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Research Article

On a New Space $m^2(M, A, \phi, p)$ of Double Sequences

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We introduce a new space $m^2(M, A, \phi, p)$ of double sequences related to p-absolute convergent double sequence space, combining an Orlicz function and an infinity double matrix. We study some properties of $m^2(M, A, \phi, p)$ and obtain some inclusion relations involving $m^2(M, A, \phi, p)$.

1. Introduction

Throughout this work, N and C denote the set of positive integers and complex numbers, respectively. A complex double sequence is a function x from $N \times N$ into C and briefly denoted by $\{x_{k,l}\}$. Throughout this work, w and w^2 denote the spaces of single complex sequences and double complex sequences, respectively. If, for all $\varepsilon > 0$, there is $n_\varepsilon \in N$ such that $\|x_{k,l} - a\|_X < \varepsilon$ where $k > n_\varepsilon$ and $l > n_\varepsilon$, then a double sequence $\{x_{k,l}\}$ is said to be converging (in terms of Pringsheim) to $a \in C$. A real double sequence $\{x_{k,l}\}$ is nondecreasing, if $x_{k,l} \le x_{p,q}$ for $(k, l) < (p, q)$. A double series is infinity sum $\sum_{k,l=1}^{\infty} x_{k,l}$, and its convergence implies the convergence by $|\cdot|$ of partial sums sequence $\{S_{n,m}\}$, where $S_{n,m} = \sum_{k=1}^{n} \sum_{l=1}^{m} x_{k,l}$ (see [1–4]).

A double sequence space E is said to be solid if $\{x_{k,l} y_{k,l}\} \in E$ for all double sequences $\{y_{k,l}\}$ of scalars such that $|y_{k,l}| \le 1$ for all $k, l \in N$ whenever $\{x_{k,l}\} \in E$.

Now let $\wp_s$ be a family of subsets σ having most elements s in N. Also $\wp_{s,t}$ denotes the class of subsets $\sigma = \sigma_1 \times \sigma_2$ in $N \times N$ such that the element numbers of σ_1 and σ_2 are most s and t, respectively. Besides $\{\phi_{k,l}\}$ is taken as a nondecreasing double sequence of the positive real numbers such that

$$k\phi_{k+1,l} \le (k+1)\phi_{k,l}, \qquad l\phi_{k,l+1} \le (l+1)\phi_{k,l}. \tag{1}$$

An Orlicz function is a function $M : [0, \infty) \to [0, \infty)$ which is continuous, nondecreasing, and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$, and $M(x) \to \infty$ as $x \to \infty$.

An Orlicz function M can always be represented in the following integral form: $M(x) = \int_0^x \eta(t)dt$, where η is known as the kernel of M, is right differentiable for $t \ge 0$, $\eta(0) = 0$, $\eta(t) > 0$ for $t > 0$, η is nondecreasing, and $\eta(t) \to \infty$ as $t \to \infty$.

An Orlicz function is said to be satisfied Δ_2-condition for all values of u, if there exists a constant $T > 0$ such that $M(2u) \le TM(u)$ for all $u \ge 0$.

Lindenstrauss and Tzafriri [5] used the idea of Orlicz functions to construct Orlicz sequence space

$$\ell_M = \left\{ x = \{x_k\} \in w : \sum_{k\ge1} M\left(\frac{|x_k|}{\rho}\right) < \infty \text{ for some } \rho > 0 \right\}. \tag{2}$$

The sequence space ℓ_M is a Banach space according to the norm defined by

$$\|x\| = \inf\left\{ \rho > 0 : \sum_{k\ge1} M\left(\frac{|x_k|}{\rho}\right) \le 1 \right\}. \tag{3}$$

This space $(\ell_M, \|\cdot\|)$ is called an Orlicz sequence space. The space ℓ_M is closely related to the space ℓ_p, which is an Orlicz sequence space with $M(x) = x^p$ for $1 \le p < \infty$.

The double sequence spaces in the various forms defined by Orlicz functions were introduced and studied by Khan and Tabassum in [6–12] and by Khan et al. in [13].

The space $m(\phi)$, introduced by Sargent in [14], is in the form

$$m(\phi) = \left\{ x = \{x_k\} \in w : \|x\|_{m(\phi)} = \sup_{s\geq 1, \sigma\in\wp_s} \frac{1}{\phi_s} \sum_{k\in\sigma} |x_k| < \infty \right\}. \quad (4)$$

Sargent studied some properties of this space and examined relationship between this space and l_p-space. Similar sequence classes were studied by many mathematicians using Orlicz functions (see [15–17]).

Later on, this space was investigated from sequence space point of view by Rath [18], Rath and Tripathy [19], Tripathy and Sen [20], Tripathy and Mahanta [17], and others. Recently Altun and Bilgin [15] introduced and studied the following sequence space $m(M, A, \phi, p)$.

Let $A = (a_{ik})$ be an infinite matrix of complex numbers, M an Orlicz space, and $p = (p_i)$ a bounded sequence of positive real numbers such that $0 < h = \inf p_i \leq p_i \leq H = \sup p_i < \infty$. Then the space $m(M, A, \phi, p)$ is defined by

$$m^2(M, A, \phi, p) = \left\{ \{x_k\} \in w : \sup_{s\geq 1, \sigma\in\wp_s} \frac{1}{\phi_s} \sum_{k\in\sigma} M\left(\frac{|A_i(x)|}{\rho}\right)^{p_i} < \infty \text{ for some } \rho > 0 \right\}, \quad (5)$$

where $Ax = (A_i(x))$ if $A_i(x) = \sum_{k=1}^{\infty} a_{ik}x_k$ converges for each i.

Let $x = \{x_{k,l}\}$ be a double sequence. A set $S(x)$ is defined by

$$S(x) = \left\{ \left\{ x_{\pi_1(k), \pi_2(k)} \right\} : \pi_1 \text{ and } \pi_2 \text{ are permutations of } N \right\}. \quad (6)$$

If $S(x) \subseteq E$ for all $x \in E$, then E is said to be symmetric.

In this work, we introduce the following sequence space.

Let $A = (a_{ijkl})$ be an infinite double matrix of complex numbers, M an Orlicz function and $p = (p_{ij})$ bounded double sequence of positive real numbers such that $0 < h = \inf p_{ij} \leq p_i \leq H = \sup p_{ij} < \infty$. Then the space $m^2(M, A, \phi, p)$ is defined by

$$m^2(M, A, \phi, p) = \left\{ x = (x_{kl}) \in w^2 : \sup \left\{ \frac{1}{\phi_{st}} \sum_{i\in\sigma_1} \sum_{j\in\sigma_2} M\left(\frac{|A_{ij}(x)|}{\rho}\right)^{p_{ij}} : (s,t) \geq (1,1), \sigma_1 \times \sigma_2 \in \wp_{st} \right\} < \infty, \text{for some } \rho > 0 \right\}, \quad (7)$$

where $Ax = (A_{ij}(x))$ if $A_{ij}(x) = \sum_{k,l=1}^{\infty} a_{ijkl}x_{kl}$ converges for each $(i, j) \in N \times N$.

Also, we introduce and investigate the following space:

$$m^2(M, \phi, p) = \left\{ x = (x_{kl}) \in w^2 : \sup \left\{ \frac{1}{\phi_{st}} \sum_{k\in\sigma_1} \sum_{l\in\sigma_2} M\left(\frac{|x_{k,l}|}{\rho}\right)^{p_{kl}} : (s,t) \geq (1,1), \sigma_1 \times \sigma_2 \in \wp_{st} \right\} < \infty \text{ for some } \rho > 0 \right\}. \quad (8)$$

In this work, we also use the following sequence spaces:

$$l^{(2)}(M, A, \phi, p)_0 = \left\{ x = \{x_{k,l}\} \in w^2 : \sum_{i,j=1}^{\infty} M\left(\frac{|A_{ij}(x)|}{\rho}\right)^{p_{ij}} < \infty \text{ for some } \rho > 0 \right\},$$
$$l^{(2)}(M, A, \phi, p)_\infty = \left\{ x = \{x_{k,l}\} \in w^2 : \sup_{(i,j)\in N\times N} M\left(\frac{|A_{ij}(x)|}{\rho}\right)^{p_{ij}} < \infty \text{ for some } \rho > 0 \right\}. \quad (9)$$

The following inequality will be used throughout this paper:

$$|a+b|^{p_{qr}} \leq \max\left(1, 2^{H-1}\right)\left(|a|^{p_{qr}} + |b|^{p_{qr}}\right), \quad (10)$$

where $a, b \in C$ and $H = \sup\{p_{qr} : (q,r) \in N \times N\}$.

2. Main Results

Definition 1. Let $K^{(2)}$ be a set of increasing positive integer binaries, namely, $(k_1, k_2) < (l_1, l_2)$ if and only if $k_1 < l_1$ and $k_2 < l_2$, and E be a double sequence space. A $K^{(2)}$-set space is a double sequence space, defined by

$$\lambda_{K^{(2)}}^E = \left\{ \left\{ x_{k_m, l_m} \right\} \in w^2 : \{x_{k,l}\} \in E, (k_m, l_m) \in K^{(2)} \right\}. \quad (11)$$

The canonical preimage of a double sequence $\{x_{k_m,l_m}\} \in \lambda_{K^{(2)}}^E$ is a double sequence $\{y_{k,l}\} \in w^2$ with

$$y_{k,l} = \begin{cases} x_{k,l}, & (k,l) \in K^{(2)} \\ 0, & (k,l) \notin K^{(2)}. \end{cases} \quad (12)$$

The canonical preimage of a set space $\lambda^E_{K^{(2)}}$ is a set of canonical preimages of all elements in $\lambda^E_{K^{(2)}}$.

Definition 2. If a double sequence space E contains the canonical preimages of all set spaces, then E is said to be monotone.

The following lemma is an easy result of the definitions.

Lemma 3. *If a double sequence space E is solid, then E is monotone.*

Proposition 4. *The space $m^2(M, A, \phi, p)$ is a C-linear space.*

Proof. Let $x = \{x_{k,l}\}$, $y = \{y_{k,l}\}$ be in $m^2(M, A, \phi, p)$ and α, β in $C \setminus \{0\}$. Then there exist positive numbers ρ_1 and ρ_2 such that

$$\begin{aligned}
&\sup\left\{\frac{1}{\phi_{s,t}}\sum_{i\in\sigma_1}\sum_{j\in\sigma_2}M\left(\frac{|A_{ij}(x)|}{\rho_1}\right)^{p_{ij}} : (s,t) \geq (1,1), \sigma_1\times\sigma_2\in\wp_{s,t}\right\} < \infty,\\
&\sup\left\{\frac{1}{\phi_{s,t}}\sum_{i\in\sigma_1}\sum_{j\in\sigma_2}M\left(\frac{|A_{ij}(y)|}{\rho_2}\right)^{p_{ij}} : (s,t) \geq (1,1), \sigma_1\times\sigma_2\in\wp_{s,t}\right\} < \infty.
\end{aligned} \tag{13}$$

Let $\rho_3 = \max\{2|\alpha|\rho_1, 2|\beta|\rho_2\}$. Using that M is nondecreasing convex function, we have

$$\begin{aligned}
&\sum_{i\in\sigma_1}\sum_{j\in\sigma_2}M\left(\frac{|A_{ij}(\alpha x+\beta y)|}{\rho_3}\right)^{p_{ij}}\\
&\quad\leq \sum_{i\in\sigma_1}\sum_{j\in\sigma_2}M\left(\frac{\left|\sum_{k,l=1}^{\infty}\left(\alpha a_{ijkl}x_{k,l}+\beta a_{ijkl}y_{k,l}\right)\right|}{\rho_3}\right)^{p_{ij}}\\
&\quad\leq \sum_{i\in\sigma_1}\sum_{j\in\sigma_2}M\left(\frac{|\alpha A_{ij}(x)|}{2|\alpha|\rho_1}+\frac{|\beta A_{ij}(y)|}{2|\beta|\rho_2}\right)^{p_{ij}}\\
&\quad= \sum_{k\in\sigma_1}\sum_{l\in\sigma_2}M\left(\frac{|A_{ij}(x)|}{2\rho_1}+\frac{|A_{ij}(y)|}{2\rho_2}\right)^{p_{ij}}\\
&\quad\leq \max\left(1,2^{H-1}\right)\left(\sum_{i\in\sigma_1}\sum_{i\in\sigma_2}M\left(\frac{|A_{ij}(x)|}{\rho_1}\right)^{p_{ij}}\right.\\
&\qquad\left.+\sum_{k\in\sigma_1}\sum_{l\in\sigma_2}M\left(\frac{|A_{ij}(y)|}{\rho_2}\right)^{p_{ij}}\right).
\end{aligned} \tag{14}$$

Thus we can write

$$\begin{aligned}
&\sup\left\{\frac{1}{\phi_{s,t}}\sum_{i\in\sigma_1}\sum_{j\in\sigma_2}M\left(\frac{|A_{ij}(\alpha x+\beta y)|}{\rho_3}\right)^{p_{ij}} : (s,t) \geq (1,1), \sigma_1\times\sigma_2\in\wp_{s,t}\right\}\\
&\quad\leq \max\left(1,2^{H-1}\right)\\
&\qquad\times\left\{\sup\left\{\frac{1}{\phi_{s,t}}\sum_{i\in\sigma_1}\sum_{j\in\sigma_2}M\left(\frac{|A_{ij}(x)|}{\rho_1}\right)^{p_{ij}} : (s,t) \geq (1,1), \sigma_1\times\sigma_2\in\wp_{s,t}\right\}\right.\\
&\qquad\left.+\sup\left\{\frac{1}{\phi_{s,t}}\sum_{k\in\sigma_1}\sum_{l\in\sigma_2}M\left(\frac{|A_{ij}(y)|}{\rho_2}\right)^{p_{ij}} : (s,t) \geq (1,1), \sigma_1\times\sigma_2\in\wp_{s,t}\right\}\right\} < \infty.
\end{aligned} \tag{15}$$

This shows that $\{\alpha x_{k,l} + \beta y_{k,l}\} \in m^2(M, A, \phi, p)$. Hence $m^2(M, A, \phi, p)$ is a linear space. □

Proposition 5. *The space $m^2(M, A, \phi, p)$ is a paranormed space with the paranorm*

$$\begin{aligned}
&g(x)\\
&= \inf\left\{\rho^{p_{qr}/H} : \left[\sup\left\{\frac{1}{\phi_{s,t}}\sum_{i\in\sigma_1}\sum_{j\in\sigma_2}M\left(\frac{|A_{ij}(x)|}{\rho}\right)^{p_{ij}} : (s,t) \geq (1,1), \sigma_1\times\sigma_2\in\wp_{s,t}\right\} \leq 1\right]^{1/H},\right.\\
&\qquad\left. q\in N, r\in N\right\}.
\end{aligned} \tag{16}$$

Proof. It is clear that $g(x) = g(-x)$ and $g(x) = 0$ if $x = 0$. If there are $\rho_1 > 0$ and $\rho_2 > 0$ such that

$$\begin{aligned}
&\sup\left\{\frac{1}{\phi_{s,t}}\sum_{i\in\sigma_1}\sum_{j\in\sigma_2}M\left(\frac{|A_{ij}(x)|}{\rho_1}\right)^{p_{ij}} : (s,t) \geq (1,1), \sigma_1\times\sigma_2\in\wp_{s,t}\right\} \leq 1,\\
&\sup\left\{\frac{1}{\phi_{s,t}}\sum_{i\in\sigma_1}\sum_{j\in\sigma_2}M\left(\frac{|A_{ij}(y)|}{\rho_2}\right)^{p_{ij}} : (s,t) \geq (1,1), \sigma_1\times\sigma_2\in\wp_{s,t}\right\} \leq 1,
\end{aligned} \tag{17}$$

then

$$\begin{aligned}
&\sup\left\{\frac{1}{\phi_{s,t}}\sum_{i\in\sigma_1}\sum_{j\in\sigma_2}M\left(\frac{\left|A_{ij}(x+y)\right|}{\rho_1+\rho_2}\right)^{p_{ij}}:(s,t)\right.\\
&\qquad\left.\geq(1,1),\sigma_1\times\sigma_2\in\wp_{s,t}\right\}\\
&\leq\sup\left\{\frac{1}{\phi_{s,t}}\sum_{i\in\sigma_1}\sum_{j\in\sigma_2}M\left(\frac{\left|A_{ij}(x)\right|}{\rho_1+\rho_2}+\frac{\left|A_{ij}(y)\right|}{\rho_1+\rho_2}\right)^{p_{ij}}:(s,t)\right.\\
&\qquad\left.\geq(1,1),\sigma_1\times\sigma_2\in\wp_{s,t}\right\}\\
&\leq\left(\frac{\rho_1}{\rho_1+\rho_2}\right)^h\sup\left\{\frac{1}{\phi_{s,t}}\sum_{i\in\sigma_1}\sum_{j\in\sigma_2}M\left(\frac{\left|A_{ij}(x)\right|}{\rho_1}\right)^{p_{ij}}:(s,t)\right.\\
&\qquad\left.\geq(1,1),\sigma_1\times\sigma_2\in\wp_{s,t}\right\}\\
&\quad+\left(\frac{\rho_2}{\rho_1+\rho_2}\right)^h\sup\left\{\frac{1}{\phi_{s,t}}\sum_{i\in\sigma_1}\sum_{j\in\sigma_2}M\left(\frac{\left|A_{ij}(x)\right|}{\rho_1}\right)^{p_{ij}}:(s,t)\right.\\
&\qquad\left.\geq(1,1),\sigma_1\times\sigma_2\in\wp_{s,t}\right\},
\end{aligned}\tag{18}$$

where $h=\inf p_{ij}$. This shows that $g(x+y)\leq g(x)+g(y)$. Using this triangle inequality we can write

$$g(\lambda^n x^n-\lambda x)\leq g(\lambda^n x^n-\lambda^n x)+g(\lambda^n x-\lambda x).\tag{19}$$

Separately, we obtain

$$\begin{aligned}
&g(\lambda^n x^n-\lambda^n x)\\
&=\inf\left\{\rho_n^{p_{qr}/H}:\right.\\
&\qquad\left[\sup\left\{\frac{1}{\phi_{s,t}}\sum_{i\in\sigma_1}\sum_{j\in\sigma_2}M\left(\frac{\left|A_{ij}(\lambda^n x^n-\lambda^n x)\right|}{\rho_n}\right)^{p_{ij}}:\right.\right.\\
&\qquad\qquad\left.\left.(s,t)\geq(1,1),\sigma_1\times\sigma_2\in\wp_{s,t}\right\}\right.\\
&\qquad\qquad\left.\left.\leq 1\right]^{1/H}\leq 1,q\in N,r\in N\right\}\\
&=\inf\left\{\rho_n^{p_{qr}/H}:\left[\sup\left\{\frac{1}{\phi_{s,t}}\sum_{i\in\sigma_1}\sum_{j\in\sigma_2}M\left(\frac{\left|A_{ij}(x^n-x)\right|}{(\rho_n/|\lambda^n|)}\right)^{p_{ij}}:\right.\right.\right.\\
&\qquad\qquad\left.\left.\left.(s,t)\geq(1,1),\sigma_1\times\sigma_2\in\wp_{s,t}\right\}\right.\right.\\
&\qquad\qquad\left.\left.\left.\leq 1\right]^{1/H}\leq 1,q\in N,r\in N\right\}\\
&=\inf\left\{(\lambda^n\rho_n)^{p_{qr}/H}:\right.\\
&\qquad\left[\sup\left\{\frac{1}{\phi_{s,t}}\sum_{i\in\sigma_1}\sum_{j\in\sigma_2}M\left(\frac{\left|A_{ij}(x^n-x)\right|}{\rho_n}\right)^{p_{ij}}:(s,t)\right.\right.\\
&\qquad\qquad\left.\left.\geq(1,1),\sigma_1\times\sigma_2\in\wp_{s,t}\right\}\right]^{1/H}\\
&\qquad\left.\leq 1,q\in N,r\in N\right\}\\
&\leq\max\left\{|\lambda^n|^{h/H},|\lambda^n|\right\}.\\
&\inf\left\{(|\lambda^n|\rho_n)^{p_{qr}/H}:\right.\\
&\qquad\left[\sup\left\{\frac{1}{\phi_{s,t}}\sum_{i\in\sigma_1}\sum_{j\in\sigma_2}M\left(\frac{\left|A_{ij}(x^n-x)\right|}{\rho_n}\right)^{p_{ij}}:(s,t)\right.\right.\\
&\qquad\qquad\left.\left.\geq(1,1),\sigma_1\times\sigma_2\in\wp_{s,t}\right\}\right]^{1/H}\\
&\qquad\left.\leq 1,q\in N,r\in N\right\}\\
&=\max\left\{|\lambda^n|^{h/H},|\lambda^n|\right\}\cdot g(x^n-x),\\
&g(\lambda^n x-\lambda x)\\
&=\inf\left\{\rho_n^{p_{qr}/H}:\right.\\
&\qquad\left[\sup\left\{\frac{1}{\phi_{s,t}}\sum_{i\in\sigma_1}\sum_{j\in\sigma_2}M\left(\frac{\left|A_{ij}((\lambda^n-\lambda)x)\right|}{\rho_n}\right)^{p_{ij}}:\right.\right.\\
&\qquad\qquad\left.\left.(s,t)\geq(1,1),\sigma_1\times\sigma_2\in\wp_{s,t}\right\}\right]^{1/H}\\
&\qquad\left.\leq 1,q\in N,r\in N\right\}
\end{aligned}$$

$$= \inf \left\{ \rho_n^{p_{qr}/H} : \left[\sup \left\{ \frac{1}{\phi_{s,t}} \sum_{i\in\sigma_1} \sum_{j\in\sigma_2} M\left(\frac{|A_{ij}(x)|}{(\rho_n/|\lambda^n - \lambda|)} \right)^{p_{ij}} : (s,t) \geq (1,1), \sigma_1 \times \sigma_2 \in \wp_{s,t} \right\} \right]^{1/H} \leq 1, q \in N, r \in N \right\}$$

$$= \inf \left\{ (|\lambda^n - \lambda| \rho_n)^{p_{qr}/H} : \left[\sup \left\{ \frac{1}{\phi_{s,t}} \sum_{i\in\sigma_1} \sum_{j\in\sigma_2} M\left(\frac{|A_{ij}(x)|}{\rho_n} \right)^{p_{ij}} : (s,t) \geq (1,1), \sigma_1 \times \sigma_2 \in \wp_{s,t} \right\} \right]^{1/H} \leq 1, q \in N, r \in N \right\}$$

$$\leq \max \left\{ |\lambda^n - \lambda|^{h/H}, |\lambda^n - \lambda| \right\} g(x). \tag{20}$$

Hence $g(\lambda^n x^n - \lambda x) \to 0$ where $\lambda^n \to \lambda$ and $x^n \to x$ as $n \to \infty$. Consequently g is a paranom on $m^2(M, A, \phi, p)$. □

Proposition 6. *The class $m^2(M, \phi, p)$ of double sequences is solid.*

Proof. Let $\alpha = \{\alpha_{k,l}\}$ be a double sequence of scalars such that $|\alpha_{k,l}| \leq 1$ and $y = \{y_{k,l}\} \in m^2(M, \phi, p)$. Then we can write

$$\sup \left\{ \frac{1}{\phi_{s,t}} \sum_{k\in\sigma_1} \sum_{l\in\sigma_2} M\left(\frac{|\alpha_{k,l} x_{k,l}|}{\rho} \right)^{p_{kl}} : (s,t) \geq (1,1), \sigma_1 \times \sigma_2 \in \wp_{s,t} \right\}$$

$$\leq \sup \left\{ \frac{1}{\phi_{s,t}} \sum_{k\in\sigma_1} \sum_{l\in\sigma_2} M\left(\frac{|\alpha_{k,l}| |y_{k,l}|}{\rho} \right)^{p_{kl}} : (s,t) \geq (1,1), \sigma_1 \times \sigma_2 \in \wp_{s,t} \right\}$$

$$\leq \sup \left\{ \frac{1}{\phi_{s,t}} \sum_{k\in\sigma_1} \sum_{l\in\sigma_2} M\left(\frac{|y_{k,l}|}{\rho} \right)^{p_{kl}} : (s,t) \geq (1,1), \sigma_1 \times \sigma_2 \in \wp_{s,t} \right\}. \tag{21}$$

This implies that $\{\alpha_{k,l} y_{k,l}\} \in m^2(M, \phi, p)$, and hence the class $m^2(M, \phi, p)$ is solid. □

Corollary 7. *The space $m^2(M, \phi, p)$ is monotone.*

Theorem 8. *Let ψ be another double sequence like ϕ. Then $m^2(M, A, \phi, p) \subseteq m^2(M, A, \psi, p)$ if and only if $\sup_{(s,t)\geq(1,1)}(\phi_{s,t}/\psi_{s,t}) < \infty$.*

Proof. Let $K = \sup_{(s,t)\geq(1,1)}(\phi_{s,t}/\psi_{s,t}) < \infty$. Then $\phi_{s,t} \leq K \cdot \psi_{s,t}$ for all $(s,t) \geq (1,1)$. If $x = \{x_{k,l}\} \in m^2(M, A, \phi, p)$, then

$$\sup \left\{ \frac{1}{\phi_{s,t}} \sum_{i\in\sigma_1} \sum_{j\in\sigma_2} M\left(\frac{|A_{ij}(x)|}{\rho} \right)^{p_{ij}} : (s,t) \geq (1,1), \sigma_1 \times \sigma_2 \in \wp_{s,t} \right\} < \infty, \tag{22}$$

for some $\rho > 0$. Thus

$$\sup \left\{ \frac{1}{K\psi_{s,t}} \sum_{k\in\sigma_1} \sum_{l\in\sigma_2} M\left(\frac{|A_{ij}(x)|}{\rho} \right)^{p_{ij}} : (s,t) \geq (1,1), \sigma_1 \times \sigma_2 \in \wp_{s,t} \right\} < \infty, \tag{23}$$

for some $\rho > 0$, and hence $x = \{x_{k,l}\} \in m^2(M, A, \psi, p)$. This shows that $m^2(M, A, \phi, p) \subseteq m^2(M, A, \psi, p)$.

Conversely, let $m^2(M, A, \phi, p) \subseteq m^2(M, A, \psi, p)$. We say $\alpha_{s,t} = \phi_{s,t}/\psi_{s,t}$ for all $(s,t) \geq (1,1)$ and suppose $\sup_{(s,t)\geq(1,1)} \alpha_{s,t} = \infty$. Then there exists a subsequence $\{\alpha_{s_i,t_i}\}$ of $\{\alpha_{s,t}\}$ such that $\lim_{i\to\infty} \alpha_{s_i,t_i} = \infty$. If $x = \{x_{k,l}\} \in m^2(M, A, \phi, p)$, then we have

$$\sup \left\{ \frac{1}{\psi_{s,t}} \sum_{i\in\sigma_1} \sum_{j\in\sigma_2} M\left(\frac{|A_{ij}(x)|}{\rho} \right)^{p_{ij}} : (s,t) \geq (1,1), \sigma_1 \times \sigma_2 \in \wp_{s,t} \right\}$$

$$= \sup \left\{ \alpha_{s,t} \frac{1}{\phi_{s,t}} \sum_{i\in\sigma_1} \sum_{j\in\sigma_2} M\left(\frac{|A_{ij}(x)|}{\rho} \right)^{p_{ij}} : (s,t) \geq (1,1), \sigma_1 \times \sigma_2 \in \wp_{s,t} \right\}$$

$$\geq \left\{\sup_{m\geq 1} \alpha_{s_m,t_m}\right\} \cdot \sup\left\{\frac{1}{\phi_{s_m,t_m}} \sum_{i\in\sigma_1}\sum_{j\in\sigma_2} M\left(\frac{|A_{ij}(x)|}{\rho}\right)^{p_{ij}} : \sigma_1\times\sigma_2 \in \wp_{s_m,t_m}\right\} = \infty. \tag{24}$$

This is a contradiction as $x = \{x_{k,l}\} \notin m^2(M, A, \phi, p)$. This completes the proof. □

Corollary 9. *$m^2(M, A, \phi, p) = m^2(M, A, \psi, p)$ if and only if $\sup_{(s,t)\geq(1,1)}\alpha_{s,t} < \infty$ and $\sup_{(s,t)\geq(1,1)}\alpha_{s,t}^{-1} < \infty$.*

Theorem 10. *Let M, M_1, and M_2 be Orlicz functions satisfying Δ_2-condition. Then*

(a) $m^2(M_1, A, \phi, p) \subseteq m^2(M \circ M_1, A, \phi, p)$,

(b) $m^2(M_1, A, \phi, p) \cap m^2(M_2, A, \phi, p) \subseteq m^2(M_1 + M_2, A, \phi, p)$.

Proof. (a) Let $x = \{x_{k,l}\} \in m^2(M_1, A, \phi, p)$. Then there exists $\rho > 0$ such that

$$\sup\left\{\frac{1}{\phi_{s,t}} \sum_{i\in\sigma_1}\sum_{j\in\sigma_2} M_1\left(\frac{|A_{ij}(x)|}{\rho}\right)^{p_{ij}} : (s,t) \geq (1,1), \sigma_1\times\sigma_2 \in \wp_{s,t}\right\} < \infty. \tag{25}$$

By the continuity of M, we select a number δ with $0 < \delta < 1$ such that $M(t) < \varepsilon$, whenever $0 \leq t < \delta$, for arbitrary $0 < \varepsilon < 1$. Now let $y_{i,j} = M_1(|A_{ij}(x)|/\rho)$. We can write

$$\sum_{i\in\sigma_1}\sum_{j\in\sigma_2} M\left(y_{i,j}\right)^{p_{ij}} = \& \sum_{y_{i,j}\leq\delta} M\left(y_{i,j}\right)^{p_{ij}} + \sum_{y_{i,j}>\delta} M\left(y_{i,j}\right)^{p_{ij}}. \tag{26}$$

By the properties of M, we have

$$\sum_{y_{i,j}\leq\delta} M\left(y_{i,j}\right)^{p_{ij}} \leq \max\left\{1, M(1)^H\right\} \sum_{y_{i,j}\leq\delta} \left(y_{i,j}\right)^{p_{ij}}. \tag{27}$$

Again we can write

$$M\left(y_{i,j}\right) < M\left(1 + \frac{y_{i,j}}{\delta}\right) < \frac{1}{2}M(2) + \frac{1}{2}M\left(\frac{2y_{i,j}}{\delta}\right), \tag{28}$$

for $y_{i,j} > \delta$. If we use that M satisfies Δ_2-condition, then we find

$$M\left(y_{i,j}\right) < \frac{1}{2}T\frac{y_{i,j}}{\delta}M(2) + \frac{1}{2}T\frac{y_{i,j}}{\delta}M(2) = T\frac{y_{i,j}}{\delta}M(2), \tag{29}$$

and so

$$\sum_{y_{i,j}>\delta} M\left(y_{i,j}\right)^{p_{ij}} \leq \max\left(1, \left(\frac{T}{\delta}M(2)\right)^H\right) \sum_{y_{i,j}>\delta} y_{i,j}. \tag{30}$$

Hence we get

$$\begin{aligned}
&\sup\left\{\frac{1}{\phi_{s,t}} \sum_{i\in\sigma_1}\sum_{j\in\sigma_2} M\left(M_1\left(\frac{|A_{ij}(x)|}{\rho}\right)\right)^{p_{ij}} : (s,t) \geq (1,1), \sigma_1\times\sigma_2 \in \wp_{s,t}\right\} \\
&\leq \max\left\{1, M(1)^H\right\} \times \sup\left\{\frac{1}{\phi_{s,t}} \sum_{i\in\sigma_1}\sum_{j\in\sigma_2} \left(y_{i,j}\right)^{p_{ij}} : (s,t) \geq (1,1), \sigma_1\times\sigma_2 \in \wp_{s,t}\right\} \\
&+ \max\left(1, \left(\frac{T}{\delta}M(2)\right)^H\right) \times \sup\left\{\frac{1}{\phi_{s,t}} \sum_{i\in\sigma_1}\sum_{j\in\sigma_2} \left(y_{i,j}\right)^{p_{ij}} : (s,t) \geq (1,1), \sigma_1\times\sigma_2 \in \wp_{s,t}\right\}.
\end{aligned} \tag{31}$$

Finally, we have $\{x_{k,l}\} \in m^2(M \circ M_1, \phi)$, and hence $m^2(M_1, \phi) \subseteq m^2(M \circ M_1, \phi)$.

(b) Let $\{x_{k,l}\} \in m^2(M_1, A, \phi, p) \cap m^2(M_2, A, \phi, p)$. Then there exists $\rho > 0$ such that

$$\begin{aligned}
&\sup\left\{\frac{1}{\phi_{s,t}} \sum_{i\in\sigma_1}\sum_{j\in\sigma_2} M_1\left(\frac{|A_{ij}(x)|}{\rho}\right)^{p_{ij}} : (s,t) \geq (1,1), \sigma_1\times\sigma_2 \in \wp_{s,t}\right\} < \infty, \\
&\sup\left\{\frac{1}{\phi_{s,t}} \sum_{i\in\sigma_1}\sum_{j\in\sigma_2} M_2\left(\frac{|A_{ij}(x)|}{\rho}\right)^{p_{ij}} : (s,t) \geq (1,1), \sigma_1\times\sigma_2 \in \wp_{s,t}\right\} < \infty.
\end{aligned} \tag{32}$$

By the inequality

$$\begin{aligned}
&\sum_{i\in\sigma_1}\sum_{j\in\sigma_2} (M_1 + M_2)\left(\frac{|A_{ij}(x)|}{\rho}\right)^{p_{ij}} \\
&\leq \max\left(1, 2^{H-1}\right) \sum_{i\in\sigma_1}\sum_{j\in\sigma_2} M_1\left(\frac{|A_{ij}(x)|}{\rho}\right)^{p_{ij}} \\
&+ \max\left(1, 2^{H-1}\right) \sum_{i\in\sigma_1}\sum_{j\in\sigma_2} M_2\left(\frac{|A_{ij}(x)|}{\rho}\right)^{p_{ij}},
\end{aligned} \tag{33}$$

we have $m^2(M_1, A, \phi, p) \cap m^2(M_2, A, \phi, p) \subseteq m^2(M_1 + M_2, A, \phi, p)$. □

Theorem 11. (a) $l^{(2)}(M, A, \phi, p) \subseteq m^2(M, A, \phi, p) \subseteq l^{(2)}(M, A, \phi, \infty)$.

(b) $m^2(M, A, \phi, p) = l^{(2)}(M, A, \phi, p)$ *if and only if* $\sup_{(s,t)\geq(1,1)}\phi_{s,t} < \infty$.

(c) $m^2(M, A, \phi, p) = l^{(2)}(M, A, \phi, \infty)$ *if and only if* $\sup_{(s,t)\geq(1,1)}(\phi_{s,t}/s.t) > 0$.

Proof. (a) Let $x = \{x_{k,l}\} \in l^{(2)}(M, A, \phi, p)$, and let a set B be defined as follows:

$$B = \left\{ \frac{1}{\phi_{s,t}} \sum_{i\in\sigma_1} \sum_{j\in\sigma_2} M\left(\frac{\left|A_{ij}(x)\right|}{\rho} \right)^{p_{ij}} : (s,t) \geq (1,1), \sigma_1 \times \sigma_2 \in \wp_{s,t} \right\}. \tag{34}$$

Since $\{\phi_{k,l}\}$ is a nondecreasing double sequence, $\{1/\phi_{k,l}\}$ is a nonincreasing double sequence. So we obtain

$$\frac{1}{\phi_{1,1}} \sum_{i,j=1}^{\infty} M\left(\frac{\left|A_{ij}(x)\right|}{\rho} \right)^{p_{ij}} \geq b, \tag{35}$$

for all $b \in B$ and hence

$$\infty > M\left(\frac{\left|A_{ij}(x)\right|}{\rho} \right)^{p_{ij}} \geq \phi_{1,1} \cdot \sup B. \tag{36}$$

Thus we have $l^{(2)}(M, A, \phi, p) \subseteq m^2(M, A, \phi, p)$.

Conversely if $x = \{x_{k,l}\} \in m^2(M, A, \phi, p)$, then it is clear that

$$\sup B \geq \frac{1}{\phi_{1,1}} M\left(\frac{\left|A_{ij}(x)\right|}{\rho} \right)^{p_{ij}}, \tag{37}$$

for all $(i, j) \in N \times N$, and hence

$$\sup B \geq \frac{1}{\phi_{1,1}} \sup_{(i,j)\in N\times N} M\left(\frac{\left|A_{ij}(x)\right|}{\rho} \right)^{p_{ij}}. \tag{38}$$

This shows that if $x = \{x_{k,l}\} \in m^2(M, A, \phi, p)$, then $\{x_{k,l}\} \in l^{(2)}(M, A, \phi, \infty)$. Thus we have $m^2(M, A, \phi, p) \subseteq l^{(2)}(M, A, \phi, \infty)$.

(b) It is clear that $m^2(M, A, \psi, p) = l^{(2)}(M, A, \phi, \infty)$ where $\psi_{s,t} = 1$ for all $(s, t) \in N \times N$. Then we can write $\sup_{(s,t)\leq(1,1)}\phi_{s,t} = \sup_{(s,t)\leq(1,1)}(\phi_{s,t}/\psi_{s,t}) < \infty$. By Theorem 8, we have $m^2(M, A, \phi, p) \subseteq m^2(M, A, \psi, p)$, and according to alternative (a)

$$m^2(M, A, \phi, p) = l^{(2)}(M, A, \phi, p). \tag{39}$$

(c) Firstly we show that $m^2(M, A, \psi, p) = l^{(2)}(M, A, \phi, \infty)$ if $\psi_{s,t} = s.t$ for all $(s, t) \in N \times N$. Let $x = \{x_{k,l}\} \in l^{(2)}(M, A, \phi, \infty)$. Then for $(s, t) \geq (1, 1)$ and $\sigma_1 \times \sigma_2 \in \wp_{s,t}$, we can find some $\rho > 0$ such that

$$\frac{1}{s.t} \sum_{i\in\sigma_1} \sum_{j\in\sigma_2} M\left(\frac{\left|A_{ij}(x)\right|}{\rho} \right)^{p_{ij}} \leq \frac{1}{s.t} s.t. \sup_{(i,j)\in N\times N} M\left(\frac{\left|A_{ij}(x)\right|}{\rho} \right)^{p_{ij}} < \infty. \tag{40}$$

This gives the inclusion $l^{(2)}(M, A, \phi, \infty) \subseteq m^2(M, A, \psi, p)$. Conversely let $x = \{x_{k,l}\} \in m^2(M, A, \psi, p)$. Then for $(i, j) \in N \times N$, we can find some $\rho > 0$ such that

$$\begin{aligned} M\left(\frac{\left|A_{ij}(x)\right|}{\rho} \right)^{p_{ij}} &= \frac{1}{s.t} s.t. M\left(\frac{\left|A_{ij}(x)\right|}{\rho} \right)^{p_{ij}} \\ &\leq \sup \left\{ \frac{1}{\phi_{s,t}} \sum_{i\in\sigma_1} \sum_{j\in\sigma_2} M\left(\frac{\left|A_{ij}(x)\right|}{\rho} \right)^{p_{ij}} : (s,t) \geq (1,1), \sigma_1 \times \sigma_2 \in \wp_{s,t} \right\} < \infty. \end{aligned} \tag{41}$$

This shows that $m^2(M, A, \psi, p) \subseteq l^{(2)}(M, A, \phi, \infty)$. By Theorem 8 and alternative (a), we can write $m^2(M, A, \phi, p) = l^{(2)}(M, A, \phi, \infty)$ if and only if $\sup_{(s,t)\geq(1,1)}(s.t/\phi_{s,t}) < \infty$. This completes the proof. □

References

[1] V. A. Khan, "On Δ_v^m-Cesáro summable double sequences," *Thai Journal of Mathematics*, vol. 10, no. 3, pp. 535–539, 2012.

[2] V. A. Khan and S. Tabassum, "Statistically pre-Cauchy double sequences," *Southeast Asian Bulletin of Mathematics*, vol. 36, no. 2, pp. 249–254, 2012.

[3] V. A. Khan, S. Tabassum, and A. Esi, "Statistically convergent double sequence spaces in n-normed spaces," *ARPN Journal of Science and Technology*, vol. 2, no. 10, pp. 991–995, 2012.

[4] B. V. Limaye and M. Zeltser, "On the Pringsheim convergence of double series," *Proceedings of the Estonian Academy of Sciences*, vol. 58, no. 2, pp. 108–121, 2009.

[5] J. Lindenstrauss and L. Tzafriri, "On Orlicz sequence spaces," *Israel Journal of Mathematics*, vol. 10, pp. 379–390, 1971.

[6] V. A. Khan and S. Tabassum, "Statistically convergent double sequence spaces in 2-normed spaces defined by Orlicz function," *Applied Mathematics*, vol. 2, no. 4, pp. 398–402, 2011.

[7] V. A. Khan and S. Tabassum, "On some new quasi almost Δ^m-lacunary strongly P-convergent double sequences defined by Orlicz functions," *Journal of Mathematics and Applications*, vol. 34, pp. 45–52, 2011.

[8] V. A. Khan and S. Tabassum, "On ideal convergent difference double sequence spaces in 2-normed spaces defined by Orlicz functions," *JMI International Journal of Mathematics and Applications*, vol. 1, no. 2, pp. 26–34, 2010.

[9] V. A. Khan and S. Tabassum, "Some vector valued multiplier difference double sequence spaces in 2-normed spaces defined by Orlicz functions," *Journal of Mathematical and Computational Science*, vol. 1, no. 1, pp. 126–139, 2011.

[10] V. A. Khan and S. Tabassum, "On some new double sequence spaces of invariant means defined by Orlicz functions," *Communications de la Faculté des Sciences de l'Université d'Ankara A1*, vol. 60, no. 2, pp. 11–21, 2011.

[11] V. A. Khan and S. Tabassum, "The strongly summable generalized difference double sequence spaces in 2-normed spaces defined by Orlicz functions," *Journal of Mathematical Notes*, vol. 7, no. 2, pp. 45–58, 2011.

[12] V. A. Khan and S. Tabassum, "On some new almost double Lacunary Δ^m-squence spaces dened by Orlicz functions," *Journal of Mathematical Notes*, vol. 6, no. 2, pp. 80–94, 2011.

[13] V. A. Khan, S. Tabassum, and A. Esi, "A_σ double sequence spaces and statistical convergence in 2-normed spaces dened by Orlicz functions," *Theory and Applications of Mathematics and Computer Science*, vol. 2, no. 1, pp. 61–71, 2012.

[14] W. L. C. Sargent, "Some sequence spaces related to the l_p spaces," *Journal of the London Mathematical Society*, vol. 35, pp. 161–171, 1960.

[15] Y. Altun and T. Bilgin, "On a new class of sequences related to the l_p space defined by Orlicz function," *Taiwanese Journal of Mathematics*, vol. 13, no. 4, pp. 1189–1196, 2009.

[16] A. Esi, "On a class of new type difference sequence spaces related to the space l_p," *Far East Journal of Mathematical Sciences*, vol. 13, no. 2, pp. 167–172, 2004.

[17] B. C. Tripathy and S. Mahanta, "On a class of sequences related to the l_p space defined by Orlicz functions," *Soochow Journal of Mathematics*, vol. 29, no. 4, pp. 379–391, 2003.

[18] D. Rath, "Spaces of r-convex sequences and matrix transformations," *Indian Journal of Mathematics*, vol. 41, no. 2, pp. 265–280, 1999.

[19] D. Rath and B. C. Tripathy, "Characterization of certain matrix operators," *Orissa Mathematical Society*, vol. 8, pp. 121–134, 1989.

[20] B. C. Tripathy and M. Sen, "On a new class of sequences related to the space l_p," *Tamkang Journal of Mathematics*, vol. 33, no. 2, pp. 167–171, 2002.