Hindawi Publishing Corporation
Journal of Function Spaces and Applications
Volume 2013, Article ID 496361, 6 pages
http://dx.doi.org/10.1155/2013/496361

Research Article

Hyperstability of the Fréchet Equation and a Characterization of Inner Product Spaces

Anna Bahyrycz,[1] Janusz Brzdęk,[1] Magdalena Piszczek,[1] and Justyna Sikorska[2]

[1] *Department of Mathematics, Pedagogical University, Podchorazych 2, 30-084 Kraków, Poland*
[2] *Institute of Mathematics, University of Silesia, Bankowa 14, 40-007 Katowice, Poland*

Correspondence should be addressed to Anna Bahyrycz; bah@up.krakow.pl

Received 30 May 2013; Accepted 4 October 2013

Academic Editor: Bjorn Birnir

We prove some stability and hyperstability results for the well-known Fréchet equation stemming from one of the characterizations of the inner product spaces. As the main tool, we use a fixed point theorem for the function spaces. We finish the paper with some new inequalities characterizing the inner product spaces.

1. Introduction

In the literature there are many characterizations of inner product spaces. The first norm characterization of inner product space was given by Fréchet [1] in 1935. He proved that a normed space $(X, \|\cdot\|)$ is an inner product space if and only if, for all $x, y, z \in X$,

$$\begin{aligned}\|x+y+z\|^2 + \|x\|^2 + \|y\|^2 + \|z\|^2 \\ = \|x+y\|^2 + \|x+z\|^2 + \|y+z\|^2.\end{aligned} \tag{1}$$

In the same year Jordan and von Neumann [2] gave the celebrated parallelogram law characterization of an inner product space. Since then numerous further conditions, characterizing the inner product spaces among the normed spaces, have been shown. More than 300 such conditions have been collected in the book of Amir [3]. Many geometrical characterizations are presented in the book by Alsina et al. [4]; for some other see, for example, [5–8].

The results that we obtain are motivated by the notion of hyperstability of functional equations (see, e.g., [9–14]), which has been introduced in connection with the issue of stability of functional equations (for more details see, e.g., [15, 16]).

The main tool in the proof of the main theorem is a fixed point result for function spaces from [17] (for related outcomes see [18, 19]). Similar method of the proof has been already applied in [11, 20].

To present the fixed point theorem we introduce the following necessary hypotheses ($\mathbb{R}_+$ stands for the set of nonnegative reals and A^B denotes the family of all functions mapping a set $B \neq \emptyset$ into a set $A \neq \emptyset$).

(H1) S is a nonempty set, E is a Banach space, and functions $f_1, \ldots, f_k : S \to S$ and $L_1, \ldots, L_k : S \to \mathbb{R}_+$ are given.

(H2) $\mathcal{T} : E^S \to E^S$ is an operator satisfying the inequality

$$\begin{aligned}&\|\mathcal{T}\xi(x) - \mathcal{T}\mu(x)\| \\ &\quad \le \sum_{i=1}^{k} L_i(x) \|\xi(f_i(x)) - \mu(f_i(x))\|, \quad \xi, \mu \in E^X,\ x \in S.\end{aligned} \tag{2}$$

(H3) $\Lambda : \mathbb{R}_+^S \to \mathbb{R}_+^S$ is defined by

$$\Lambda\delta(x) := \sum_{i=1}^{k} L_i(x)\, \delta(f_i(x)), \quad \delta \in \mathbb{R}_+^S,\ x \in S. \tag{3}$$

Now we are in a position to present the above mentioned fixed point theorem for function spaces (see [17]).

Theorem 1. *Let hypotheses (H1)–(H3) be valid and functions $\varepsilon : S \to \mathbb{R}_+$ and $\varphi : S \to E$ fulfil the following two conditions:*

$$\begin{aligned} &\left\|\mathcal{T}\varphi(x) - \varphi(x)\right\| \le \varepsilon(x), \quad x \in S, \\ &\varepsilon^*(x) := \sum_{n=0}^{\infty} \Lambda^n \varepsilon(x) < \infty, \quad x \in S. \end{aligned} \tag{4}$$

Then there exists a unique fixed point ψ of $\mathcal{T}$ with

$$\left\|\varphi(x) - \psi(x)\right\| \le \varepsilon^*(x), \quad x \in S. \tag{5}$$

Moreover,

$$\psi(x) := \lim_{n\to\infty} \mathcal{T}^n \varphi(x), \quad x \in S. \tag{6}$$

We start our considerations from the functional equation

$$\begin{aligned} &f(x+y+z) + f(x) + f(y) + f(z) \\ &\quad = f(x+y) + f(x+z) + f(y+z), \end{aligned} \tag{7}$$

that is patterned on (1) and therefore quite often named after Fréchet (see, e.g., [21]).

Note that (7) can be written in the form

$$\Delta_{x,y,z} f(0) = 0, \tag{8}$$

where Δ denotes the Fréchet difference operator defined (for functions mapping a commutative semigroup $(S,+)$ into a group) by

$$\begin{aligned} &\Delta_y f(x) = \Delta_y^1 f(x) := f(x+y) - f(x), \quad x, y \in S, \\ &\Delta_{t,z} := \Delta_t \circ \Delta_z, \quad \Delta_t^2 := \Delta_{t,t}, \quad t, z \in S, \\ &\Delta_{t,u,z} := \Delta_t \circ \Delta_u \circ \Delta_z, \quad \Delta_t^3 := \Delta_{t,t,t}, \quad t, u, z \in S. \end{aligned} \tag{9}$$

It is easy to check that

$$\begin{aligned} \Delta_{t,z} f(x) &= f(x+t+z) - f(x+t) - f(x+z) \\ &\quad + f(x), \quad x, t, z \in S, \\ \Delta_{t,z,u} f(x) &= f(x+t+z+u) - f(x+t+z) \\ &\quad - f(x+t+u) - f(x+z+u) \\ &\quad + f(x+t) + f(x+z) + f(x+u) \\ &\quad - f(x), \quad x, t, z, u \in S. \end{aligned} \tag{10}$$

Such operators were first considered by Fréchet in [22, 23] (we refer to [24] for more information and further references concerning this subject); so, it is still another motivation for (7) to be called the Fréchet equation.

Let us yet observe (see [25]) that, alternatively, (7) can be written in the form

$$C^2 f(x,y,z) = 0, \tag{11}$$

where $C^2 f(x,y,z) = Cf(x,y+z) - Cf(x,y) - Cf(x,z)$ and $Cf(x,y) = f(x+y) - f(x) - f(y)$; that is, $C^2 f$ is the Cauchy difference of f of the second order.

We prove the subsequent theorem, which corresponds to [26, Theorem 3.1], where the equation

$$\Delta_z^3 f(x) = 0 \tag{12}$$

has been investigated (the author has named it the superstability result, which is not a precise description, because according to the terminology applied in [10–14] it should be rather called the hyperstability result). For some analogous investigations see [27–29]. Let us mention yet that stability of (7) has been already studied in [30–33] and our results complement the outcomes included there.

It is easy to show that every solution f of (7), mapping a commutative group $(G,+)$ into a real linear space X, must be of the form $f = a + q$ with some additive $a : G \to X$ and quadratic $q : G \to X$ (see, e.g., [21]). Namely, with $x = y = z = 0$ from (7) we deduce that $f(0) = 0$, and, next, taking $z = -y$ in (7), we obtain that the even part of f is quadratic while the odd part is a solution of the Jensen equation, whence it is additive.

2. Main Results

The next theorem and corollary are the main results of the paper ($\mathbb{N}$ and $\mathbb{Z}$ stand, as usual, for the sets of all positive integers and integers, respectively; moreover, $\mathbb{Z}_0 := \mathbb{Z} \setminus \{0\}$).

Theorem 2. *Let $(X,+)$ be a commutative group, $X_0 := X \setminus \{0\}$, Y a Banach space, and $f : X \to Y$, $c : \mathbb{Z}_0 \to [0,\infty)$ and $L : X_0^3 \to [0,\infty)$ satisfying the following three conditions:*

$$\begin{aligned} \mathcal{M} := \{&m \in \mathbb{Z}_0 : c(-2m) + 2c(m+1) + 2c(-m) \\ &+ c(2m+1) < 1\} \ne \emptyset, \end{aligned} \tag{13}$$

$$\begin{aligned} &L(kx, ky, kz) \le c(k) L(x,y,z), \quad x, y, z \in X_0, \\ &k \in \{-2m, m+1, -m, 2m+1\}, \quad m \in \mathcal{M}, \end{aligned} \tag{14}$$

$$\begin{aligned} &\left\|f(x+y+z) + f(x) + f(y) + f(z)\right. \\ &\quad \left. - f(x+y) - f(x+z) - f(y+z)\right\| \\ &\quad \le L(x,y,z), \quad x, y, z \in X_0. \end{aligned} \tag{15}$$

Then there is a unique function $F : X \to Y$ satisfying (7) for all $x, y, z \in X$ and such that

$$\left\|f(x) - F(x)\right\| \le \rho_L(x), \quad x \in X_0, \tag{16}$$

where

$$\begin{aligned} &\rho_L(x) \\ &:= \inf_{m\in\mathcal{M}} \frac{L((2m+1)x, -mx, -mx)}{1 - c(-2m) - 2c(m+1) - 2c(-m) - c(2m+1)}, \\ &\qquad x \in X_0. \end{aligned} \tag{17}$$

Proof. Replacing x by $(2m+1)x$ and taking $y = z = -mx$ in (15) we get

$$\begin{aligned}&\|f(x) + f((2m+1)x) + 2f(-mx)\\&\quad -2f((m+1)x) - f(-2mx)\|\\&\le L((2m+1)x, -mx, -mx) =: \varepsilon_m(x), \quad x \in X_0,\ m \in \mathbb{Z}_0.\end{aligned} \tag{18}$$

Further, put

$$\begin{aligned}\mathcal{T}_m\xi(x) &:= \xi(-2mx) + 2\xi((m+1)x) - 2\xi(-mx)\\&\quad - \xi((2m+1)x), \quad \xi \in Y^X,\ x \in X,\ m \in \mathbb{Z}_0.\end{aligned} \tag{19}$$

Then,

$$\mathcal{T}_m^n f(0) = 0, \quad n \in \mathbb{N},\ m \in \mathbb{Z}_0, \tag{20}$$

and inequality (18) takes the form

$$\|\mathcal{T}_m f(x) - f(x)\| \le \varepsilon_m(x), \quad x \in X_0,\ m \in \mathbb{Z}_0. \tag{21}$$

Define an operator $\Lambda_m : \mathbb{R}_+^{X_0} \to \mathbb{R}_+^{X_0}$ for $m \in \mathbb{Z}_0$ by

$$\begin{aligned}\Lambda_m\eta(x) &:= \eta(-2mx) + 2\eta((m+1)x)\\&\quad + 2\eta(-mx) + \eta((2m+1)x)\end{aligned} \tag{22}$$

for $\eta \in \mathbb{R}_+^{X_0}$ and $x \in X_0$. Then it is easily seen that, for each $m \in \mathbb{Z}_0$, the operator $\Lambda := \Lambda_m$ has the form described in (H3) with $k = 4$, $S = X_0$, $E = Y$, and

$$\begin{aligned}&f_1(x) = -2mx, \qquad f_2(x) = (m+1)x,\\&f_3(x) = -mx, \qquad f_4(x) = (2m+1)x,\\&L_1(x) = L_4(x) = 1, \quad L_2(x) = L_3(x) = 2, \quad x \in X_0.\end{aligned} \tag{23}$$

Moreover, for every $\xi, \mu \in Y^{X_0}$, $x \in X_0$, $m \in \mathbb{Z}_0$,

$$\begin{aligned}&\|\mathcal{T}_m\xi(x) - \mathcal{T}_m\mu(x)\|\\&\quad = \|\xi(-2mx) + 2\xi((m+1)x) - 2\xi(-mx)\\&\qquad - \xi((2m+1)x) - \mu(-2mx) - 2\mu((m+1)x)\\&\qquad + 2\mu(-mx) + \mu((2m+1)x)\|\\&\quad \le \|(\xi-\mu)(-2mx)\| + 2\|(\xi-\mu)((m+1)x)\|\\&\qquad + 2\|(\xi-\mu)(-mx)\| + \|(\xi-\mu)((2m+1)x)\|\\&\quad = \sum_{i=1}^{4} L_i(x)\|(\xi-\mu)(f_i(x))\|,\end{aligned} \tag{24}$$

where $(\xi-\mu)(y) = \xi(y) - \mu(y)$ for $y \in X_0$. It is easy to check that, in view of (14),

$$\begin{aligned}\Lambda_m\varepsilon_k(x) &\le (c(-2m) + 2c(m+1) + 2c(-m)\\&\quad + c(2m+1))\varepsilon_k(x), \quad k, m \in \mathbb{Z}_0,\ x \in X_0.\end{aligned} \tag{25}$$

Therefore, since the operator Λ_m is linear, we have

$$\begin{aligned}\varepsilon_m^*(x) &:= \sum_{n=0}^{\infty} \Lambda_m{}^n \varepsilon_m(x)\\&\le \sum_{n=0}^{\infty} (c(-2m) + 2c(m+1)\\&\qquad +2c(-m) + c(2m+1))^n \varepsilon_m(x)\\&= \frac{\varepsilon_m(x)}{1 - c(-2m) - 2c(m+1) - 2c(-m) - c(2m+1)},\\&\qquad m \in \mathcal{M},\ x \in X_0.\end{aligned} \tag{26}$$

Thus, by Theorem 1 (with $S = X_0$ and $E = Y$), for each $m \in \mathcal{M}$ there exists a function $F_m' : X_0 \to Y$ with

$$\begin{aligned}&F_m'(x) = F_m'(-2mx) + 2F_m'((m+1)x)\\&\qquad - 2F_m'(-mx) - F_m'((2m+1)x), \quad x \in X_0,\\&\|f(x) - F_m'(x)\|\\&\quad \le \frac{\varepsilon_m(x)}{1 - c(-2m) - 2c(m+1) - 2c(-m) - c(2m+1)},\\&\qquad x \in X_0.\end{aligned} \tag{27}$$

Moreover,

$$F_m'(x) = \lim_{n\to\infty} \mathcal{T}_m^n f(x), \quad x \in X_0,\ m \in \mathcal{M}. \tag{28}$$

Define $F_m : X \to Y$ by $F_m(0) = 0$ and $F_m(x) := F_m'(x)$ for $x \in X_0$ and $m \in \mathcal{M}$. Then it is easily seen that, by (20),

$$F_m(x) = \lim_{n\to\infty} \mathcal{T}_m^n f(x), \quad x \in X,\ m \in \mathcal{M}. \tag{29}$$

Next, we show that

$$\begin{aligned}&\|\mathcal{T}_m^n f(x+y+z) + \mathcal{T}_m^n f(x) + \mathcal{T}_m^n f(y) + \mathcal{T}_m^n f(z)\\&\quad -\mathcal{T}_m^n f(x+y) - \mathcal{T}_m^n f(x+z) - \mathcal{T}_m^n f(y+z)\|\\&\quad \le (c(-2m) + 2c(m+1) + 2c(-m) + c(2m+1))^n\\&\qquad \times L(x, y, z)\end{aligned} \tag{30}$$

for every $x, y, z \in X_0$, $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $m \in \mathcal{M}$.

Fix $m \in \mathscr{M}$. For $n = 0$, the condition (30) is simply (15). So, take $r \in \mathbb{N}_0$ and suppose that (30) holds for $n = r$ and $x, y, z \in X_0$. Then,

$$\begin{aligned}
&\left\|\mathscr{T}_m^{r+1} f(x+y+z) + \mathscr{T}_m^{r+1} f(x) + \mathscr{T}_m^{r+1} f(y) + \mathscr{T}_m^{r+1} f(z)\right.\\
&\quad\left. -\mathscr{T}_m^{r+1} f(x+y) - \mathscr{T}_m^{r+1} f(x+z) - \mathscr{T}_m^{r+1} f(y+z)\right\|\\
&\quad= \left\|\mathscr{T}_m^{r} f(-2m(x+y+z))\right.\\
&\qquad + 2\mathscr{T}_m^{r} f((m+1)(x+y+z))\\
&\qquad - 2\mathscr{T}_m^{r} f(-m(x+y+z))\\
&\qquad - \mathscr{T}_m^{r} f((2m+1)(x+y+z))\\
&\qquad + \mathscr{T}_m^{r} f(-2mx) + 2\mathscr{T}_m^{r} f((m+1)x)\\
&\qquad - 2\mathscr{T}_m^{r} f(-mx) - \mathscr{T}_m^{r} f((2m+1)x)\\
&\qquad + \mathscr{T}_m^{r} f(-2my) + 2\mathscr{T}_m^{r} f((m+1)y)\\
&\qquad - 2\mathscr{T}_m^{r} f(-my) - \mathscr{T}_m^{r} f((2m+1)y)\\
&\qquad + \mathscr{T}_m^{r} f(-2mz) + 2\mathscr{T}_m^{r} f((m+1)z)\\
&\qquad - 2\mathscr{T}_m^{r} f(-mz) - \mathscr{T}_m^{r} f((2m+1)z)\\
&\qquad - \mathscr{T}_m^{r} f(-2m(x+y)) - 2\mathscr{T}_m^{r} f((m+1)(x+y))\\
&\qquad + 2\mathscr{T}_m^{r} f(-m(x+y)) + \mathscr{T}_m^{r} f((2m+1)(x+y))\\
&\qquad - \mathscr{T}_m^{r} f(-2m(x+z)) - 2\mathscr{T}_m^{r} f((m+1)(x+z))\\
&\qquad + 2\mathscr{T}_m^{r} f(-m(x+z)) + \mathscr{T}_m^{r} f((2m+1)(x+z))\\
&\qquad - \mathscr{T}_m^{r} f(-2m(y+z)) - 2\mathscr{T}_m^{r} f((m+1)(y+z))\\
&\qquad \left. + 2\mathscr{T}_m^{r} f(-m(y+z)) + \mathscr{T}_m^{r} f((2m+1)(y+z))\right\|\\
&\quad\le (c(-2m) + 2c(m+1) + 2c(-m) + c(2m+1))^r\\
&\qquad \times (L(-2mx, -2my, -2mz)\\
&\qquad + 2L((m+1)x, (m+1)y, (m+1)z)\\
&\qquad + 2L(-mx, -my, -mz)\\
&\qquad + L((2m+1)x, (2m+1)y, (2m+1)z))\\
&\quad\le (c(-2m) + 2c(m+1) + 2c(-m)\\
&\qquad + c(2m+1))^{r+1} L(x, y, z)
\end{aligned} \tag{31}$$

for every $x, y, z \in X_0$, which completes the proof of (30).

Letting $n \to \infty$ in (30), we obtain that

$$\begin{aligned}
&F_m(x+y+z) + F_m(x) + F_m(y) + F_m(z)\\
&\quad= F_m(x+y) + F_m(x+z) + F_m(y+z), \quad x, y, z \in X_0.
\end{aligned} \tag{32}$$

So, we have proved that for each $m \in \mathscr{M}$ there exists a function $F_m : X \to Y$ satisfying (7) for $x, y, z \in X_0$ and such that

$$\begin{aligned}
&\|f(x) - F_m(x)\|\\
&\quad\le \frac{\varepsilon_m(x)}{1 - c(-2m) - 2c(m+1) - 2c(-m) - c(2m+1)},\\
&\qquad x \in X_0.
\end{aligned} \tag{33}$$

Now, we show that $F_m = F_k$ for all $m, k \in \mathscr{M}$. So, fix $m, k \in \mathscr{M}$. Note that F_k satisfies (32) with m replaced by k. Hence, replacing x by $(2m+1)x$ and taking $y = z = -mx$ in (32), we obtain that $\mathscr{T}_m F_j = F_j$ for $j = m, k$ and

$$\begin{aligned}
&\|F_m(x) - F_k(x)\|\\
&\quad\le \frac{\varepsilon_m(x)}{1 - c(-2m) - 2c(m+1) - 2c(-m) - c(2m+1)}\\
&\qquad + \frac{\varepsilon_k(x)}{1 - c(-2k) - 2c(k+1) - 2c(-k) - c(2k+1)},\\
&\qquad x \in X_0,
\end{aligned} \tag{34}$$

whence, by the linearity of Λ and (25),

$$\begin{aligned}
&\|F_m(x) - F_k(x)\|\\
&= \|\mathscr{T}_m^n F_m(x) - \mathscr{T}_m^n F_k(x)\|\\
&\le \frac{\Lambda_m^n \varepsilon_m(x)}{1 - c(-2m) - 2c(m+1) - 2c(-m) - c(2m+1)}\\
&\quad + \frac{\Lambda_m^n \varepsilon_k(x)}{1 - c(-2k) - 2c(k+1) - 2c(-k) - c(2k+1)}\\
&\le \frac{(c(-2m) + 2c(m+1) + 2c(-m) + c(2m+1))^n \varepsilon_m(x)}{1 - c(-2m) - 2c(m+1) - 2c(-m) - c(2m+1)}\\
&\quad + \frac{(c(-2m) + 2c(m+1) + 2c(-m) + c(2m+1))^n \varepsilon_k(x)}{1 - c(-2k) - 2c(k+1) - 2c(-k) - c(2k+1)}
\end{aligned} \tag{35}$$

for every $x \in X_0$ and $n \in \mathbb{N}$. Now, letting $n \to \infty$ we get $F_m = F_k =: F$.

Thus, in view of (33), we have proved that

$$\begin{aligned}
&\|f(x) - F(x)\|\\
&\quad\le \frac{\varepsilon_m(x)}{1 - c(-2m) - 2c(m+1) - 2c(-m) - c(2m+1)},\\
&\qquad x \in X_0,\ m \in \mathscr{M},
\end{aligned} \tag{36}$$

whence we derive (16).

Since (in view of (32)) it is easy to notice that F is a solution to (7) (i.e., (7) holds for all $x, y, z \in X$), it remains to

prove the statement concerning the uniqueness of F. So, let $G : X \to Y$ be also a solution of (7) and $\|f(x) - G(x)\| \le \rho_L(x)$ for $x \in X_0$. Then,

$$\|G(x) - F(x)\| \le 2\rho_L(x), \quad x \in X_0. \tag{37}$$

Further, $\mathcal{T}_m G = G$ for each $m \in \mathbb{Z}_0$. Consequently, with a fixed $m \in \mathcal{M}$,

$$\begin{aligned}\|G(x) - F(x)\| &= \left\|\mathcal{T}_m^n G(x) - \mathcal{T}_m^n F(x)\right\| \le 2\Lambda_m^n \rho_L(x) \\ &\le \frac{2\Lambda_m^n \varepsilon_m(x)}{1 - c(-2m) - 2c(m+1) - 2c(-m) - c(2m+1)}\end{aligned} \tag{38}$$

for $x \in X_0$ and $n \in \mathbb{N}$. Next, analogously as (25), by induction we get

$$\begin{aligned}\Lambda_m^n \varepsilon_m(x) \le (&c(-2m) + 2c(m+1) \\ &+ 2c(-m) + c(2m+1))^n \varepsilon_m(x)\end{aligned} \tag{39}$$

for $x \in X_0$, $n \in \mathbb{N}$. This implies that $G = F$. □

Theorem 2 yields at once the following hyperstability result.

Corollary 3. *Let $(X, +)$ be a commutative group, $X_0 := X \setminus \{0\}$, let Y be a normed space, $f : X \to Y$, $c : \mathbb{Z}_0 \to [0, \infty)$, $L : X_0^3 \to [0, \infty)$, and let the conditions* (13), (14), *and* (15) *be valid. Assume that*

$$\inf_{m \in \mathcal{M}} L((2m+1)x, -mx, -mx) = 0, \quad x \in X_0. \tag{40}$$

Then f satisfies (7) *for all $x, y, z \in X$.*

Proof. Note that without loss of generality we may assume that Y is complete, because otherwise we can replace it by its completion. Next, in view of (40), $\rho_L(x) = 0$ for each $x \in X_0$, where ρ_L is defined by (17). Hence, from Theorem 2, we easily derive that f is a solution to (7). □

3. Final Remarks

Remark 4. Note that if, in Theorem 2,

$$\liminf_{m \to \infty} c(-2m) + 2c(m+1) + 2c(-m) + c(2m+1) = 0 \tag{41}$$

(this is the case when, e.g., $\lim_{|m| \to \infty} c(m) = 0$), then (13) holds and

$$\rho_L(x) := \inf_{m \in \mathcal{M}} \varepsilon_m(x), \quad x \in X_0. \tag{42}$$

Further, let X be a normed space and

$$L(x, y, z) = \varepsilon\left(\|x\|^p + \|y\|^p + \|z\|^p\right), \quad x, y, z \in X_0 \tag{43}$$

with some reals $\varepsilon > 0$ and $p < 0$. Then, the condition (14) is valid, for instance, with $c(k) = |k|^p$ for $k \in \mathbb{Z}_0$. Obviously, (40) holds, and there exists $m_0 \in \mathbb{N}$ such that

$$|2m|^p + 2|m+1|^p + 2|m|^p + |2m+1|^p < 1, \quad m \ge m_0; \tag{44}$$

so we obtain (13), as well. Consequently, by Corollary 3, every function $f : X \to Y$, fulfilling the inequality (15), satisfies (7) for all $x, y, z \in X$. In this way we have obtained a hyperstability result that corresponds to the recent hyperstability outcomes in [11, 20] and some classical stability results concerning the Cauchy equation (see, e.g., [9, page 3], [15, page 15, 16], and [16, page 2]).

Below, we provide two further simple and natural examples of functions L and c satisfying the conditions (13) and (14). The first one, clearly, includes the case just described.

(a) $L(x, y, z) = (\alpha_1\|x\|^{s_1} + \alpha_2\|y\|^{s_2} + \alpha_3\|z\|^{s_3})^w$ for $x, y, z \in X_0$ with some $w, \alpha_i, s_i \in \mathbb{R}$ such that $\alpha_i > 0$ and $ws_i < 0$ for $i = 1, 2, 3$, and $c(m) \equiv |m|^{-\nu|w|}$, where $\nu := \min\{|s_1|, |s_2|, |s_3|\}$.

(b) $L(x, y, z) = \alpha\|x\|^s\|y\|^t\|z\|^u$ for $x, y, z \in X_0$ with some reals $\alpha > 0$ and s, t, u such that $s + t + u < 0$.

Clearly, if two functions L satisfy the condition (14), then so do their sum and product, with suitable functions c. Therefore, we can easily produce numerous examples of such functions. Of course there are some other such examples that are a bit more artificial; for instance, $L(x, y, z) = B(\|A(x)\|^{-n})$ for $x \in X_0$, where $n \in \mathbb{N}$, $A : X \to X$ and $B : \mathbb{R} \to \mathbb{R}$ are functions with $A(xm) = mA(x)$ and $B(tm) = mB(t)$ for $x \in X$, $t \in \mathbb{R}$ and $m \in \mathbb{Z}$.

We end the paper with a simple example of applications of our main result.

Corollary 5. *Let X be a normed space and $X_0 := X \setminus \{0\}$. Write*

$$\begin{aligned}D(x, y, z) := \Big|&\|x + y + z\|^2 + \|x\|^2 + \|y\|^2 + \|z\|^2 \\ &- \|x + y\|^2 - \|x + z\|^2 - \|y + z\|^2\Big|\end{aligned} \tag{45}$$

for $x, y, z \in X$. Assume that one of the following two hypotheses is valid.

(i) *There exist $w_0, \alpha_i, s_i \in \mathbb{R}$ such that $\alpha_i, w_0 s_i > 0$ for $i = 1, 2, 3$ and*

$$\sup_{x,y,z \in X_0} \frac{D(x, y, z)}{\left(\alpha_1\|x\|^{s_1} + \alpha_2\|y\|^{s_2} + \alpha_3\|z\|^{s_3}\right)^{w_0}} < \infty. \tag{46}$$

(ii) *There exist reals s, t, u such that $s + t + u > 0$ and*

$$\sup_{x,y,z \in X_0} \frac{D(x, y, z)}{\|x\|^s\|y\|^t\|z\|^u} < \infty. \tag{47}$$

Then X is an inner product space.

Proof. Write $f(x) = \|x\|^2$ for $x \in X$. Then, with L and c of the forms described in Remark 4 (with $w = -w_0$), from Corollary 3 we easily derive that f is a solution to (7), which yields the statement. □

References

[1] M. Fréchet, "Sur la définition axiomatique d'une classe d'espaces vectoriels distanciés applicables vectoriellement sur l'espace de Hilbert," *Annals of Mathematics. Second Series*, vol. 36, no. 3, pp. 705–718, 1935.

[2] P. Jordan and J. von Neumann, "On inner products in linear, metric spaces," *Annals of Mathematics. Second Series*, vol. 36, no. 3, pp. 719–723, 1935.

[3] D. Amir, *Characterizations of Inner Product Spaces, Operator Theory: Advances and Applications*, vol. 20, Birkhäuser, Basel, Switzerland, 1986.

[4] C. Alsina, J. Sikorska, and M. S. Tomás, *Norm Derivatives and Characterizations of Inner Product Spaces*, World Scientific Publishing, 2010.

[5] S. S. Dragomir, "Some characterizations of inner product spaces and applications," *Universitatis Babeş-Bolyai*, vol. 34, no. 1, pp. 50–55, 1989.

[6] M. S. Moslehian and J. M. Rassias, "A characterization of inner product spaces concerning an Euler-Lagrange identity," *Communications in Mathematical Analysis*, vol. 8, no. 2, pp. 16–21, 2010.

[7] K. Nikodem and Z. Páles, "Characterizations of inner product spaces by strongly convex functions," *Banach Journal of Mathematical Analysis*, vol. 5, no. 1, pp. 83–87, 2011.

[8] T. M. Rassias, "New characterizations of inner product spaces," *Bulletin des Sciences Mathématiques. 2nd Série*, vol. 108, no. 1, pp. 95–99, 1984.

[9] N. Brillouët-Belluot, J. Brzdęk, and K. Ciepliński, "On some recent developments in Ulam's type stability," *Abstract and Applied Analysis*, vol. 2012, Article ID 716936, 41 pages, 2012.

[10] J. Brzdęk, "Remarks on hyperstability of the Cauchy functional equation," *Aequationes Mathematicae*, vol. 86, no. 3, pp. 255–267, 2013.

[11] J. Brzdęk, "Hyperstability of the Cauchy equation on restricted domains," *Acta Mathematica Hungarica*, vol. 141, no. 1-2, pp. 58–67, 2013.

[12] E. Gselmann, "Hyperstability of a functional equation," *Acta Mathematica Hungarica*, vol. 124, no. 1-2, pp. 179–188, 2009.

[13] G. Maksa, "The stability of the entropy of degree alpha," *Journal of Mathematical Analysis and Applications*, vol. 346, no. 1, pp. 17–21, 2008.

[14] G. Maksa and Z. Páles, "Hyperstability of a class of linear functional equations," *Acta Mathematica*, vol. 17, no. 2, pp. 107–112, 2001.

[15] D. H. Hyers, G. Isac, and T. M. Rassias, *Stability of Functional Equations in Several Variables*, Birkhäuser, Boston, Mass, USA, 1998.

[16] S.-M. Jung, *Hvers-Ulam-Rassias Stability of Functional Equations in Nonlinear Analysis*, vol. 48 of *Springer Optimization and Its Applications*, Springer, New York, NY, USA, 2011.

[17] J. Brzdęk, J. Chudziak, and Z. Páles, "A fixed point approach to stability of functional equations," *Nonlinear Analysis: Theory, Methods & Applications A*, vol. 74, no. 17, pp. 6728–6732, 2011.

[18] J. Brzdęk and K. Ciepliński, "A fixed point approach to the stability of functional equations in non-Archimedean metric spaces," *Nonlinear Analysis: Theory, Methods & Applications A*, vol. 74, no. 18, pp. 6861–6867, 2011.

[19] L. Cădariu, L. Găvruţa, and P. Găvruţa, "Fixed points and generalized Hyers-Ulam stability," *Abstract and Applied Analysis*, vol. 2012, Article ID 712743, 10 pages, 2012.

[20] M. Piszczek, "Remark on hyperstability of the general linear equation," *Aequationes Mathematicae*, 2013.

[21] Pl. Kannappan, *Functional Equations and Inequalities with Applications*, Springer Monographs in Mathematics, Springer, New York, NY, USA, 2009.

[22] M. Fréchet, "Une definition fonctionnelles des polvnomes," *Nouvelles Annales de Mathematique. 4th série*, vol. 9, pp. 145–162, 1909.

[23] M. Fréchet, "Les polvnomes abstraits," *Journal de Mathematiques Pures et Appliquees. 9th série*, vol. 8, pp. 71–92, 1929.

[24] M. Kuczma, *An Introduction to the Theory of Functional Equations and Inequalities: Cauchy's Equation and Jensen's Inequality*, Birkhäuser, Basel, Switzerland, 2nd edition, 2009.

[25] H. Stetkaer, *Functional Equations on Groups*, World Scientific Publishing, Singapore, 2013.

[26] Y. H. Lee, "On the Hvers-Ulam-Rassias stability of the generalized polynomial function of degree 2," *Journal of the Chungcheong Mathematical Society*, vol. 22, no. 2, pp. 201–209, 2009.

[27] M. Albert and J. A. Baker, "Functions with bounded m-th differences," *Annales Polonici Mathematici*, vol. 43, no. 1, pp. 93–103, 1983.

[28] D. H. Hyers, "Transformations with bounded m-th differences," *Pacific Journal of Mathematics*, vol. 11, pp. 591–602, 1961.

[29] Y.-H. Lee, "On the stability of the monomial functional equation," *Bulletin of the Korean Mathematical Society*, vol. 45, no. 2, pp. 397–403, 2008.

[30] M. Chudziak, *On solutions and stability of functional equations connected to the Popoviciu inequality [Ph.D. thesis]*, Pedagogical University of Cracow, Cracow, Poland, 2012, (Polish).

[31] W. Fechner, "On the Hyers-Ulam stability of functional equations connected with additive and quadratic mappings," *Journal of Mathematical Analysis and Applications*, vol. 322, no. 2, pp. 774–786, 2006.

[32] S.-M. Jung, "On the Hyers-Ulam stability of the functional equations that have the quadratic property," *Journal of Mathematical Analysis and Applications*, vol. 222, no. 1, pp. 126–137, 1998.

[33] J. Sikorska, "On a direct method for proving the Hyers-Ulam stability of functional equations," *Journal of Mathematical Analysis and Applications*, vol. 372, no. 1, pp. 99–109, 2010.