 21. *Let j, l , and n be integers, $l \geq 0$, $n \geq 2^{l+2} + 1$, and $1 \leq j \leq 2^{l+1} - 1$. Suppose that $V \in (RH)_{n/2}$. Assume also that Theorem 9 is true for $m = l$. Then there exists a positive constant C such that*

$$\|\rho(\cdot, V)^{2^{l+1}-j} \nabla^j H_{2^l}^{-1} f\|_{L^2(\mathbf{R}^n)} \leq C \|f\|_{L^2(\mathbf{R}^n)}. \tag{48}$$

4. Proof of Theorem 9 (Case $m = l + 1$)

In this section, we assume that Theorem 9 is true for $m = l$ and prove the case $m = l + 1$ in Theorem 9. It follows easily from the following lemma.

Lemma 22. *Let j, l , and n be integers, $l \geq 0$, and $n \geq 2^{l+2} + 1$. Suppose that $V \in (RH)_{n/2}$. Assume also that $(-\Delta)^{2^{l+1}} u + V^{2^{l+1}} u = 0$ in $B_R(x_0)$ for some $x_0 \in \mathbf{R}^n$ and Theorem 9 is true for $m = l$. Then for any positive integer N there exists a positive constant C_N such that*

$$\sup_{y \in B_{R/2}(x_0)} |u(y)| \leq \frac{C_N}{\{1 + R\rho(x_0, V)\}^N} \sup_{y \in B_R(x_0)} |u(y)|. \tag{49}$$

Assuming Lemma 22 for the moment, we give

Proof of Theorem 9 (Case $m = l + 1$). Fix $x_0, y_0 \in \mathbf{R}^n$ and put $R = |x_0 - y_0|$. Then $u(x) = \Gamma_{H_{2^{l+1}}}(x, y_0)$ is a solution of $(-\Delta)^{2^{l+1}} u + V^{2^{l+1}} u = 0$ on $B_{R/2}(x_0)$. Using the estimate $0 \leq \Gamma_{H_{2^{l+1}}}(x, y) \leq C|x - y|^{2^{l+2}-n}$ and (49), we arrive at the desired estimate. $\square$

To prove Lemma 22, we need some lemmas.

Lemma 23. *Let j, l , and n be integers, $l \geq 0$, $n \geq 2^{l+2} + 1$, and $0 \leq j \leq 2^{l+1} - 1$. Suppose that $V \in (RH)_{n/2}$. Assume also that Theorem 9 is true for $m = l$. Then there exists a positive constant C such that*

$$\begin{aligned}
& \sum_{j=0}^{2^{l+1}-1} \int_{\mathbf{R}^n} \rho(x, V)^{2(2^{l+1}-j)} |\nabla^j u(x)|^2 dx \\
& \leq C \left(\int_{\mathbf{R}^n} |\Delta^{2^l} u(x)|^2 dx + \int_{\mathbf{R}^n} V(x)^{2^{l+1}} |u(x)|^2 dx \right), \tag{50}
\end{aligned}$$

where $u \in C_0^\infty(\mathbf{R}^n)$.

Proof. By the case $p = 2$ in Lemma 19 and Corollary 21, we have

$$\begin{aligned}
& \sum_{j=0}^{2^{l+1}-1} \int_{\mathbf{R}^n} \rho(x, V)^{2(2^{l+1}-j)} |\nabla^j u(x)|^2 dx \\
& \leq C \int_{\mathbf{R}^n} \left| \{(-\Delta)^{2^l} + V^{2^l}\} u(x) \right|^2 dx \\
& \leq C \left(\int_{\mathbf{R}^n} |\Delta^{2^l} u(x)|^2 dx + \int_{\mathbf{R}^n} V(x)^{2^{l+1}} |u(x)|^2 dx \right). \tag{51}
\end{aligned}$$

$\square$

If $n \geq 2^{l+2} + 1$ and $1 \leq j \leq 2^{l+1} - 1$, then letting $k = 2^{l+1}$ and $q_0 = n/2$ in Lemma 18 we have the following.

Corollary 24. Let j, l , and n be integers, $l \geq 0$, $n \geq 2^{l+2} + 1$, and $1 \leq j \leq 2^{l+1} - 1$. Suppose that $V \in (RH)_{n/2}$. Assume also that $(-\Delta)^{2^{l+1}} u + V^{2^{l+1}} u = 0$ in $B_R(x_0)$ for some $x_0 \in \mathbf{R}^n$. Then there exists a positive constant C such that

$$\begin{aligned} & \left(\frac{1}{|B_{R/2}(x_0)|} \int_{B_{R/2}(x_0)} |\nabla^j u(x)|^2 dx \right)^{1/2} \\ & \leq \frac{C\{1 + R\rho(x_0, V)\}^{2^{l+2}}}{R^j} \sup_{y \in B_R(x_0)} |u(y)|. \end{aligned} \quad (52)$$

Now we are ready to give the following.

Proof of Lemma 22. Let $\eta \in C_0^\infty(B_{R/2}(x_0))$ such that $\eta \equiv 1$ on $B_{R/4}(x_0)$, $|\nabla^j \eta| \leq C/R^j$, where j is an integer satisfying $1 \leq j \leq 2^{l+1} - 1$. Applying Lemma 23 to ηu and using Corollary 14 we have

$$\begin{aligned} & \sum_{j=0}^{2^{l+1}-1} \int_{B_{R/4}(x_0)} \rho(x, V)^{2(2^{l+1}-j)} |\nabla^j u(x)|^2 dx \\ & \leq C \sum_{j=0}^{2^{l+1}-1} \frac{1}{R^{2(2^{l+1}-j)}} \int_{B_R(x_0)} |\nabla^j u(x)|^2 dx. \end{aligned} \quad (53)$$

From Lemma 12 (2) it follows for each integer j satisfying $1 \leq j \leq 2^{l+1} - 1$ that

$$\begin{aligned} & \int_{B_{R/4}(x_0)} |\nabla^j u(x)|^2 dx \\ & \leq \frac{C\{1 + R\rho(x_0, V)\}^{2(2^{l+1}-j)k_0/(k_0+1)}}{\{R\rho(x_0, V)\}^{2(2^{l+1}-j)}} \cdot \frac{R^{2^{l+2}}}{R^{2j}} \\ & \cdot \sum_{i=0}^{2^{l+1}-1} \frac{R^{2i}}{R^{2^{l+2}}} \int_{B_R(x_0)} |\nabla^i u(x)|^2 dx. \end{aligned} \quad (54)$$

Then we have

$$\begin{aligned} & \sum_{j=0}^{2^{l+1}-1} R^j \left(\frac{1}{|B_{R/4}(x_0)|} \int_{B_{R/4}(x_0)} |\nabla^j u(x)|^2 dx \right)^{1/2} \\ & \leq \frac{C}{\{1 + R\rho(x_0, V)\}^{1/(k_0+1)}} \sum_{i=0}^{2^{l+1}-1} R^i \\ & \cdot \left(\frac{1}{|B_R(x_0)|} \int_{B_R(x_0)} |\nabla^i u(x)|^2 dx \right)^{1/2}. \end{aligned} \quad (55)$$

Repeating the above argument, for any positive integer K we have

$$\sum_{j=0}^{2^{l+1}-1} R^j \left(\frac{1}{|B_{R/4^K}(x_0)|} \int_{B_{R/4^K}(x_0)} |\nabla^j u(x)|^2 dx \right)^{1/2}$$

$$\begin{aligned} & \leq \frac{C_K}{\{1 + R\rho(x_0, V)\}^{K/(k_0+1)}} \sum_{j=0}^{2^{l+1}-1} R^j \\ & \cdot \left(\frac{1}{|B_R(x_0)|} \int_{B_R(x_0)} |\nabla^j u(x)|^2 dx \right)^{1/2}. \end{aligned} \quad (56)$$

Then using Lemma 17, Corollary 24, and estimate (56), we arrive at the desired estimate. $\square$

5. Proof of Theorem 10

In this section, we prove Theorem 10 which states derivative estimates of the fundamental solution for H_{2^m} . We arrive at Theorem 10 combining Lemmas 25 with 22.

Lemma 25. Let j, m , and n be integers, $m \geq 0$, $n \geq 2^{m+1} + 1$, and $1 \leq j \leq 2^{m+1} - 1$. Suppose that $V \in (RH)_{2^m n/(2^{m+1}-j)}$. Assume also that $(-\Delta)^{2^m} u + V^{2^m} u = 0$ in $B_R(x_0)$ for some $x_0 \in \mathbf{R}^n$. Then there exist positive constants C_j and C'_j such that

$$\sup_{y \in B_{R/2}(x_0)} |\nabla^j u(y)| \leq \frac{C_j \{1 + R\rho(x_0, V)\}^{C'_j}}{R^j} \sup_{y \in B_R(x_0)} |u(y)|. \quad (57)$$

Proof. Let $\eta \in C_0^\infty(B_R(x_0))$ such that $\eta \equiv 1$ on $B_{3R/4}(x_0)$ and $|\nabla^b(\Delta^a \eta)| \leq C/R^{2a+b}$, where a and b are nonnegative integers. We use (29); integrating by parts, we have

$$\begin{aligned} |\nabla^j u(x_0)| & \leq C \int_{B_R(x_0)} \frac{V(y)^{2^m} |u(y)|}{|x_0 - y|^{n-2^{m+1}+j}} dy \\ & + \frac{C}{R^{n+j}} \int_{B_R(x_0)} |u(y)| dy. \end{aligned} \quad (58)$$

Since $V \in (RH)_{2^m n/(2^{m+1}-j)}$, it follows that $V \in (RH)_q$ for some $q > 2^m n/(2^{m+1} - j)$. We choose r such that $2^m/q + 1/r = 1$ and $r > 1$. By Hölder's inequality we have

$$\begin{aligned} & |\nabla^j u(x_0)| \\ & \leq CR^n \left(\frac{1}{R^n} \int_{B_R(x_0)} V(y)^q dy \right)^{2^m/q} \\ & \cdot \left(\frac{1}{R^n} \int_{B_R(x_0)} \frac{dy}{|x_0 - y|^{(n-2^{m+1}+j)r}} \right)^{1/r} \sup_{y \in B_R(x_0)} |u(y)| \\ & + \frac{C}{R^{n+j}} \int_{B_R(x_0)} |u(y)| dy \\ & \leq \frac{C\{1 + R\rho(x_0, V)\}^{2^{m+1}}}{R^j} \sup_{y \in B_R(x_0)} |u(y)|, \end{aligned} \quad (59)$$

where we have used Remark 4 (1). From (59) we have, for all $y \in B_{R/2}(x_0)$,

$$|\nabla^j u(y)| \leq \frac{C\{1 + R\rho(y, V)\}^{2^{m+1}}}{R^j} \sup_{x \in B_{R/4}(y)} |u(x)|. \quad (60)$$

Using Lemma 12 (1), we have

$$\begin{aligned} \sup_{y \in B_{R/2}(x_0)} |\nabla^j u(y)| \\ \leq \frac{C\{1 + R\rho(x_0, V)\}^{2^{m+1}(k_0+1)}}{R^j} \sup_{y \in B_R(x_0)} |u(y)|. \end{aligned} \quad (61)$$

Then the proof is complete. $\square$

6. Proofs of Theorems 5 and 8

In this section, we prove Theorems 5 and 8.

Proof of Theorem 5. We assume that $V \in (RH)_{q_0}$ for some $q_0 \geq 2^m n / (2^{m+1} - j)$. Then $V \in (RH)_{q_1}$ for some $q_1 > q_0$. Let

$$T_{2^m} f(x) = V(x)^{(2^{m+1}-j)/2} \int_{\mathbb{R}^n} \nabla_x^j \Gamma_{H_{2^m}}(x, y) f(y) dy. \quad (62)$$

The adjoint of T_{2^m} is given by

$$T_{2^m}^* f(x) = \int_{\mathbb{R}^n} \nabla_y^j \Gamma_{H_{2^m}}(y, x) V(y)^{(2^{m+1}-j)/2} f(y) dy. \quad (63)$$

Let $p_0 = 2q_0 / (2^{m+1} - j)$. By duality, it suffices to show that

$$\|T_{2^m}^* f\|_{L^p(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)} \quad \text{for } p'_0 \leq p \leq \infty, \quad (64)$$

where $1/p_0 + 1/p'_0 = 1$. Let $r = 1/\rho(x, V)$. By Theorems 9 and 10 and Hölder's inequality we have

$$\begin{aligned} |T_{2^m}^* f(x)| \\ \leq \sum_{i=-\infty}^{\infty} \int_{2^{i-1}r < |y-x| \leq 2^i r} |\nabla_y^j \Gamma_{H_{2^m}}(y, x)| \\ \times V(y)^{(2^{m+1}-j)/2} |f(y)| dy \\ \leq CC_N \sum_{i=-\infty}^{\infty} \frac{1}{(1+2^i)^N} \cdot \frac{1}{(2^i r)^{n-2^{m+1}+j}} \\ \times \int_{B_{2^i r}(x)} V(y)^{(2^{m+1}-j)/2} |f(y)| dy \\ \leq CC_N \sum_{i=-\infty}^{\infty} \frac{(2^i r)^{2^{m+1}-j}}{(1+2^i)^N} \\ \times \left\{ \frac{1}{(2^i r)^n} \int_{B_{2^i r}(x)} V(y)^{q_1} dy \right\}^{(2^{m+1}-j)/(2q_1)} \\ \cdot \left\{ \frac{1}{(2^i r)^n} \int_{B_{2^i r}(x)} |f(y)|^{p_1} dy \right\}^{1/p_1}, \end{aligned} \quad (65)$$

where $1/p_1 = 1 - (2^{m+1} - j)/(2q_1)$. Since $V \in (RH)_{q_1}$ we have

$$\begin{aligned} |T_{2^m}^* f(x)| \\ \leq CC_N \{M(|f|^{p_1})(x)\}^{1/p_1} \\ \cdot \sum_{i=-\infty}^{\infty} \frac{1}{(1+2^i)^N} \left\{ \frac{1}{(2^i r)^{n-2}} \int_{B_{2^i r}(x)} V(y) dy \right\}^{(2^{m+1}-j)/2}. \end{aligned} \quad (66)$$

For the case $j \leq 0$, by Lemma 11 and (4), we have

$$\begin{aligned} \frac{1}{(2^i r)^{n-2}} \int_{B_{2^i r}(x)} V(y) dy \\ \leq C \left(\frac{r}{2^i r} \right)^{n/q_1-2} \cdot \frac{1}{r^{n-2}} \int_{B_r(x)} V(y) dy \\ = C(2^i)^{2-n/q_1}. \end{aligned} \quad (67)$$

For the case $j \geq 1$, by the doubling condition (9) and (4), we have

$$\begin{aligned} \frac{1}{(2^i r)^{n-2}} \int_{B_{2^i r}(x)} V(y) dy \\ \leq 2^{2i} \cdot 2^{-in} C_0^i \cdot \frac{1}{r^{n-2}} \int_{B_r(x)} V(y) dy \\ \leq C(2^i)^{k_1}, \end{aligned} \quad (68)$$

where $k_1 = 2 - n + \log_2 C_0$. Hence, choosing $N \geq k_1(2^{m+1} - j)/2 + 1$, we obtain

$$|T_{2^m}^* f(x)| \leq C \{M(|f|^{p_1})(x)\}^{1/p_1}. \quad (69)$$

It follows that

$$\|T_{2^m}^* f\|_{L^p(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)} \quad \text{for } p_1 < p \leq \infty. \quad (70)$$

Then (64) follows since $p_1 = \{2q_1/(2^{m+1} - j)\}' < \{2q_0/(2^{m+1} - j)\}' = p'_0$. $\square$

Proof of Theorem 8. We assume that $V \in (RH)_{q_0}$ for some q_0 , $n/2 \leq q_0 < 2^m n / (2^{m+1} - j)$. Then $V \in (RH)_{q_1}$ for some q_1 , $2^m n / (2^{m+1} - j) > q_1 > q_0$. We use (62) and (63) again and it suffices to show (64) for $1/p_0 = (2^{m+2} - j)/(2q_0) - (2^{m+1} - j)/n$ and $1/p_0 + 1/p'_0 = 1$. We choose t and p_1 such that $1/t = 2^m/q_1 - (2^{m+1} - j)/n$ and $1/p_1 = 1 - (2^{m+2} - j)/(2q_1) + (2^{m+1} - j)/n$. Thus,

$$\frac{1}{t} + \frac{2^{m+1} - j}{2q_1} + \frac{1}{p_1} = 1. \quad (71)$$

Let $r = 1/\rho(x, V)$. By Hölder's inequality we have

$$\begin{aligned} |T_{2^m}^* f(x)| &\leq \sum_{i=-\infty}^{\infty} \left(\int_{2^{i-1}r < |y-x| \leq 2^i r} |\nabla_y^j \Gamma_{H_2^m}(y, x)|^t dy \right)^{1/t} \\ &\quad \cdot \left(\int_{|y-x| \leq 2^i r} V(y)^{q_1} dy \right)^{(2^{m+1}-j)/(2q_1)} \\ &\quad \times \left(\int_{|y-x| \leq 2^i r} |f(y)|^{p_1} dy \right)^{1/p_1}. \end{aligned} \quad (72)$$

It follows from Lemma 18 and Theorem 9 that

$$\begin{aligned} &\left(\int_{2^{i-1}r < |y-x| \leq 2^i r} |\nabla_y^j \Gamma_{H_2^m}(y, x)|^t dy \right)^{1/t} \\ &\leq CC_N \frac{(2^i r)^{2^m n/q_1 - n}}{(1 + 2^i)^N}. \end{aligned} \quad (73)$$

Thus, as in the proof of Theorem 5, we have

$$\begin{aligned} |T_{2^m}^* f(x)| &\leq CC_N \{M(|f|^{p_1})(x)\}^{1/p_1} \\ &\quad \cdot \sum_{i=-\infty}^{\infty} \frac{1}{(1 + 2^i)^N} \left\{ \frac{1}{(2^i r)^{n-2}} \int_{B_{2^i r}(x)} V(y) dy \right\}^{(2^{m+1}-j)/2} \\ &\leq C \{M(|f|^{p_1})(x)\}^{1/p_1}. \end{aligned} \quad (74)$$

Then (64) follows since $p'_0 > p_1$. $\square$

7. Remarks

We show estimate (57) under another assumption.

Lemma 26. Let k, l , and n be integers, $m \geq 0$, $n \geq 2^{m+1} + 1$, and $1 \leq j \leq 2^{m+1} - 1$. Suppose that $V \in (RH)_{n/2}$ and there exists a positive constant C such that $V(x) \leq C\rho(x, V)^2$. Assume also that $(-\Delta)^{2^m} u + V^{2^m} u = 0$ in $B_R(x_0)$ for some $x_0 \in \mathbb{R}^n$. Then there exist constants C_j and C'_j such that

$$\begin{aligned} &\sup_{y \in B_{R/2}(x_0)} |\nabla^j u(y)| \\ &\leq \frac{C_j \{1 + R\rho(x_0, V)\}^{C'_j}}{R^j} \sup_{y \in B_R(x_0)} |u(y)|. \end{aligned} \quad (75)$$

Proof. We use (58). From Lemma 12 (1) we have

$$\begin{aligned} |\nabla^j u(x_0)| &\leq C \{1 + R\rho(x_0, V)\}^{2^{m+1}k_0} \\ &\quad \times \rho(x_0, V)^{2^{m+1}} R^{2^{m+1}-j} \sup_{y \in B_R(x_0)} |u(y)| \\ &\quad + \frac{C}{R^j} \left(\frac{1}{R^n} \int_{B_R(x_0)} |u(x)|^2 dx \right)^{1/2} \\ &\leq \frac{C \{1 + R\rho(x_0, V)\}^{2^{m+1}(k_0+1)}}{R^j} \sup_{y \in B_R(x_0)} |u(y)|. \end{aligned} \quad (76)$$

Then by the same argument as in the proof of Lemma 25, we arrive at the desired estimate. $\square$

The following theorem immediately follows from Lemmas 22 and 26.

Theorem 27. Let m and n be integers, $m \geq 0$, $n \geq 2^{m+1} + 1$, and $1 \leq j \leq 2^{m+1} + 1$. Suppose $V \in (RH)_{n/2}$ and there exists a positive constant C such that $V(x) \leq C\rho(x, V)^2$. Then for any positive integer N there exists a constant C_N such that

$$|\nabla_x^j \Gamma_{H_2^m}(x, y)| \leq \frac{C_N}{\{1 + \rho(x, V)|x - y|\}^N} \cdot \frac{1}{|x - y|^{n-2^{m+1}+j}}. \quad (77)$$

Using Theorem 27 we can prove the following theorem by the same way as the proof of Lemma 19.

Theorem 28. Let j, m , and n be integers, $m \geq 0$, $n \geq 2^{m+1} + 1$, and $0 \leq j \leq 2^{m+1} - 1$. Suppose that $V \in (RH)_{n/2}$ and there exists a positive constant C such that $V(x) \leq C\rho(x, V)^2$. Then there exist positive constants C_j such that (6) holds, where $1 \leq p \leq \infty$.

We obtain the following pointwise estimate stated with the Hardy-Littlewood maximal operator.

Theorem 29. Let j, m , and n be integers, $m \geq 0$, $n \geq 2^{m+1} + 1$, and $0 \leq j \leq 2^{m+1} - 1$. Suppose that $V \in (RH)_{n/2}$ and there exists a positive constant C such that $V(x) \leq C\rho(x, V)^2$. Then there exists a constant C' such that

$$|V^{2^m-j/2} \nabla^j H_2^{-1} f(x)| \leq C' Mf(x). \quad (78)$$

In Theorem 29, the cases $m = 0$ and $m = 1$ were shown in [5, Theorem 1.3] and [8, Lemma 10], respectively.

Proof of Theorem 29. Let $r = 1/\rho(x, V)$. Since $V(x) \leq C\rho(x, V)^2$, it follows from Theorems 9 and 27 that

$$\begin{aligned}
 & \left| V(x)^{2^m-j/2} \nabla^j H_{2^m}^{-1} f(x) \right| \\
 & \leq CC_N \int_{\mathbb{R}^n} \frac{\rho(x, V)^{2^{m+1}-j} |f(y)|}{\{1 + \rho(x, V) |x - y|\}^N |x - y|^{n-2^{m+1}+j}} dy \\
 & \leq CC_N \sum_{i=-\infty}^{\infty} \int_{2^{i-1}r < |x-y| \leq 2^i r} \frac{|f(y)| dy}{r^{2^{m+1}-j} (1 + r^{-1} |x - y|)^N |x - y|^{n-2^{m+1}+j}} \\
 & \leq CC_N \sum_{i=-\infty}^{\infty} \frac{(2^{i-1})^{2^{m+1}-j}}{(1 + 2^{i-1})^N} \\
 & \quad \cdot \frac{1}{(2^{i-1}r)^n} \int_{|x-y| \leq 2^i r} |f(y)| dy \\
 & \leq CC_N \sum_{i=-\infty}^{\infty} \frac{(2^i)^{2^{m+1}-j}}{(1 + 2^i)^N} Mf(x).
 \end{aligned} \tag{79}$$

Then choosing $N \geq 2^{m+1} - j + 1$ we arrive at the desired estimate. $\square$

Remark 30 (see [5, Remark 1.6]). Once we obtain the estimate (78), we can obtain boundedness of the operator $V^{2^m-j/2} \nabla^j H_{2^m}^{-1}$ on any space on which M is bounded, for example, Morrey spaces (see [12]).

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The author is grateful to the referee(s) for the helpful comment. The author is also grateful to Professor Kazuhiro Kurata, who read the first version of this paper and gave useful comment. This work was supported by Grant-in-Aid for Scientific Research (C) (no. 25400152), Japan Society for the Promotion of Science.

References

- [1] J. Zhong, *Harmonic analysis for some Schrödinger type operators* [Ph.D. thesis], Princeton University, 1993.
- [2] Y. Liu and J.-Z. Huang, " L^p estimates for the Schrödinger type operators," *Applied Mathematics-A Journal of Chinese Universities*, vol. 26, no. 4, pp. 412–424, 2011.
- [3] Z. W. Shen, " L^p estimates for Schrödinger operators with certain potentials," *Annales de l'Institut Fourier*, vol. 45, no. 2, pp. 513–546, 1995.

- [4] J. Dziubański and J. Zienkiewicz, "Hardy space H^1 associated to Schrödinger operator with potential satisfying reverse Hölder inequality," *Revista Matemática Iberoamericana*, vol. 15, no. 2, pp. 279–296, 1999.
- [5] K. Kurata and S. Sugano, "A remark on estimates for uniformly elliptic operators on weighted L^p spaces and Morrey spaces," *Mathematische Nachrichten*, vol. 209, pp. 137–150, 2000.
- [6] K. Kurata and S. Sugano, "Estimates of the fundamental solution for magnetic Schrödinger operators and their applications," *The Tohoku Mathematical Journal*, vol. 52, no. 3, pp. 367–382, 2000.
- [7] S. Sugano, "Estimates for the operators $V^\alpha(-\Delta + V)^{-\beta}$ and $V^\alpha \nabla(-\Delta + V)^{-\beta}$ with certain nonnegative potentials V ," *Tokyo Journal of Mathematics*, vol. 21, no. 2, pp. 441–452, 1998.
- [8] S. Sugano, " L^p estimates for some Schrödinger type operators and a Calderón-Zygmund operator of Schrödinger type," *Tokyo Journal of Mathematics*, vol. 30, no. 1, pp. 179–197, 2007.
- [9] Y. Liu and J. Dong, "Some estimates of higher order Riesz transform related to Schrödinger type operators," *Potential Analysis*, vol. 32, no. 1, pp. 41–55, 2010.
- [10] C. L. Fefferman, "The uncertainty principle," *Bulletin of the American Mathematical Society*, vol. 9, no. 2, pp. 129–206, 1983.
- [11] F. W. Gehring, "The L^p -integrability of the partial derivatives of a quasiconformal mapping," *Acta Mathematica*, vol. 130, pp. 265–277, 1973.
- [12] F. Chiarenza and M. Frasca, "Morrey spaces and Hardy-Littlewood maximal function," *Rendiconti di Matematica e delle sue Applicazioni*, vol. 7, no. 3-4, pp. 273–279, 1987.