Research Article

On a New Space $m^2(M, A, \phi, p)$ of Double Sequences

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1. Introduction

Throughout this work, $N$ and $C$ denote the set of positive integers and complex numbers, respectively. A complex double sequence is a function $x$ from $N \times N$ into $C$ and briefly denoted by $\{x_{k,l}\}$. Throughout this work, $w$ and $w^2$ denote the spaces of single complex sequences and double complex sequences, respectively. If, for all $e > 0$, there is $n_e \in N$ such that $\|x_{k,l} - a\|_\infty < e$ where $k > n_e$ and $l > n_e$, then a double sequence $\{x_{k,l}\}$ is said to be converging (in terms of Pringsheim) to $a \in C$. A real double sequence $\{x_{k,l}\}$ is nondecreasing, if $x_{k,l} \leq x_{k,l+1}$ for $(k, l) < (p, q)$. A double series is infinity sum $\sum_{k,l=1}^{\infty} x_{k,l}$, and its convergence implies the convergence by $|\cdot|$ of partial sums sequence $\{S_{n,m}\}$, where $S_{n,m} = \sum_{k=1}^{n} \sum_{l=1}^{m} x_{k,l}$ (see [1–4]).

A double sequence space $E$ is said to be solid if $\{x_{k,j}y_{k,j}\} \in E$ for all double sequences $\{y_{k,j}\}$ of scalars such that $\|y_{k,j}\| \leq 1$ for all $k, l \in N$ whenever $\{x_{k,j}\} \in E$.

Now let $\mathcal{S}$ be a family of subsets $\sigma$ having most elements $s$ in $N$. Also $\mathcal{S}_{s,t}$ denotes the class of subsets $\sigma = \sigma_1 \times \sigma_2$ in $N \times N$ such that the element numbers of $\sigma_1$ and $\sigma_2$ are most $s$ and $t$, respectively. Besides $\mathcal{S}_{s,t}$ is taken as a nondecreasing double sequence of the positive real numbers such that

\[ k\phi_{k+1} \leq (k + 1)\phi_k, \]
\[ l\phi_{k+1} \leq (l + 1)\phi_l. \]

An Orlicz function is a function $M : [0, \infty) \rightarrow [0, \infty)$ which is continuous, nondecreasing, and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$, and $M(x) \rightarrow \infty$ as $x \rightarrow \infty$.

An Orlicz function $M$ can always be represented in the following integral form: $M(x) = \int_0^x \eta(t)dt$, where $\eta$ is known as the kernel of $M$, is right differentiable for $t \geq 0$, $\eta(0) = 0$, $\eta(t) > 0$ for $t > 0$, $\eta$ is nondecreasing, and $\eta(t) \rightarrow \infty$ as $t \rightarrow \infty$.

An Orlicz function is said to be satisfied $\Delta_2$-condition for all values of $\eta$, if there exists a constant $T > 0$ such that $M(2u) \leq TM(u)$ for all $u \geq 0$.

Lindenstrauss and Tzafriri [5] used the idea of Orlicz functions to construct Orlicz sequence space

\[ \ell_M = \left\{ x = \{x_k\} \in w: \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty \right\}. \]

The sequence space $\ell_M$ is a Banach space according to the norm defined by

\[ \|x\| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}. \]

This space $(\ell_M, \|\|)$ is called an Orlicz sequence space. The space $\ell_M$ is closely related to the space $\ell_p$, which is an Orlicz sequence space with $M(x) = x^p$ for $1 \leq p < \infty$.

The double sequence spaces in the various forms defined by Orlicz functions were introduced and studied by Khan and Tabassum in [6–12] and by Khan et al. in [13].
The space $m(\phi)$, introduced by Sargent in [14], is in the form
\[
m(\phi) = \left\{ x = \{x_k\} \in \omega : \|x\|_{m(\phi)} \right\} = \sup_{\varepsilon > 0} \frac{1}{\varepsilon} \sum_{k \in \sigma} |x_k| < \infty .
\] (4)

Sargent studied some properties of this space and examined relationship between this space and $l_p$ space. Similar sequence classes were studied by many mathematicians using Orlicz functions (see [15–17]).

Later on, this space was investigated from sequencespace point of view by Rath [18], Rath and Tripathy [19], Tripathy and Sen [20], Tripathy and Mahanta [17], and others. Recently Altun and Bilgin [15] introduced and studied the following sequence classes were studied by many mathematicians using Orlicz functions (see [15–17]).

Let $A = (a_{ij})$ be an infinite matrix of complex numbers, $M$ an Orlicz function and $p = (p_i)$ a bounded sequence of positive real numbers such that $0 < h = \inf p_i \leq p_i \leq H = \sup p_i < \infty$. Then the space $m(M, A, \phi, p)$ is defined by
\[
m^2(M, A, \phi, p) = \left\{ x = \{x_{k, l}\} \in \omega : \sup_{\varepsilon > 0} \frac{1}{\varepsilon} \sum_{i, j}(A_{ij}(x))^\rho < \infty \right\}
\]
where $Ax = (A_i(x))$ if $A_i(x) = \sum_{k=1}^{\infty} a_{ik}x_k$ converges for each $i$.

Let $x = \{x_{k, l}\}$ be a double sequence. A set $S(x)$ is defined by
\[
S(x) = \left\{ \{x_{\pi_1(k), \pi_2(l)}\} : \pi_1\text{ and }\pi_2\text{ are permutations of }N \right\} .
\] (6)

If $S(x) \subseteq E$ for all $x \in E$, then $E$ is said to be symmetric.

In this work, we introduce the following sequence space. Let $A = (a_{ijk})$ be an infinite double matrix of complex numbers, $M$ an Orlicz function and $p = (p_{ij})$ bounded double sequence of positive real numbers such that $0 < h = \inf p_{ij} \leq p_{ij} \leq H = \sup p_{ij} < \infty$. Then the space $m^2(M, A, \phi, p)$ is defined by
\[
m^2(M, A, \phi, p) = \left\{ x = \{x_{k, l}\} \in \omega : \sup_{(s, t) \geq (1, 1), \sigma_1 \times \sigma_2 \in \mathcal{P}_{st}} \frac{1}{\sigma_1 \times \sigma_2} \sum_{i, j}(A_{ij}(x))^\rho < \infty \right\}
\] (7)

where $Ax = (A_{ij}(x))$ if $A_{ij}(x) = \sum_{k=1}^{\infty} a_{ijk}x_{k, l}$ converges for each $(i, j) \in N \times N$.

Also, we introduce and investigate the following space:
\[
m^2(M, \phi, p) = \left\{ x = \{x_{k, l}\} \in \omega^2 : \sup_{(s, t) \geq (1, 1), \sigma_1 \times \sigma_2 \in \mathcal{P}_{st}} \frac{1}{\sigma_1 \times \sigma_2} \sum_{i, j}(M(\phi_{ij}(x))^\rho < \infty \right\}
\] (8)

In this work, we also use the following sequence spaces:
\[
\begin{align*}
l^{2}(M, A, \phi, p)_0 &= \left\{ x = \{x_{k, l}\} \in \omega^2 : \sum_{i, j=1}^{\infty} M(\frac{A_{ij}(x))^\rho < \infty \right\} , \\
l^{2}(M, A, \phi, p)_\infty &= \left\{ x = \{x_{k, l}\} \in \omega^2 : \sup_{(s, t) \geq (1, 1), \sigma_1 \times \sigma_2 \in \mathcal{P}_{st}} \frac{1}{\sigma_1 \times \sigma_2} \sum_{i, j}(A_{ij}(x))^\rho < \infty \right\} .
\end{align*}
\] (9)

The following inequality will be used throughout this paper:
\[
|a + b|^{p_{\sigma}} \leq \max \{1, 2^{H-1} \} (|a|^{p_{\sigma}} + |b|^{p_{\sigma}} ) ,
\] (10)

where $a, b \in C$ and $H = \sup\{p_{\sigma} : (q, r) \in N \times N\}$.

2. Main Results

Definition 1. Let $K^{(2)}$ be a set of increasing positive integer binaries, namely, $(k_1, k_2) < (l_1, l_2)$ if and only if $k_1 < l_1$ and $k_2 < l_2$, and $E$ be a double sequence space. A $K^{(2)}$-set space is a double sequence space, defined by
\[
\Lambda_{K^{(2)}}^E = \left\{ x_{k,m} \in \omega^2 : x_{k,m} \in E, (k_m, l_m) \in K^{(2)} \right\} .
\] (11)

The canonical preimage of a double sequence $\{x_{k,m}\} \in \Lambda_{K^{(2)}}^E$ is a double sequence $\{y_{k,l}\} \in \omega^2$ with
\[
y_{k,l} = \begin{cases} x_{k,j}, & (k, l) \in K^{(2)} \\ 0, & (k, l) \notin K^{(2)} . \end{cases}
\] (12)
The canonical preimage of a set space $\lambda_{K}^E$ is a set of canonical preimages of all elements in $\lambda_{K}^E$.

**Definition 2.** If a double sequence space $E$ contains the canonical preimages of all set spaces, then $E$ is said to be monotone.

The following lemma is an easy result of the definitions.

**Lemma 3.** If a double sequence space $E$ is solid, then $E$ is monotone.

**Proposition 4.** The space $m^2(M,A,\phi,p)$ is a C-linear space.

**Proof.** Let $x = \{x_{kj}\}$, $y = \{y_{kj}\}$ be in $m^2(M,A,\phi,p)$ and $\alpha, \beta$ in $C \setminus \{0\}$. Then there exist positive numbers $\rho_1$ and $\rho_2$ such that

\[
\sup \left\{ \frac{1}{\phi_{s,t}} \sum_{i \in \sigma_1, j \in \sigma_2} M \left( \frac{A_{ij}(x)}{\rho_1} \right)^{p_0} : (s,t) \right\} \geq (1,1), \sigma_1 \times \sigma_2 \in \wp_{s,t} \] \leq \infty,

\[
\sup \left\{ \frac{1}{\phi_{s,t}} \sum_{i \in \sigma_1, j \in \sigma_2} M \left( \frac{A_{ij}(y)}{\rho_2} \right)^{p_0} : (s,t) \right\} \geq (1,1), \sigma_1 \times \sigma_2 \in \wp_{s,t} \] \leq \infty.

Let $\rho_3 = \max\{2|\alpha|\rho_1, 2|\beta|\rho_2\}$. Using that $M$ is nondecreasing convex function, we have

\[
\sum_{i \in \sigma_1, j \in \sigma_2} M \left( \frac{A_{ij}(ax + \beta y)}{\rho_3} \right)^{p_0} \leq \sum_{i \in \sigma_1, j \in \sigma_2} M \left( \frac{\sum_{k,l=1}^{\infty} (\alpha a_{ijk} x_{kl} + \beta a_{ijkl} y_{kl})}{\rho_3} \right)^{p_0}
\]

\[
\leq \sum_{i \in \sigma_1, j \in \sigma_2} M \left( \frac{\alpha A_{ij}(x)}{2|\alpha| \rho_1} + \frac{\beta A_{ij}(y)}{2|\beta| \rho_2} \right)^{p_0}
\]

\[
= \sum_{i \in \sigma_1, j \in \sigma_2} M \left( \frac{A_{ij}(x)}{2\rho_1} + \frac{A_{ij}(y)}{2\rho_2} \right)^{p_0}
\]

\[
\leq \max\left\{1, 2^H-1\right\} \left( \sum_{i \in \sigma_1, j \in \sigma_2} M \left( \frac{A_{ij}(x)}{\rho_1} \right)^{p_0} \right) + \sum_{i \in \sigma_1, j \in \sigma_2} M \left( \frac{A_{ij}(y)}{\rho_2} \right)^{p_0}.
\]

Thus we can write

\[
\sup \left\{ \frac{1}{\phi_{s,t}} \sum_{i \in \sigma_1, j \in \sigma_2} M \left( \frac{A_{ij}(ax + \beta y)}{\rho_3} \right)^{p_0} : (s,t) \right\} \geq (1,1), \sigma_1 \times \sigma_2 \in \wp_{s,t} \] \leq \max\left\{1, 2^H-1\right\} \times \left\{ \sup \left\{ \frac{1}{\phi_{s,t}} \sum_{i \in \sigma_1, j \in \sigma_2} M \left( \frac{A_{ij}(x)}{\rho_1} \right)^{p_0} : (s,t) \right\} \geq (1,1), \sigma_1 \times \sigma_2 \in \wp_{s,t} \right\} + \sup \left\{ \frac{1}{\phi_{s,t}} \sum_{i \in \sigma_1, j \in \sigma_2} M \left( \frac{A_{ij}(y)}{\rho_2} \right)^{p_0} : (s,t) \right\} \geq (1,1), \sigma_1 \times \sigma_2 \in \wp_{s,t} \right\} < \infty.

This shows that $\{\alpha x_{kj} + \beta y_{kj}\} \in m^2(M,A,\phi,p)$. Hence $m^2(M,A,\phi,p)$ is a linear space.

**Proposition 5.** The space $m^2(M,A,\phi,p)$ is a paranormed space with the paranorm

\[
g(x) = \inf \left\{ \rho^{p_{\phi}/H} : \left\{ \sup \left\{ \frac{1}{\phi_{s,t}} \sum_{i \in \sigma_1, j \in \sigma_2} M \left( \frac{A_{ij}(x)}{\rho} \right)^{p_0} : (s,t) \right\} \geq (1,1), \sigma_1 \times \sigma_2 \in \wp_{s,t} \right\} \leq 1 \right\}.
\]

**Proof.** It is clear that $g(x) = g(-x)$ and $g(x) = 0$ if $x = 0$. If there are $\rho_1 > 0$ and $\rho_2 > 0$ such that

\[
\sup \left\{ \frac{1}{\phi_{s,t}} \sum_{i \in \sigma_1, j \in \sigma_2} M \left( \frac{A_{ij}(x)}{\rho_1} \right)^{p_0} : (s,t) \right\} \geq (1,1), \sigma_1 \times \sigma_2 \in \wp_{s,t} \right\} \leq 1,
\]

\[
\sup \left\{ \frac{1}{\phi_{s,t}} \sum_{i \in \sigma_1, j \in \sigma_2} M \left( \frac{A_{ij}(y)}{\rho_2} \right)^{p_0} : (s,t) \right\} \geq (1,1), \sigma_1 \times \sigma_2 \in \wp_{s,t} \right\} \leq 1.
\]
then
\[
\sup \left\{ \frac{1}{\phi_{s,t}} \sum_{i \in \sigma_1} \sum_{j \in \sigma_2} M \left( \frac{A_{ij}(x+y)}{\rho_1 + \rho_2} \right)^{p_{ij}} : (s,t) \right\} \\
\geq (1,1), \sigma_1 \times \sigma_2 \in p_{s,t}
\]
\[
\leq \sup \left\{ \frac{1}{\phi_{s,t}} \sum_{i \in \sigma_1} \sum_{j \in \sigma_2} M \left( \frac{A_{ij}(x)}{\rho_1} + \frac{A_{ij}(y)}{\rho_1} \right)^{p_{ij}} : (s,t) \right\} \\
\geq (1,1), \sigma_1 \times \sigma_2 \in p_{s,t}
\]
\[
\leq \left( \frac{\rho_1}{\rho_1 + \rho_2} \right)^h \sup \left\{ \frac{1}{\phi_{s,t}} \sum_{i \in \sigma_1} \sum_{j \in \sigma_2} M \left( \frac{A_{ij}(x)}{\rho_1} \right)^{p_{ij}} : (s,t) \right\} \\
\geq (1,1), \sigma_1 \times \sigma_2 \in p_{s,t}
\]
\[
+ \left( \frac{\rho_2}{\rho_1 + \rho_2} \right)^h \sup \left\{ \frac{1}{\phi_{s,t}} \sum_{i \in \sigma_1} \sum_{j \in \sigma_2} M \left( \frac{A_{ij}(x)}{\rho_1} \right)^{p_{ij}} : (s,t) \right\} \\
\geq (1,1), \sigma_1 \times \sigma_2 \in p_{s,t},
\]
(18)
where \( h = \inf p_{ij} \). This shows that \( g(x+y) \leq g(x) + g(y) \).
Using this triangle inequality we can write
\[
g(\lambda^n x^n - \lambda x) \leq g(\lambda^n x^n - \lambda^n x) + g(\lambda^n x - \lambda x). 
\]
(19)
Separately, we obtain
\[
g(\lambda^n x^n - \lambda^n x) \\
= \inf \left\{ \rho_n^{p_{n/H}} : \sup \left\{ \frac{1}{\phi_{s,t}} \sum_{i \in \sigma_1} \sum_{j \in \sigma_2} M \left( \frac{A_{ij}(\lambda^n x^n - \lambda^n x)}{\rho_n} \right)^{p_{ij}} : (s,t) \right\} \\
\geq (1,1), \sigma_1 \times \sigma_2 \in p_{s,t}
\right\}^{1/H} \\
\leq 1 \leq 1, q \in N, r \in N \}
\]
= \max \left\{ |\lambda^n|^{p_{n/H}}, |\lambda^n| \right\} \cdot g(x^n - x),
\]
\[
g(\lambda^n x - \lambda x) \\
= \inf \left\{ \rho_n^{p_{n/H}} : \sup \left\{ \frac{1}{\phi_{s,t}} \sum_{i \in \sigma_1} \sum_{j \in \sigma_2} M \left( \frac{A_{ij}(\lambda^n - \lambda) x}{\rho_n} \right)^{p_{ij}} : (s,t) \right\} \\
\geq (1,1), \sigma_1 \times \sigma_2 \in p_{s,t}
\right\}^{1/H} \\
\leq 1, q \in N, r \in N \}
\]
\[
\inf \rho_n^{p \nu / H} : \left[ \sup \left\{ \frac{1}{\phi_{s,t}^{i \sigma_1, j \sigma_2}} \sum \sum M \left( \frac{|A_{ij}(x)|}{\rho_n^{i |\lambda^s - \lambda|}} \right)^{p_{ij}} : (s,t) \right\} \right]^{1/H} \geq (1,1), \sigma_1 \times \sigma_2 \in \varphi_{s,t} \}\]

\[
\leq \sup \left\{ \frac{1}{\phi_{s,t}^{i \sigma_1, j \sigma_2}} \sum \sum M \left( \frac{|A_{ij}(x)|}{\rho} \right)^{p_{ij}} : (s,t) \right\} \geq (1,1), \sigma_1 \times \sigma_2 \in \varphi_{s,t} \}
\]

This implies that \( \{\alpha_{s,t},y_{s,t}\} \in m^2(M,\phi,p) \), and hence the class \( m^2(M,\phi,p) \) is solid.

\[\square\]

**Corollary 7.** The space \( m^2(M,\phi,p) \) is monotone.

**Theorem 8.** Let \( \psi \) be another double sequence like \( \phi \). Then \( m^2(M,\lambda,\phi,p) \subseteq m^2(M,\lambda,\psi,p) \) if and only if \( \sup_{(s,t) \geq (1,1)} \phi_{s,t}^{\psi_{s,t}} < \infty \).

**Proof.** Let \( K = \sup_{(s,t) \geq (1,1)} \phi_{s,t}^{\psi_{s,t}} < \infty \). Then \( \phi_{s,t} \leq K \cdot \psi_{s,t} \) for all \( (s,t) \geq (1,1) \). If \( x = \{x_{s,t}\} \in m^2(M,\lambda,\phi,p) \), then

\[
\sup \left\{ \frac{1}{\phi_{s,t}^{i \sigma_1, j \sigma_2}} \sum \sum M \left( \frac{|A_{ij}(x)|}{\rho} \right)^{p_{ij}} : (s,t) \right\} \geq (1,1), \sigma_1 \times \sigma_2 \in \varphi_{s,t} \}
\]

for some \( \rho > 0 \). Thus

\[
\leq \sup \left\{ \frac{1}{K \psi_{s,t}^{i \sigma_1, j \sigma_2}} \sum \sum M \left( \frac{|A_{ij}(x)|}{\rho} \right)^{p_{ij}} : (s,t) \right\} \geq (1,1), \sigma_1 \times \sigma_2 \in \varphi_{s,t} \}
\]

for some \( \rho > 0 \), and hence \( x = \{x_{s,t}\} \in m^2(M,\lambda,\psi,p) \). This shows that \( m^2(M,\lambda,\phi,p) \subseteq m^2(M,\lambda,\psi,p) \).

Conversely, let \( m^2(M,\lambda,\phi,p) \subseteq m^2(M,\lambda,\psi,p) \). We say \( \alpha_{s,t} = \phi_{s,t}^{\psi_{s,t}} \) for all \( (s,t) \geq (1,1) \) and suppose \( \sup_{(s,t) \geq (1,1)} \alpha_{s,t} = \infty \). Then there exists a subsequence \( \{\alpha_{s,t}\} \) of \( \{\alpha_{s,t}\} \) such that \( \lim_{n \to \infty} \alpha_{s,t} = \infty \). If \( x = \{x_{s,t}\} \in m^2(M,\lambda,\phi,p) \), then we have

\[
\sup \left\{ \frac{1}{\phi_{s,t}^{i \sigma_1, j \sigma_2}} \sum \sum M \left( \frac{|A_{ij}(x)|}{\rho} \right)^{p_{ij}} : (s,t) \right\} \geq (1,1), \sigma_1 \times \sigma_2 \in \varphi_{s,t} \}
\]

for some \( \rho > 0 \), and hence \( x = \{x_{s,t}\} \in m^2(M,\lambda,\phi,p) \). This shows that \( m^2(M,\lambda,\phi,p) \subseteq m^2(M,\lambda,\psi,p) \).

Proposition 6. The class \( m^2(M,\phi,p) \) of double sequences is solid.

**Proof.** Let \( \alpha = \{\alpha_{s,t}\} \) be a double sequence of scalars such that \( |\alpha_{s,t}| \leq 1 \) and \( y = \{y_{s,t}\} \in m^2(M,\phi,p) \). Then we can write

\[
\sup \left\{ \frac{1}{\phi_{s,t}^{i \sigma_1, j \sigma_2}} \sum \sum M \left( \frac{|A_{ij}(x)|}{\rho} \right)^{p_{ij}} : (s,t) \right\} \geq (1,1), \sigma_1 \times \sigma_2 \in \varphi_{s,t} \}
\]

\[
\leq \sup \left\{ \frac{1}{\phi_{s,t}^{i \sigma_1, j \sigma_2}} \sum \sum M \left( \frac{|\alpha_{s,t} y_{s,t}|}{\rho} \right)^{p_{ij}} : (s,t) \right\} \geq (1,1), \sigma_1 \times \sigma_2 \in \varphi_{s,t} \}
\]
\[ \sum_{i \in \sigma_1, j \in \sigma_2} M \left( \frac{|A_{ij}(x)|}{\rho} \right)^{p_{ij}} : (s, t) \geq (1, 1), \sigma_1 \times \sigma_2 \in \varrho_{s,t} \right] \leq \max \left\{ 1, M(1)^H \right\} \times \sup \left\{ \frac{1}{\phi_{s,t}} \sum_{i \in \sigma_1, j \in \sigma_2} \sum_{y_{ij} \in \delta} \frac{M(y_{ij})^{p_{ij}}}{\rho} : (s, t) \right\} \geq (1, 1), \sigma_1 \times \sigma_2 \in \varrho_{s,t} \right\} . \]

Hence we get
\[ \sup \left\{ \frac{1}{\phi_{s,t}} \sum_{i \in \sigma_1, j \in \sigma_2} \sum_{y_{ij} \in \delta} \frac{M(y_{ij})^{p_{ij}}}{\rho} : (s, t) \right\} \geq (1, 1), \sigma_1 \times \sigma_2 \in \varrho_{s,t} \right\} , \]
Theorem 11. (a) \( l^{(2)}(M, A, \psi, p) \subseteq m^2(M, A, \psi, p) \subseteq l^{(2)}(M, A, \phi, \infty) \).

(b) \( m^2(M, A, \phi, p) = l^{(2)}(M, A, \phi, p) \) if and only if 
\[ \sup_{(s, t)}(s, t) \phi_{i,j} < \infty. \]

(c) \( m^2(M, A, \phi, p) = l^{(2)}(M, A, \phi, \infty) \) if and only if 
\[ \sup_{(s, t)}(s, t) \phi_{i,j}/s.t > 0. \]

Proof. (a) Let \( x = \{x_{k,l}\} \in l^{(2)}(M, A, \phi, p) \), and let a set \( B \) be defined as follows:
\[
B = \left\{ \frac{1}{\phi_{i,j}} \sum_{i,j=1}^\infty M \left( \frac{A_{ij}(x)}{\rho} \right)^{p_j} : (s, t) \right\}
\]
\[ \geq (1, 1), \sigma_1 \times \sigma_2 \in \varphi_{s,t} \}
\]
Since \( \{\phi_{i,j}\} \) is a nondecreasing double sequence, \( \{1/\phi_{i,j}\} \) is a nonincreasing double sequence. So we obtain
\[
\frac{1}{\phi_{i,1}} \sum_{i,j=1}^\infty M \left( \frac{A_{ij}(x)}{\rho} \right)^{p_j} \geq b,
\] for all \( b \in B \) and hence
\[
\infty > M \left( \frac{A_{ij}(x)}{\rho} \right)^{p_j} \geq \phi_{i,1} \cdot \sup B. \]
Thus we have \( l^{(2)}(M, A, \phi, p) \subseteq m^2(M, A, \phi, p) \).

Conversely if \( x = \{x_{k,l}\} \in m^2(M, A, \phi, p) \), then it is clear that
\[
\sup B \geq \frac{1}{\phi_{i,1}} M \left( \frac{A_{ij}(x)}{\rho} \right)^{p_j} \geq (s, t), \quad \forall (i, j) \in N \times N,
\]
and hence
\[
\sup B \geq \frac{1}{\phi_{i,1}} \sup_{(i,j) \in N \times N} M \left( \frac{A_{ij}(x)}{\rho} \right)^{p_j} \geq (s, t). \]
This shows that if \( x = \{x_{k,l}\} \in m^2(M, A, \phi, p) \), then \( \{x_{k,l}\} \in l^{(2)}(M, A, \phi, \infty) \). Thus we have \( m^2(M, A, \phi, p) \subseteq l^{(2)}(M, A, \phi, \infty) \).

(b) It is clear that \( m^2(M, A, \psi, p) = l^{(2)}(M, A, \phi, \infty) \) where \( \psi_{s,t} = 1 \) for all \( s, t \in N \times N \). Then we can write
\[
\sup_{(s, t)}(s, t) \psi_{i,j} \geq \sup_{(s, t)}(s, t) \phi_{i,j} \psi_{i,j} < \infty. \]
By Theorem 8, we have \( m^2(M, A, \phi, p) \subseteq l^{(2)}(M, A, \phi, p) \), and according to alternative (a)
\[
m^2(M, A, \phi, p) = l^{(2)}(M, A, \phi, p). \] (39)

(c) Firstly we show that \( m^2(M, A, \psi, p) = l^{(2)}(M, A, \phi, \infty) \) if \( \psi_{s,t} = s.t \) for all \( s, t \in N \times N \). Let \( x = \{x_{k,l}\} \in l^{(2)}(M, A, \phi, \infty) \). Then for \( (s, t) \geq (1, 1) \) and \( \sigma_1 \times \sigma_2 \in \varphi_{s,t} \), we can find some \( \rho > 0 \) such that
\[
\frac{1}{s.t \rho} \sum_{(i,j) \in \sigma_{s,t}} M \left( \frac{A_{ij}(x)}{\rho} \right)^{p_j} \leq \frac{1}{s.t \rho} \sup_{(i,j) \in N \times N} M \left( \frac{A_{ij}(x)}{\rho} \right)^{p_j} < \infty. \]
This gives the inclusion \( l^{(2)}(M, A, \phi, p) \subseteq m^2(M, A, \psi, p) \).

Conversely let \( x = \{x_{k,l}\} \in m^2(M, A, \psi, p) \). Then for \( (i, j) \in \sigma_{s,t} \), we can find some \( \rho > 0 \) such that
\[
M \left( \frac{A_{ij}(x)}{\rho} \right)^{p_j} \leq \frac{1}{s.t \rho} \sup_{(i,j) \in N \times N} M \left( \frac{A_{ij}(x)}{\rho} \right)^{p_j} < \infty. \]
This shows that \( m^2(M, A, \psi, p) \subseteq l^{(2)}(M, A, \phi, \infty) \). By Theorem 8 and alternative (a), we can write \( m^2(M, A, \phi, p) = l^{(2)}(M, A, \phi, p) \) if and only if \( \sup_{(s, t) \supseteq (1, 1)}(s, t) \phi_{i,j} < \infty \). This completes the proof. \[ \square \]

References


