Research Article

Evaluation Formulas for Generalized Conditional Wiener Integrals with Drift on a Function Space

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Let $C[0, t]$ denote a generalized Wiener space, the space of real-valued continuous functions on the interval $[0, t]$ and define a stochastic process $Y : C[0, t] \times [0, t] \rightarrow \mathbb{R}$ by $Y(x, s) = \int_0^s h(u)dx(u) + a(s)$ for $x \in C[0, t]$ and $s \in [0, t]$, where $h \in L_2[0, t]$ with $h \neq 0$ a.e. and a is continuous on $[0, t]$. Let random vectors $Y_n : C[0, t] \rightarrow \mathbb{R}^n$ and $Y_{n+1} : C[0, t] \rightarrow \mathbb{R}^{n+1}$ be given by $Y_n(x) = (Y(x, t_1), \dots, Y(x, t_n))$ and $Y_{n+1}(x) = (Y(x, t_1), \dots, Y(x, t_n), Y(x, t_{n+1}))$, where $0 < t_1 < \dots < t_n < t_{n+1} = t$ is a partition of $[0, t]$. In this paper we derive a translation theorem for a generalized Wiener integral and then prove that Y is a generalized Brownian motion process with drift a . Furthermore, we derive two simple formulas for generalized conditional Wiener integrals of functions on $C[0, t]$ with the drift and the conditioning functions Y_n and Y_{n+1} . As applications of these simple formulas, we evaluate the generalized conditional Wiener integrals of various functions on $C[0, t]$.

1. Introduction

Let $C_0[0, t]$ denote the Wiener space, the space of real-valued continuous functions x on $[0, t]$ with $x(0) = 0$. On the space, Yeh [1] introduced an inversion formula that a conditional expectation can be found by a Fourier-transform. But Yeh's inversion formula is very complicated in its application when the conditioning function is vector-valued. In [2], Park and Skoug derived a simple formula for conditional Wiener integrals on $C_0[0, t]$ with a vector-valued conditioning function $X : C_0[0, t] \rightarrow \mathbb{R}^n$ given by $X(x) = (x(t_1), \dots, x(t_{n+1}))$, where $0 = t_0 < t_1 < \dots < t_n < t_{n+1} = t$ is a partition of the interval $[0, t]$. In their simple formula, they expressed the conditional Wiener integral directly in terms of an ordinary Wiener integral. Using the simple formula in [2], Chang and Skoug [3] investigated the effect that drift has on the conditional Fourier-Feynman transform, the conditional convolution product, and various relationships that occur between them.

On the other hand, let $C[0, t]$ denote the space of real-valued continuous functions on the interval $[0, t]$. Im and Ryu [4] introduced a probability measure w_φ on $C[0, t]$, where φ is

a probability measure on the Borel class of $\mathbb{R}$. When $\varphi = \delta_0$, the Dirac measure concentrated at 0, w_φ is exactly the Wiener measure on $C_0[0, t]$. On the space $C[0, t]$, the author [5, 6] derived two simple formulas for the conditional Wiener w_φ -integral of functions on $C[0, t]$ with the vector-valued conditioning functions $X_n : C[0, t] \rightarrow \mathbb{R}^{n+1}$ and $X_{n+1} : C[0, t] \rightarrow \mathbb{R}^{n+2}$ given by $X_n(x) = (x(t_0), x(t_1), \dots, x(t_n))$ and $X_{n+1}(x) = (x(t_0), x(t_1), \dots, x(t_n), x(t_{n+1}))$ which generalize the Park and Skoug's formula in [2]. Using these formulas with the conditioning functions X_n and X_{n+1} , he evaluated the conditional Wiener w_φ -integral of function of the form $F_m(x) = \int_0^t (x(s))^m ds$ for any positive integer m .

Let $h \in L_2[0, t]$ with $h \neq 0$ a.e. on $[0, t]$, and let a be a continuous function on $[0, t]$. Define a stochastic process $Y : C[0, t] \times [0, t] \rightarrow \mathbb{R}$ by $Y(x, s) = \int_0^s h(u)dx(u) + a(s)$ for $x \in C[0, t]$ and $s \in [0, t]$. Let $Y_n : C[0, t] \rightarrow \mathbb{R}^n$ and $Y_{n+1} : C[0, t] \rightarrow \mathbb{R}^{n+1}$ be given by

$$Y_n(x) = (Y(x, t_1), \dots, Y(x, t_n)), \quad (1)$$

$$Y_{n+1}(x) = (Y(x, t_1), \dots, Y(x, t_n), Y(x, t_{n+1})).$$

In this paper, we derive a translation theorem for a generalized Wiener w_φ -integral, and then prove that Y is a generalized Brownian motion process with drift a and variance $\int_0^s (h(u))^2 du$ for $s \in [0, t]$. Furthermore, we derive two simple formulas for generalized conditional Wiener integrals of functions on $C[0, t]$ with the drift and the conditioning functions Y_n and Y_{n+1} . As applications of these simple formulas, we evaluate the generalized conditional Wiener integrals of functions of the forms

$$F_m(Y(x, \cdot)), \quad Y(x, s_1)Y(x, s_2), \quad (2)$$

$$\int_{L_2[0, t]} \exp\{i(v, Y(x, \cdot))\} d\sigma(v)$$

for $x \in C[0, t]$, where $0 < s_1 < s_2 \leq t$ and σ is a complex Borel measure on $L_2[0, t]$.

2. A Generalized Brownian Motion Process with Drift

In this section, we introduce a generalized Brownian motion process with drift which generalizes the generalized Wiener space $C_{a,b}[0, T]$ as given in [7].

Let $\mathbb{C}$ and $\mathbb{C}_+$ denote the sets of complex numbers and complex numbers with positive real parts, respectively. Let $(C[0, t], \mathcal{B}(C[0, t]), w_\varphi)$ be the analogue of Wiener space associated with a probability measure φ on the Borel class of $\mathbb{R}$, where $\mathcal{B}(C[0, t])$ denotes the Borel class of $C[0, t]$ [4]. Let $\alpha \in \mathbb{R}$, and let φ_α be a measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that $\varphi_\alpha(B) = \varphi(B + \alpha)$ for $B \in \mathcal{B}(\mathbb{R})$. For v in $L_2[0, t]$ and x in $C[0, t]$, let $(v, x) = \int_0^t v(s)dx(s)$ denote the Paley-Wiener-Zygmund integral of v according to x [4]. Let $h \in L_2[0, t]$ with $h \neq 0$ a.e. on $[0, t]$, and let a be a continuous function on $[0, t]$. Define stochastic processes $Z, Y : C[0, t] \times [0, t] \rightarrow \mathbb{R}$ by $Z(x, s) = \int_0^s h(u)dx(u)$ and $Y(x, s) = \int_0^s h(u)dx(u) + a(s)$ for $x \in C[0, t]$ and $s \in [0, t]$. Let $b(s) = \int_0^s (h(u))^2 du$ for $0 \leq s \leq t$.

By (3.1) in [4, Theorem 3.1], we have the following translation theorem for a generalized Wiener integral on the analogue of Wiener space.

Theorem 1. *Let g be of bounded variation on $[0, t]$, let $g_0(s) = \int_0^s h(u)g(u)du$ for $s \in [0, t]$, and let F be a measurable function on $C[0, t]$. Then, $F(Y(x, \cdot) + g_0)$ is also measurable function of x on $C[0, t]$ and*

$$\int_{C[0, t]} F(Y(x, \cdot)) dw_\varphi(x) = e^{-\|g\|^2/2} \int_{C[0, t]} F(Y(x, \cdot) + g_0) \exp\{-(g, x)\} dw_{\varphi_\alpha}(x), \quad (3)$$

where $\stackrel{*}{=}$ means that if one side exists then both sides exist, and they are equal.

Letting $g = \chi_{[0, s]}h$, we have the following corollary by Theorem 1.

Corollary 2. *Let h be of bounded variation on $[0, t]$ and F a measurable function on $C[0, t]$. Then, for $s \in [0, t]$ $F(Y(x, \cdot) + b(\min\{s, \cdot\}))$ is also measurable on $C[0, t]$ and*

$$\int_{C[0, t]} F(Y(x, \cdot)) dw_\varphi(x) = \exp\left\{-\frac{1}{2}b(s)\right\} \int_{C[0, t]} F(Y(x, \cdot) + b(\min\{s, \cdot\})) \times \exp\{-Z(x, s)\} dw_{\varphi_\alpha}(x). \quad (4)$$

Letting $F \equiv 1$ in Theorem 1 and Corollary 2, we have the following corollary.

Corollary 3. *Under the assumptions as given in Theorem 1 and Corollary 2*

$$\int_{C[0, t]} \exp\{-(g, x)\} dw_{\varphi_\alpha}(x) = \exp\left\{\frac{1}{2}\|g\|^2\right\}. \quad (5)$$

In particular for $s \in [0, t]$

$$\int_{C[0, t]} \exp\{-Z(x, s)\} dw_{\varphi_\alpha}(x) = \exp\left\{\frac{1}{2}b(s)\right\}. \quad (6)$$

Using the same method as used in the proof of Corollary 3.3 in [4], we have the following corollary.

Corollary 4. *Under the assumptions as given in Theorem 1, one has for $\lambda \in \mathbb{C}$*

$$\int_{C[0, t]} \exp\{-\lambda(g, x)\} dw_{\varphi_\alpha}(x) = \exp\left\{\frac{\lambda^2}{2}\|g\|^2\right\}. \quad (7)$$

In particular for $s \in [0, t]$

$$\int_{C[0, t]} \exp\{-\lambda Z(x, s)\} dw_{\varphi_\alpha}(x) = \exp\left\{\frac{\lambda^2}{2}b(s)\right\}. \quad (8)$$

Letting $\lambda = -i\xi$ ($\xi \in \mathbb{R}$) in the equations of Corollary 4, we have the following corollary by Fourier transforms on $(C[0, t], w_{\varphi_\alpha})$.

Corollary 5. *Under the assumptions as given in Theorem 1, let $X(x) = (g, x)$. Then, X is normally distributed with mean zero and variance $\|g\|^2$ on $(C[0, t], w_{\varphi_\alpha})$ if $\|g\| > 0$. Moreover, for $s \in (0, t]$ $Y(\cdot, s)$ is normally distributed with mean $a(s)$ and variance $b(s)$ on $(C[0, t], w_{\varphi_\alpha})$. In particular, $Z(\cdot, s)$ is normally distributed on $(C[0, t], w_{\varphi_\alpha})$ with mean 0 and variance $b(s)$.*

Using the same method as used in the proof of Theorem 3.5 in [4], we have the following theorem.

Theorem 6. *Let $\{g_1, \dots, g_m\}$ be an orthonormal subset of $L_2[0, t]$ such that each g_j is of bounded variation. For $j = 1, \dots, m$, let $X_j(x) = (g_j, x)$. Then, $X_1, \dots, X_m$ are independent,*

and each X_j has the standard normal distribution. Moreover if $f : \mathbb{R}^m \rightarrow \mathbb{C}$ is Borel measurable, then

$$\begin{aligned} & \int_{C[0,t]} f(X_1(x), \dots, X_m(x)) dw_{\varphi_\alpha}(x) \\ &= \left(\frac{1}{2\pi}\right)^{m/2} \int_{\mathbb{R}^m} f(u_1, u_2, \dots, u_m) \\ & \times \exp\left\{-\frac{1}{2} \sum_{j=1}^m u_j^2\right\} d(u_1, u_2, \dots, u_m). \end{aligned} \quad (9)$$

Theorem 7. Let h be of bounded variation on $[0, t]$, and let $0 = s_0 < s_1 < \dots < s_m \leq t$. Then, the random vector $(Y(\cdot, s_1), \dots, Y(\cdot, s_m))$ has a joint density function W_m with respect to w_{φ_α} given by

$$\begin{aligned} & W_m(s_1, \dots, s_m; u_1, \dots, u_m) \\ &= \prod_{j=1}^m \left[\frac{1}{2\pi(b(s_j) - b(s_{j-1}))} \right]^{1/2} \\ & \times \exp\left\{-\sum_{j=1}^m \frac{(u_j - a(s_j) - (u_{j-1} - a(s_{j-1})))^2}{2(b(s_j) - b(s_{j-1}))}\right\}, \end{aligned} \quad (10)$$

where $u_0 = 0$, $a(0) = 0$, and $(u_1, \dots, u_m) \in \mathbb{R}^m$.

Proof. For $j = 1, \dots, m$, let $h_j = \chi_{[s_{j-1}, s_j]} h$, where $\chi_{[s_{j-1}, s_j]}$ denotes the indicator function on $[s_{j-1}, s_j]$. Let $g_j = (1/\sqrt{b(s_j) - b(s_{j-1})})h_j$. Then, $g_1, \dots, g_m$ are orthonormal, and $Y(\cdot, s_j) = \sum_{l=1}^j (h_l, \cdot) + a(s_j) = \sum_{l=1}^j \sqrt{b(s_l) - b(s_{l-1})}(g_l, \cdot) + a(s_j)$. Let B be a Borel subset of $\mathbb{R}^m$. By Theorem 6,

$$\begin{aligned} & w_{\varphi_\alpha}((Y(\cdot, s_1), \dots, Y(\cdot, s_m)) \in B) \\ &= \left(\frac{1}{2\pi}\right)^{m/2} \int_{\mathbb{R}^m} \chi_B\left(\sqrt{b(s_1) - b(s_0)}w_1 + a(s_1), \dots, \right. \\ & \quad \left. \sum_{l=1}^m \sqrt{b(s_l) - b(s_{l-1})}w_l + a(s_m)\right) \\ & \times \exp\left\{-\frac{1}{2} \sum_{j=1}^m w_j^2\right\} d(w_1, \dots, w_m). \end{aligned} \quad (11)$$

For $j = 1, \dots, m$, let $u_j = \sum_{l=1}^j \sqrt{b(s_l) - b(s_{l-1})}w_l + a(s_j)$. Then, by the change of variable theorem,

$$\begin{aligned} & w_{\varphi_\alpha}((Y(\cdot, s_1), \dots, Y(\cdot, s_m)) \in B) \\ &= \int_B w_m(s_1, \dots, s_m; u_1, \dots, u_m) d(u_1, \dots, u_m) \end{aligned} \quad (12)$$

which completes the proof. $\square$

Remark 8. The condition $a(0) = 0$ in Theorem 7 does not mean that the mean function a is continuous on $[0, t]$ with

$a(0) = 0$. In fact, it is only needed to express formally the exponential function $\exp\{-(u_1 - a(s_1))^2/(2b(s_1))\}$ in W_m .

By Theorem 7 and the change of variable theorem, we have the following corollary.

Corollary 9. Y is a Gaussian process with respect to w_{φ_α} , and its covariance function is given by

$$\begin{aligned} & E[(Y(\cdot, s_1) - a(s_1))(Y(\cdot, s_2) - a(s_2))] \\ &= b(\min\{s_1, s_2\}) \quad \text{for } s_1, s_2 \in [0, t], \end{aligned} \quad (13)$$

so that

$$E[Y(\cdot, s_1)Y(\cdot, s_2)] = b(\min\{s_1, s_2\}) + a(s_1)a(s_2). \quad (14)$$

Using Fourier transforms on $(C[0, t], w_{\varphi_\alpha})$, we have the following corollary by Theorem 7.

Corollary 10. Let $0 \leq s_1 < s_2 \leq t$. Then, $Y(\cdot, s_2) - Y(\cdot, s_1)$ is normally distributed on $(C[0, t], w_{\varphi_\alpha})$ with mean $a(s_2) - a(s_1)$ and variance $b(s_2) - b(s_1)$.

By Corollary 10, we can prove the following corollary.

Corollary 11. Let $0 \leq s_1 < s_2 \leq s_3 < s_4 \leq t$. Then $Y(\cdot, s_2) - Y(\cdot, s_1)$ and $Y(\cdot, s_4) - Y(\cdot, s_3)$ are independent on $(C[0, t], w_{\varphi_\alpha})$.

Theorem 12. Y is a generalized Brownian motion process on $(C[0, t], w_{\varphi_\alpha})$ determined by the mean function a and variance function b .

Remark 13. (1) A generalized Brownian motion process determined by the mean function a and variance function b is a Gaussian process whose mean and covariance functions are given by a and $b(\min\{s_1, s_2\}) - b(0)$, respectively, for $s_1, s_2 \in [0, t]$ [1].

(2) In [8], the author proved that $Y - a$ is a generalized Brownian motion process using Theorem 6. Comparing with [8], we proved the results in this paper using only the translation theorem [4, Theorem 3.1] which is another approach to prove Y being a generalized Brownian motion process.

3. Simple Formulas for Generalized Conditional Wiener Integrals

In this section, we derive two simple formulas for generalized conditional Wiener integrals on $(C[0, t], w_{\varphi_\alpha})$. For this purpose, we start with this section defining a conditional w_{φ_α} -integral.

Let $F : C[0, t] \rightarrow \mathbb{C}$ be integrable on w_{φ_α} and X a random vector on $C[0, t]$ assuming that the value space of X is a normed space equipped with the Borel σ -algebra. Then, we have the conditional expectation $E[F | X]$ of F given X from a well-known probability theory. Furthermore, there exists a P_X -integrable $\mathbb{C}$ -valued function ψ on the value space of X such that $E[F | X](x) = (\psi \circ X)(x)$ for w_{φ_α} a.e. $x \in C[0, t]$, where P_X is the probability distribution of X . The function ψ is called the conditional w_{φ_α} -integral of F given X , and it is

also denoted by $E[F | X]$. Let $0 = t_0 < t_1 < \dots < t_n < t_{n+1} = t$ be a partition of $[0, t]$. For $j = 1, \dots, n+1$, let $\alpha_j(s) = (b(t_j) - b(s))/(b(t_j) - b(t_{j-1}))$, $\beta_j(s) = (b(s) - b(t_{j-1}))/b(t_j) - b(t_{j-1})$ and $\gamma_j(s) = \alpha_j(s)\beta_j(s)(b(t_j) - b(t_{j-1}))$, where $s \in [t_{j-1}, t_j]$. Define random vectors $Y_n : C[0, t] \rightarrow \mathbb{R}^n$ and $Y_{n+1} : C[0, t] \rightarrow \mathbb{R}^{n+1}$ by $Y_n(x) = (Y(x, t_1), \dots, Y(x, t_n))$ and $Y_{n+1}(x) = (Y(x, t_1), \dots, Y(x, t_n), Y(x, t_{n+1}))$ for $x \in C[0, t]$. For any x in $C[0, t]$, define a polygonal function $[Y(x, \cdot)]_{a,b}$ of $Y(x, \cdot)$ by

$$\begin{aligned} [Y(x, \cdot)]_{a,b}(s) &= \sum_{j=1}^{n+1} \chi_{(t_{j-1}, t_j]}(s) (\alpha_j(s) Y(x, t_{j-1}) + \beta_j(s) Y(x, t_j)) \\ &\quad + \chi_{\{0\}}(s) a(0) \end{aligned} \quad (15)$$

for $s \in [0, t]$, where $\chi_{(t_{j-1}, t_j]}$ and $\chi_{\{0\}}$ denote indicator functions. For $x \in C[0, t]$ and $\vec{\xi}_{n+1} = (\xi_1, \dots, \xi_n, \xi_{n+1}) \in \mathbb{R}^{n+1}$, define polygonal functions $[Z(x, \cdot)]_b$ and $[\vec{\xi}_{n+1}]_b$ of $Z(x, \cdot)$ and $\vec{\xi}_{n+1}$ by (15), where $Y(x, t_j)$ is replaced by $Z(x, t_j)$ and ξ_j , respectively, for $j = 1, \dots, n+1$ ($\xi_0 = 0$ and $a(0) = 0$ formally). Moreover, define a polygonal function $[a]_b$ of a by (15), where $Y(x, t_j)$ is replaced by $a(t_j)$. For $\vec{\xi}_n = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$, the symbol $[\vec{\xi}_n]_b$ is understood as $\chi_{[0, t_n]}[\vec{\xi}_{n+1}]_b$ on $[0, t]$.

By Theorems 2.4 and 2.9 in [8], we can easily prove the following theorem.

Theorem 14. Let $t_{j-1} < s < t_j$ for some $j \in \{1, \dots, n+1\}$. Then, $Y(\cdot, s) - [Y(\cdot, \cdot)]_{a,b}(s)$ is normally distributed on $(C[0, t], w_{\varphi_\alpha})$ with mean $a(s) - [a]_b(s)$ and variance $\gamma_j(s)$.

Theorem 15. The process $\{Y(\cdot, s) - [Y(\cdot, \cdot)]_{a,b}(s) : 0 \leq s \leq t\}$ and Y_{n+1} are stochastically independent on $(C[0, t], w_{\varphi_\alpha})$.

Proof. Let $t_{l-1} < s < t_l$ for some $l \in \{1, \dots, n+1\}$. Then, for $j = 1, \dots, n+1$,

$$\begin{aligned} E[(Y(\cdot, s) - [Y(\cdot, \cdot)]_{a,b}(s)) Y(\cdot, t_j)] &= E[(Z(\cdot, s) - [Z(\cdot, \cdot)]_b(s) + a(s) \\ &\quad - [a]_b(s))(Z(\cdot, t_j) + a(t_j))] \\ &= E[Z(\cdot, s) - [Z(\cdot, \cdot)]_b(s)] E[Z(\cdot, t_j)] \\ &\quad + (a(s) - [a]_b(s)) a(t_j) \end{aligned} \quad (16)$$

by Corollary 2.4 and Theorems 2.9 and 2.10 in [8]. By Corollary 5 and Theorem 14 we have

$$\begin{aligned} E[(Y(\cdot, s) - [Y(\cdot, \cdot)]_{a,b}(s)) Y(\cdot, t_j)] &= (a(s) - [a]_b(s)) a(t_j) \\ &= E[Y(\cdot, s) - [Y(\cdot, \cdot)]_{a,b}(s)] E[Y(\cdot, t_j)]. \end{aligned} \quad (17)$$

Since $Y(\cdot, s) - [Y(\cdot, \cdot)]_{a,b}(s)$ and $Y(\cdot, t_j)$ are normally distributed on w_{φ_α} , they are independent. The proofs of remainder cases follow easily. $\square$

Theorem 16. The processes $\{Y(\cdot, s) - [Y(\cdot, \cdot)]_{a,b}(s) : t_{j-1} \leq s \leq t_j\}$, where $j = 1, \dots, n+1$, are stochastically independent on $(C[0, t], w_{\varphi_\alpha})$.

Proof. Let $s_1 \in [t_{l-1}, t_l]$ and $s_2 \in [t_{j-1}, t_j]$ with $l < j$. If $s_1 \in \{0, t_1, \dots, t_{n+1}\}$ or $s_2 \in \{0, t_1, \dots, t_{n+1}\}$, we can prove easily the independence of $Y(\cdot, s_1) - [Y(\cdot, \cdot)]_{a,b}(s_1)$ and $Y(\cdot, s_2) - [Y(\cdot, \cdot)]_{a,b}(s_2)$. Suppose that $t_{l-1} < s_1 < t_l \leq t_{j-1} < s_2 < t_j$. By Theorems 3.1 and 2.11 in [8], we have

$$\begin{aligned} E[(Y(\cdot, s_1) - [Y(\cdot, \cdot)]_{a,b}(s_1))(Y(\cdot, s_2) - [Y(\cdot, \cdot)]_{a,b}(s_2))] &= E[(Z(\cdot, s_1) - [Z(\cdot, \cdot)]_b(s_1) + a(s_1) - [a]_b(s_1)) \\ &\quad \times (Z(\cdot, s_2) - [Z(\cdot, \cdot)]_b(s_2) + a(s_2) - [a]_b(s_2))] \\ &= E[Z(\cdot, s_1) - [Z(\cdot, \cdot)]_b(s_1)] E[Z(\cdot, s_2) - [Z(\cdot, \cdot)]_b(s_2)] \\ &\quad + (a(s_1) - [a]_b(s_1))(a(s_2) - [a]_b(s_2)) \\ &= E[Y(\cdot, s_1) - [Y(\cdot, \cdot)]_{a,b}(s_1)] E[Y(\cdot, s_2) - [Y(\cdot, \cdot)]_{a,b}(s_2)] \end{aligned} \quad (18)$$

which completes the proof. $\square$

Applying the same method as used in the proof of Theorem 2 in [2, page 383] with Problem 4 of [9, page 216], we have the following theorem by Theorem 15.

Theorem 17. Let $F : C[0, t] \rightarrow \mathbb{C}$ be a function, and let $F(Y(x, \cdot))$ be w_{φ_α} -integrable over the variable x . Then, for a Borel subset B of $\mathbb{R}^n$

$$\begin{aligned} \int_{Y_{n+1}^{-1}(B)} F(Y(x, \cdot)) dw_{\varphi_\alpha}(x) &= \int_B E[F(Y(x, \cdot) - [Y(x, \cdot)]_{a,b} + [\vec{\xi}_{n+1}]_b)] dP_{Y_{n+1}}(\vec{\xi}_{n+1}), \end{aligned} \quad (19)$$

where the expectation is taken over w_{φ_α} and $P_{Y_{n+1}}$ is the probability distribution of Y_{n+1} on $(\mathbb{R}^{n+1}, \mathcal{B}(\mathbb{R}^{n+1}))$. Moreover, for $P_{Y_{n+1}}$ a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+1}$ (hence for a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+1}$)

$$\begin{aligned} E[F(Y(x, \cdot)) | Y_{n+1}](\vec{\xi}_{n+1}) &= E[F(Y(x, \cdot) - [Y(x, \cdot)]_{a,b} + [\vec{\xi}_{n+1}]_b)]. \end{aligned} \quad (20)$$

Using the same method as used in the proof of Theorem 2.5 in [6], we can prove the following theorem.

Theorem 18. Let $F : C[0, t] \rightarrow \mathbb{C}$ be a function and $F(Y(x, \cdot))$ w_{φ_α} -integrable over the variable x . Moreover, let P_{Y_n}

be a probability distribution of Y_n on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$. Then, for P_{Y_n} a.e. $\vec{\xi}_n = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$ (hence for a.e. $\vec{\xi}_n \in \mathbb{R}^n$)

$$\begin{aligned} E[F | Y_n](\vec{\xi}_n) &= \left[\frac{1}{2\pi(b(t) - b(t_n))} \right]^{1/2} \\ &\times \int_{\mathbb{R}} E[F(Y(x, \cdot) - [Y(x, \cdot)]_{a,b} + [\vec{\xi}_{n+1}]_b)] \\ &\times \exp \left\{ -\frac{(\xi_{n+1} - a(t) - (\xi_n - a(t_n)))^2}{2(b(t) - b(t_n))} \right\} d\xi_{n+1}, \end{aligned} \quad (21)$$

where $\vec{\xi}_{n+1} = (\xi_1, \dots, \xi_n, \xi_{n+1})$ and the expectation is taken over w_{φ_α} .

Note that the conditioning functions Y_{n+1} and Y_n describe the positions of paths at the times $t_1, \dots, t_n, t_{n+1} = t$ (the present time). Y_{n+1} contains the present position $Y(x, t)$ of the path $Y(x, \cdot)$ for $x \in C[0, t]$, while Y_n does not. Moreover if we let $h = 1$ a.e., $\alpha = 0$, and $\varphi = \delta_0$, the Dirac measure concentrated at 0, then we can obtain Theorems 1 and 2 in [10] by the translation theorem (Theorem 1). If $a(0) = 0$, $h = \sqrt{b'}$, $\varphi_\alpha = \delta_0$, and $t = T$, then we can obtain the space $C_{a,b}[0, T]$ in [7] by Theorem 12. Furthermore, if Y is replaced by the generalized Brownian motion process $x(s)$ on $C_{a,b}[0, T] \times [0, T]$, and we let $\varphi_\alpha = \delta_0$, then we can also obtain Theorem 3.4 in [7] by Theorem 17. If we let $a \equiv 0$ and $\varphi_\alpha = \delta_0$, then we can obtain Theorem 3 in [11] by Theorem 17. If we let $a \equiv 0$, then we can obtain Theorem 2.12 in [8] by Theorem 17. If we let $n = 0$ and $\varphi_\alpha = \delta_0$, then we can obtain Remark 2.2 in [3] by Theorem 17. Finally, if we let $h = 1$ a.e., $\varphi_\alpha = \delta_0$ and $a \equiv 0$, then we can obtain Theorem 2 in [2] by Theorem 17 which is among the first result expressing the conditional Wiener integrals of functions on $C_0[0, t]$ as ordinary Wiener integrals.

Remark 19. Note that Theorems 17 and 18 are not generalizations of Theorem 2.9 in [5] and Theorem 2.5 in [6]. In Theorem 2.9 of [5] and Theorem 2.5 of [6], the conditioning functions have initial distributions φ , while Y_{n+1} and Y_n in Theorems 17 and 18 have no initial distributions.

4. Evaluation Formulas Using the Simple Formulas

In this section, we derive evaluation formulas for the generalized conditional Wiener integrals of various functions which are of interest in Feynman integration theories and quantum mechanics. For a function $F : C[0, t] \rightarrow \mathbb{C}$ let $F_Y(x) = F(Y(x, \cdot))$ for $x \in C[0, t]$.

Lemma 20. Let $F_m(x) = \int_0^t (x(s))^m ds$ ($m \in \mathbb{N}$) for $x \in C[0, t]$. Then, $(F_m)_Y$ is w_{φ_α} -integrable and

$$E[(F_m)_Y] = \sum_{l=0}^{[m/2]} \frac{m!}{2^l l! (m-2l)!} \int_0^t (a(s))^{m-2l} (b(s))^l ds, \quad (22)$$

where $[\cdot]$ denotes the greatest integer function.

Proof. By Corollary 5, the change of variable theorem and the binomial expansion,

$$\begin{aligned} &\int_0^t \int_{C[0,t]} |Y(x, s)|^m dw_{\varphi_\alpha}(x) ds \\ &\leq \left(\frac{1}{\pi}\right)^{1/2} \sum_{l=0}^m 2^{l/2} \Gamma\left(\frac{l+1}{2}\right) \binom{m}{l} \int_{(0,t]} |a(s)|^{m-l} (b(s))^{l/2} ds, \end{aligned} \quad (23)$$

where Γ denote the gamma function, so that $(F_m)_Y$ is integrable over $(C[0, t], w_{\varphi_\alpha})$. Furthermore, we have

$$\begin{aligned} &\int_{C[0,t]} \int_0^t (Y(x, s))^m ds dw_{\varphi_\alpha}(x) \\ &= 2 \sum_{l=0}^{[m/2]} \binom{m}{2l} \int_{(0,t]} (a(s))^{m-2l} \left(\frac{1}{2\pi b(s)}\right)^{1/2} \\ &\times \int_0^\infty u^{2l} \exp\left\{-\frac{u^2}{2b(s)}\right\} du ds \\ &= \sum_{l=0}^{[m/2]} \frac{m!}{2^l l! (m-2l)!} \int_0^t (a(s))^{m-2l} (b(s))^l ds \end{aligned} \quad (24)$$

which completes the proof. $\square$

Theorem 21. Let the assumptions be as given in Lemma 20. Then, for a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+1}$ (hence, $P_{Y_{n+1}}$ a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+1}$)

$$\begin{aligned} &E[(F_m)_Y | Y_{n+1}](\vec{\xi}_{n+1}) \\ &= \sum_{j=1}^{n+1} \sum_{l=0}^{[m/2]} \frac{m!}{2^l l! (m-2l)!} \\ &\times \int_{t_{j-1}}^{t_j} (a(s) - [a]_b(s) + [\vec{\xi}_{n+1}]_b(s))^{m-2l} (\gamma_j(s))^l ds, \end{aligned} \quad (25)$$

where the expectation is taken over w_{φ_α} .

Proof. For $P_{Y_{n+1}}$ a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+1}$, we have by Theorems 14 and 17

$$\begin{aligned} &E[(F_m)_Y | Y_{n+1}](\vec{\xi}_{n+1}) \\ &= \sum_{j=1}^{n+1} \int_{(t_{j-1}, t_j)} \left(\frac{1}{2\pi \gamma_j(s)}\right)^{1/2} \\ &\times \int_{\mathbb{R}} (u + a(s) - [a]_b(s) + [\vec{\xi}_{n+1}]_b(s))^m \\ &\times \exp\left\{-\frac{u^2}{2\gamma_j(s)}\right\} du ds \\ &= \sum_{j=1}^{n+1} \sum_{l=0}^{[m/2]} \frac{m!}{2^l l! (m-2l)!} \\ &\times \int_{t_{j-1}}^{t_j} (a(s) - [a]_b(s) + [\vec{\xi}_{n+1}]_b(s))^{m-2l} (\gamma_j(s))^l ds \end{aligned} \quad (26)$$

by the same method as used in the proof of Lemma 20. $\square$

Theorem 22. Let the assumptions be as given in Theorem 21, and for $\vec{\xi}_n = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$ let

$$\Xi(\vec{\xi}_n) = \sum_{j=1}^n \sum_{l=0}^{[m/2]} \frac{m!}{2^l l! (m-2l)!} \times \int_{t_{j-1}}^{t_j} (a(s) - [a]_b(s) + [\vec{\xi}_n]_b(s))^{m-2l} (\gamma_j(s))^l ds. \quad (27)$$

Then, for a.e. $\vec{\xi}_n \in \mathbb{R}^n$ (hence P_{Y_n} a.e. $\vec{\xi}_n \in \mathbb{R}^n$)

$$\begin{aligned} E[(F_m)_Y | Y_n](\vec{\xi}_n) \\ = \Xi(\vec{\xi}_n) + m! \\ \times \sum_{l=0}^{[m/2]} \sum_{k=0}^{m-2l} \sum_{p=0}^k \sum_{q=0}^{[p/2]} \frac{\xi_n^{k-p} (a(t) - a(t_n))^{p-2q} (b(t) - b(t_n))^q}{2^{l+q} l! q! (k-p)! (p-2q)! (m-2l-k)!} \\ \times \int_{t_n}^t (\gamma_{n+1}(s))^l (\beta_{n+1}(s))^p (a(s) - [a]_b(s))^{m-2l-k} ds, \end{aligned} \quad (28)$$

where the expectation is taken over w_{ϕ_α} .

Proof. For $\vec{\xi}_n = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$, let $\vec{\xi}_{n+1} = (\xi_1, \dots, \xi_n, \xi_{n+1})$. By the binomial expansion, we have for $t_n \leq s \leq t$

$$\begin{aligned} (a(s) - [a]_b(s) + [\vec{\xi}_{n+1}]_b(s))^{m-2l} \\ = \sum_{k=0}^{m-2l} \sum_{p=0}^k \frac{\xi_n^{k-p} (m-2l)!}{p! (k-p)! (m-2l-k)!} (\beta_{n+1}(s))^p \\ \times (a(s) - [a]_b(s))^{m-2l-k} (\xi_{n+1} - \xi_n)^p, \end{aligned} \quad (29)$$

so that by Theorems 18 and 21, we have for a.e. $\vec{\xi}_n = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$

$$\begin{aligned} E[(F_m)_Y | Y_n](\vec{\xi}_n) \\ = \Xi(\vec{\xi}_n) \\ + \sum_{l=0}^{[m/2]} \sum_{k=0}^{m-2l} \sum_{p=0}^k \sum_{q=0}^{[p/2]} \frac{m! \xi_n^{k-p} (a(t) - a(t_n))^{p-2q}}{2^{l+q} l! (2q)! (k-p)! (p-2q)! (m-2l-k)!} \\ \times \int_{t_n}^t (\gamma_{n+1}(s))^l (\beta_{n+1}(s))^p \\ \times (a(s) - [a]_b(s))^{m-2l-k} \left[\frac{1}{2\pi (b(t) - b(t_n))} \right]^{1/2} \\ \times \int_0^\infty u^{2q} \exp \left\{ -\frac{u^2}{2(b(t) - b(t_n))} \right\} du ds \end{aligned} \quad (30)$$

by the change of variable theorem. Using the same method as used in the proof of Lemma 20, we have the desired result. $\square$

From now on, we assume that every expectation is taken over w_{ϕ_α} unless otherwise specified. By Corollary 9, Theorems 14 and 17, and Theorem 3.4 in [8], we have the following theorem.

Theorem 23. Let $0 < s_1 < s_2 \leq t$, $s_1 \in [t_{l-1}, t_l]$ and $s_2 \in [t_{j-1}, t_j]$. For $x \in C[0, t]$, let $G(x) = x(s_1)x(s_2)$.

(1) If $l \neq j$, then for a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+1}$ (hence, $P_{Y_{n+1}}$ a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+1}$)

$$\begin{aligned} E[G_Y | Y_{n+1}](\vec{\xi}_{n+1}) \\ = (a(s_1) - [a]_b(s_1) + [\vec{\xi}_{n+1}]_b(s_1)) \\ \times (a(s_2) - [a]_b(s_2) + [\vec{\xi}_{n+1}]_b(s_2)). \end{aligned} \quad (31)$$

(2) If $l = j$, then for a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+1}$ (hence, $P_{Y_{n+1}}$ a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+1}$)

$$\begin{aligned} E[G_Y | Y_{n+1}](\vec{\xi}_{n+1}) \\ = \alpha_j(s_2) (b(s_1) - b(t_{j-1})) \\ + (a(s_1) - [a]_b(s_1) + [\vec{\xi}_{n+1}]_b(s_1)) \\ \times (a(s_2) - [a]_b(s_2) + [\vec{\xi}_{n+1}]_b(s_2)). \end{aligned} \quad (32)$$

Theorem 24. Let the assumptions be as given in Theorem 23.

(1) If $l \leq n$, $j \leq n$ and $l \neq j$, then for a.e. $\vec{\xi}_n \in \mathbb{R}^n$ (hence, P_{Y_n} a.e. $\vec{\xi}_n \in \mathbb{R}^n$)

$$\begin{aligned} E[G_Y | Y_n](\vec{\xi}_n) = (a(s_1) - [a]_b(s_1) + [\vec{\xi}_n]_b(s_1)) \\ \times (a(s_2) - [a]_b(s_2) + [\vec{\xi}_n]_b(s_2)). \end{aligned} \quad (33)$$

(2) If $l \leq n$, $j \leq n$ and $l = j$, then for a.e. $\vec{\xi}_n \in \mathbb{R}^n$ (hence P_{Y_n} a.e. $\vec{\xi}_n \in \mathbb{R}^n$)

$$\begin{aligned} E[G_Y | Y_n](\vec{\xi}_n) = \alpha_j(s_2) (b(s_1) - b(t_{j-1})) \\ + (a(s_1) - [a]_b(s_1) + [\vec{\xi}_n]_b(s_1)) \\ \times (a(s_2) - [a]_b(s_2) + [\vec{\xi}_n]_b(s_2)). \end{aligned} \quad (34)$$

(3) If $l = n+1$ and $j \leq n$, then for a.e. $\vec{\xi}_n = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$ (hence P_{Y_n} a.e. $\vec{\xi}_n \in \mathbb{R}^n$)

$$\begin{aligned} E[G_Y | Y_n](\vec{\xi}_n) \\ = (a(s_1) - [a]_b(s_1) + \xi_n + \beta_{n+1}(s_1) (a(t) - a(t_n))) \\ \times (a(s_2) - [a]_b(s_2) + [\vec{\xi}_n]_b(s_2)). \end{aligned} \quad (35)$$

(4) If $l \leq n$ and $j = n + 1$, then for a.e. $\vec{\xi}_n = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$ (hence P_{Y_n} a.e. $\vec{\xi}_n \in \mathbb{R}^n$)

$$\begin{aligned} E[G_Y | Y_n](\vec{\xi}_n) \\ = (a(s_2) - [a]_b(s_2) + \xi_n + \beta_{n+1}(s_2)(a(t) - a(t_n))) \\ \times (a(s_1) - [a]_b(s_1) + [\vec{\xi}_n]_b(s_1)). \end{aligned} \quad (36)$$

(5) If $l = n+1$ and $j = n+1$, then for a.e. $\vec{\xi}_n = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$ (hence P_{Y_n} a.e. $\vec{\xi}_n \in \mathbb{R}^n$)

$$\begin{aligned} E[G_Y | Y_n](\vec{\xi}_n) \\ = \xi_n^2 + \xi_n(a(t) - a(t_n)) \\ \times (a(s_1) - [a]_b(s_1) + a(s_2) - [a]_b(s_2) \\ + \beta_{n+1}(s_1) + \beta_{n+1}(s_2)) \\ + (a(s_1) - [a]_b(s_1) + \beta_{n+1}(s_1)(a(t) - a(t_n))) \\ \times (a(s_2) - [a]_b(s_2) + \beta_{n+1}(s_2)(a(t) - a(t_n))) \\ + \beta_{n+1}(s_1)\beta_{n+1}(s_2)(b(t) - b(t_n)) \\ + \alpha_{n+1}(s_2)(b(s_1) - b(t_n)). \end{aligned} \quad (37)$$

Proof. For $\vec{\xi}_n = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$, let $\vec{\xi}_{n+1} = (\xi_1, \dots, \xi_n, \xi_{n+1})$. (1) and (2) follow immediately from Theorem 23. If $s \in [t_n, t]$, then

$$\begin{aligned} & \left[\frac{1}{2\pi(b(t) - b(t_n))} \right]^{1/2} \\ & \times \int_{\mathbb{R}} [\vec{\xi}_{n+1}]_b(s) \\ & \times \exp \left\{ -\frac{(\xi_{n+1} - \xi_n - (a(t) - a(t_n)))^2}{2(b(t) - b(t_n))} \right\} d\xi_{n+1} \\ & = \left[\frac{1}{2\pi(b(t) - b(t_n))} \right]^{1/2} \\ & \times \int_{\mathbb{R}} (\xi_n + \beta_{n+1}(s)(\xi_{n+1} - \xi_n)) \\ & \times \exp \left\{ -\frac{(\xi_{n+1} - \xi_n - (a(t) - a(t_n)))^2}{2(b(t) - b(t_n))} \right\} d\xi_{n+1} \\ & = \xi_n + \beta_{n+1}(s)(a(t) - a(t_n)) \end{aligned} \quad (38)$$

by which (3) and (4) follow. If $s_1, s_2 \in [t_n, t]$, then

$$\begin{aligned} & \left[\frac{1}{2\pi(b(t) - b(t_n))} \right]^{1/2} \\ & \times \int_{\mathbb{R}} [\vec{\xi}_{n+1}]_b(s_1) [\vec{\xi}_{n+1}]_b(s_2) \\ & \times \exp \left\{ -\frac{(\xi_{n+1} - \xi_n - (a(t) - a(t_n)))^2}{2(b(t) - b(t_n))} \right\} d\xi_{n+1} \\ & = \left[\frac{1}{2\pi(b(t) - b(t_n))} \right]^{1/2} \\ & \times \int_{\mathbb{R}} (\xi_n + \beta_{n+1}(s_1)(\xi_{n+1} - \xi_n)) \\ & \times (\xi_n + \beta_{n+1}(s_2)(\xi_{n+1} - \xi_n)) \\ & \times \exp \left\{ -\frac{(\xi_{n+1} - \xi_n - (a(t) - a(t_n)))^2}{2(b(t) - b(t_n))} \right\} d\xi_{n+1} \\ & = \xi_n^2 + \xi_n(a(t) - a(t_n))(\beta_{n+1}(s_1) + \beta_{n+1}(s_2)) \\ & + \beta_{n+1}(s_1)\beta_{n+1}(s_2)((a(t) - a(t_n))^2 + b(t) - b(t_n)) \end{aligned} \quad (39)$$

by which we have (5). $\square$

For $j = 1, \dots, n + 1$, let $h_j = \chi_{(t_{j-1}, t_j]} h$ and $g_j = h_j / \sqrt{b(t_j) - b(t_{j-1})}$. Let V be the subspace of $L_2[0, t]$ generated by $\{g_1, \dots, g_{n+1}\}$, $V^\perp$ the orthogonal complement of V , and $\mathcal{P}^\perp : L_2[0, t] \rightarrow V^\perp$ the orthogonal projection. Let $\mathcal{M}(L_2[0, t])$ be the class of all $\mathbb{C}$ -valued Borel measures of bounded variation over $L_2[0, t]$, and let $\mathcal{S}_{w_{\varphi_\alpha}}$ be the space of all functions F which for $\sigma \in \mathcal{M}(L_2[0, t])$ have the form

$$F(x) = \int_{L_2[0, t]} \exp\{i(v, x)\} d\sigma(v) \quad (40)$$

for w_{φ_α} a.e. $x \in C[0, t]$. Note that $\mathcal{S}_{w_{\varphi_\alpha}}$ is a Banach algebra [4].

Now, we have the following theorems which are our final results.

Theorem 25. Let h be of bounded variation and a absolutely continuous on $[0, t]$. Let $F \in \mathcal{S}_{w_{\varphi_\alpha}}$ and $\sigma \in \mathcal{M}(L_2[0, t])$ be related by (40). Then,

$$E[F_Y] = \int_{L_2[0, t]} \exp \left\{ -\frac{1}{2} \|vh\|^2 \right\} d\sigma_a(v), \quad (41)$$

where $(d\sigma_a/d\sigma)(v) = \exp\{i(v, a)\}$ for $v \in L_2[0, t]$. Moreover for a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+1}$ (hence for $P_{Y_{n+1}}$ a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+1}$) $E[F_Y | Y_{n+1}](\vec{\xi}_{n+1})$ is given by

$$\begin{aligned} E[F_Y | Y_{n+1}](\vec{\xi}_{n+1}) \\ = \int_{L_2[0, t]} \exp \left\{ i(v, a - [a]_b + [\vec{\xi}_{n+1}]_b) - \frac{1}{2} \|\mathcal{P}^\perp(vh)\|^2 \right\} d\sigma(v). \end{aligned} \quad (42)$$

Proof. By Corollary 5 and the Fubini's theorem

$$\begin{aligned} E[F_Y] &= \int_{L_2[0,t]} \int_{C[0,t]} \exp\{i(v, Y(x, \cdot))\} dw_{\varphi_\alpha}(x) d\sigma(v) \\ &= \int_{L_2[0,t]} \exp\{i(v, a)\} \exp\left\{-\frac{1}{2}\|vh\|^2\right\} d\sigma(v), \end{aligned} \quad (43)$$

where the last equality follows from the following integral formula:

$$\int_{\mathbb{R}} \exp\{-au^2 + ibu\} du = \left(\frac{\pi}{a}\right)^{1/2} \exp\left\{-\frac{b^2}{4a}\right\} \quad (44)$$

for $a \in \mathbb{C}_+$ and any real b . Furthermore, for a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+1}$, we have by Theorems 3.4 and 3.6 in [8]

$$\begin{aligned} E[F_Y | Y_{n+1}](\vec{\xi}_{n+1}) &= \int_{L_2[0,t]} \exp\{i(v, a - [a]_b + [\vec{\xi}_{n+1}]_b)\} \\ &\quad \times \int_{C[0,t]} \exp\{i(v, Z(x, \cdot) - [Z(x, \cdot)]_b)\} \\ &\quad dw_{\varphi_\alpha}(x) d\sigma(v) \\ &= \int_{L_2[0,t]} \exp\{i(v, a - [a]_b + [\vec{\xi}_{n+1}]_b)\} \\ &\quad \times \exp\left\{-\frac{1}{2}\|\mathcal{P}^\perp(vh)\|^2\right\} d\sigma(v) \end{aligned} \quad (45)$$

which completes the proof. $\square$

Theorem 26. Let the assumptions be as given in Theorem 25 and for $\vec{\xi}_n \in \mathbb{R}^n$ let

$$A(v, \vec{\xi}_n) = \exp\left\{i(v, a - [a]_b + [\vec{\xi}_n]_b) - \frac{1}{2}\|\mathcal{P}^\perp(vh)\|^2\right\}. \quad (46)$$

Then, for a.e. $\vec{\xi}_n \in \mathbb{R}^n$ (hence for P_{Y_n} a.e. $\vec{\xi}_n \in \mathbb{R}^n$) $E[F_Y | Y_n](\vec{\xi}_n)$ is given by

$$\begin{aligned} E[F_Y | Y_n](\vec{\xi}_n) &= \int_{L_2[0,t]} A(v, \vec{\xi}_n) \exp\left\{i \frac{\langle vh, g_{n+1} \rangle (a(t) - a(t_n))}{\sqrt{b(t) - b(t_n)}} \right. \\ &\quad \left. - \frac{1}{2} \langle vh, g_{n+1} \rangle^2\right\} d\sigma(v), \end{aligned} \quad (47)$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product on $L_2[0, t]$.

Proof. Let $\vec{\xi}_n = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$ and let $\vec{\xi}_{n+1} = (\xi_1, \dots, \xi_n, \xi_{n+1})$. By the definition of the Paley-Wiener-Zygmund integral it is not difficult to show

$$(v, [\vec{\xi}_{n+1}]_b) = (v, [\vec{\xi}_n]_b) + \frac{\langle vh, g_{n+1} \rangle}{\sqrt{b(t) - b(t_n)}} (\xi_{n+1} - \xi_n) \quad (48)$$

so that we have for a.e. $\vec{\xi}_n = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$

$$\begin{aligned} E[F_Y | Y_n](\vec{\xi}_n) &= \left[\frac{1}{2\pi(b(t) - b(t_n))} \right]^{1/2} \\ &\quad \times \int_{L_2[0,t]} A(v, \vec{\xi}_n) \\ &\quad \times \int_{\mathbb{R}} \exp\left\{i \frac{\langle vh, g_{n+1} \rangle}{\sqrt{b(t) - b(t_n)}} (\xi_{n+1} - \xi_n) \right. \\ &\quad \left. - \frac{(\xi_{n+1} - \xi_n - (a(t) - a(t_n)))^2}{2(b(t) - b(t_n))} \right\} d\xi_{n+1} d\sigma(v) \\ &= \int_{L_2[0,t]} A(v, \vec{\xi}_n) \exp\left\{i \frac{\langle vh, g_{n+1} \rangle (a(t) - a(t_n))}{\sqrt{b(t) - b(t_n)}} \right. \\ &\quad \left. - \frac{1}{2} \langle vh, g_{n+1} \rangle^2\right\} d\sigma(v) \end{aligned} \quad (49)$$

by Theorem 25 and the change of variable theorem. $\square$

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