Research Article

Nonlinear Kato Class and Unique Continuation of Eigenfunctions for p -Laplacian Operator

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We study some basic properties of nonlinear Kato class $M_p(\mathbb{R}^n)$ and $\widetilde{M}_p(\mathbb{R}^n)$, respectively, for $1 < p < n$. Also, we study the problem $-\operatorname{div}(|\nabla u|^{p-2}\nabla u) + V|u|^{p-2}u = 0$ in Ω , where Ω is a bounded domain in $\mathbb{R}^n$ and the weight function V is assumed to be not equivalent to zero and lies in $\widetilde{M}_p(\Omega)$, in the case where $p < n$. Finally, we establish the strong unique continuation property of the eigenfunction for the p -Laplacian operator in the case where $V \in \widetilde{M}_p(\Omega)$.

1. Introduction

The Kato class K_n was introduced and studied by Aizenman and Simon (see [1]). For $n \geq 3$, it consists of locally integrable functions f on $\mathbb{R}^n$, such that

$$\lim_{r \rightarrow 0} \sup_{x \in \mathbb{R}^n} \int_{B(x,r)} \frac{|f(y)|}{|x-y|^{n-2}} dy = 0. \quad (1)$$

For $1 < p < n$, the following classes were defined by Zamboni (see [2]): the class $\widetilde{M}_p$ of functions f , such that

$$\sup_{x \in \mathbb{R}^n} \left\{ \int_{B(x,r)} \frac{1}{|x-y|^{n-1}} \left(\int_{B(x,r)} \frac{|f(z)|}{|y-z|^{n-1}} dz \right)^{1/(p-1)} dy \right\}^{p-1} < \infty, \quad (2)$$

and the class M_p of functions f such that $f \in \widetilde{M}_p$ and

$$\lim_{r \rightarrow 0} \sup_{x \in \mathbb{R}^n} \left\{ \int_{B(x,r)} \frac{1}{|x-y|^{n-1}} \times \left(\int_{B(x,r)} \frac{|f(z)|}{|y-z|^{n-1}} dz \right)^{1/(p-1)} dy \right\}^{p-1} = 0. \quad (3)$$

Section 2 of the present paper is devoted to the study of some basic properties of the nonlinear Kato class $M_p(\mathbb{R}^n)$ and $\widetilde{M}_p(\mathbb{R}^n)$, respectively, for $1 < p < n$.

Among other things, we show that the Lorentz space $L(n/2, 1)$ is embedded into $M_p(\mathbb{R}^n)$ (see Lemma 10) as well as that $\widetilde{M}_p(\mathbb{R}^n)$ is a complete topological vector space (see Remark 11 and Lemma 13).

The p -Laplacian operator is a generalization of the Laplace operator, where p is allowed to range over $1 < p < \infty$;

in our case where $1 < p < n$, it is written as

$$\begin{aligned} & \operatorname{div}(|\nabla u|^{p-2} \nabla u) \\ &= \nabla \cdot (|\nabla u|^{p-2} \nabla u) \\ &= |\nabla u|^{p-4} \left(|\nabla u|^2 \Delta u + (p-2) \sum_{i,j=1}^n \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \frac{\partial^2 u}{\partial x_i \partial x_j} \right), \end{aligned} \quad (4)$$

where

$$\nabla u = \left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n} \right), \quad |\nabla u| = \left(\sum_{i=1}^n \left(\frac{\partial u}{\partial x_i} \right)^2 \right)^{1/2}, \quad (5)$$

and $u \in C_0^\infty(\Omega)$, with Ω bounded domain in $\mathbb{R}^n$.

We are concerned with the following problem:

$$-\operatorname{div}(|\nabla u|^{p-2} \nabla u) + V|u|^{p-2} u = 0, \quad \text{in } \Omega, \quad (6)$$

and the weight function V is assumed to be not equivalent to zero and lies in $\widetilde{M}_p(\mathbb{R}^n)$ in the case $p < n$.

Specifically, we are interested in studying a family of functions which enjoys the strong unique continuation property, that is, functions besides the possible zero functions which have zero of infinite order.

Definition 1. We say that a function $u \in L_{\text{loc}}^p(\Omega)$ vanishes of infinite order at point x_0 if for any natural number N there exists a constant C_N , such that

$$\int_{B(x_0, r)} |u(x)|^p dx \leq C_N r^N, \quad (7)$$

for all $N \in \mathbb{N}$ and for small positive number r . Here,

$$B(x_0, r) = \{y \in \mathbb{R}^n : |y - x_0| < r\}. \quad (8)$$

Definition 2. We say that (6) has a strong unique continuation property if and only if any solution u of (6) in Ω is identically zero in Ω provided that u vanishes of infinite order at a point in Ω .

There is an extensive literature on unique continuation. We refer to the work of Zamboni on unique continuation for nonnegative solutions of quasilinear elliptic equation [3], also the work of Jerison-Kenig on the unique continuation for Schrödinger operators [4]. The same work is done by Chiarenza and Frasca, but for linear elliptic operator in the case where $V \in L^{n/2}$ when $n > 2$ [5].

Let us recall some known results concerning Fefferman's inequality as follows:

$$\int_{\mathbb{R}^n} |u(x)|^p |V(x)| dx \leq C \int_{\mathbb{R}^n} |\nabla u(x)|^p dx \quad \forall u \in C_0^\infty(\mathbb{R}^n). \quad (9)$$

In [6], de Figueiredo and Gossez prove (9) in the case where $p = 2$, assuming that $V \in L^{r, n-2r}(\mathbb{R}^n)$ with $1 < r \leq n/2$. Later in [7], Jerison and Kening showed the same result taking V in the Stummel-Kato class $S(\mathbb{R}^n)$. We point out that it is not possible to compare the assumptions $V \in L^{r, n-2r}(\mathbb{R}^n)$ and $f \in S(\mathbb{R}^n)$. Chiarenza and Frasca [5] generalized Fefferman's result proving (9) under the assumption that $V \in L^{r, n-2r}(\mathbb{R}^n)$ with $r \in (1, n/p)$ and $p \in (1, n)$. In [3], Schechter gave a new proof of (9) assuming that $V \in \widetilde{M}_p(\mathbb{R}^n)$.

2. Definitions and Notation

In this section, we gather definitions and notations that will be used throughout the paper. We also include several simple lemmas. By $L_{\text{loc}}^1(\mathbb{R}^n)$, we will denote the space of functions which are locally integrable on $\mathbb{R}^n$, and by $L_{\text{loc}, u}^1$ the space of functions f , such that

$$\sup_{x \in \mathbb{R}^n} \int_{B(x, 1)} |f(y)| dy < \infty. \quad (10)$$

Definition 3. Let $f \in L_{\text{loc}}^1(\mathbb{R}^n)$. For any $1 < p < n$ and $r > 0$, we set

$$\begin{aligned} & \Phi(r) \\ &= \sup_{x \in \mathbb{R}^n} \left(\int_{B(x, r)} \frac{1}{|x - y|^{n-1}} \left(\int_{B(x, r)} \frac{|f(z)| dz}{|z - y|^{n-1}} \right)^{1/(p-1)} dy \right)^{p-1}, \end{aligned} \quad (11)$$

where $B(x, r) = \{y : |x - y| < r\}$. We say that f belongs to the space $\widetilde{M}_p(\mathbb{R}^n)$ if $\Phi(r) < \infty$ for all $r > 0$.

Definition 4. We say that a function $f \in M_p(\mathbb{R}^n)$ if

$$\lim_{r \rightarrow 0} \Phi(r) = 0. \quad (12)$$

We are now ready to formulate some simple properties of the classes M_p and $\widetilde{M}_p$.

Lemma 5 (see [3], page 152). For $1 < p < n$, one has

$$(i) \quad M_p(\mathbb{R}^n) \subset \widetilde{M}_p(\mathbb{R}^n),$$

$$(ii) \quad M_2(\mathbb{R}^n) = K_n.$$

From Lemma 5 we conclude that both $M_p(\mathbb{R}^n)$ and $\widetilde{M}_p(\mathbb{R}^n)$ are generalizations of K_n .

Remark 6. The following example shows that K_n is properly contained in $M_p(\mathbb{R}^n)$ for $p > 2$. It is known that the function

$f(x) = |x|^{-2}$ is not in the Kato class K_n . However, $f \in M_p$. Indeed,

$$\lim_{r \rightarrow 0} \sup_x \left\{ \int_{B(x,r)} \frac{1}{|x-y|^{n-2}} \times \left(\int_{B(x,r)} \frac{dz}{|z|^2 |z-y|^{n-1}} \right)^{1/(p-1)} dy \right\}^{p-1} = 0. \quad (13)$$

This can be shown by splitting the domain of integration in the interior integral into the following three parts: $B(x,r) \cap \{|z| < (1/2)|y|\}$, $B(x,r) \cap \{(1/2)|y| \leq |z| \leq (3/2)|y|\}$, and $B(x,r) \cap \{|z| > (3/2)|y|\}$.

After routine calculations, we can see that

$$\int_{B(x,r)} \frac{dz}{|z|^2 |z-y|^{n-1}} \quad (14)$$

is majorized by $C|y|^{-1}$. Finally, we have

$$C \sup_x \left\{ \int_{B(x,r)} \frac{dy}{|y|^{1/(p-1)} |x-y|^{n-1}} \right\}^{p-1} \rightarrow 0 \quad \text{as } r \rightarrow 0. \quad (15)$$

This shows that (13) holds. Thus, $f \in \bigcap_{p>2} M_p$.

Definition 7. The distribution function D_f of a measurable function f is given by

$$D_f(\lambda) = m(\{x \in \mathbb{R}^n : |f(x)| > \lambda\}), \quad (16)$$

where m denotes the Lebesgue measure on $\mathbb{R}^n$. The distribution function D_f provides information about the size of f but not about the behavior of f itself near any given point. For instance, a function on $\mathbb{R}^n$ and each of its translates have the same distribution function. It follows from Definition 7 that D_f is a decreasing function of λ (not strictly necessary).

Definition 8. Let f be a measurable function in $\mathbb{R}^n$. The decreasing rearrangement of f is the function f^* defined on $[0, \infty)$ by

$$f^*(t) = \inf \{ \lambda : D_f(\lambda) \leq t \} \quad (t \geq 0). \quad (17)$$

We use here the convention that $\inf \emptyset = \infty$.

Definition 9 (Lorentz space). Let f be a measurable function; we say that f belongs to $L_{(n/2,1)}$ if

$$\|f\|_{(n/2,1)} = \int_0^\infty t^{2/n-1} f^*(t) dt < \infty. \quad (18)$$

And it belongs to $L(n/(n-2), \infty)$ if

$$\|f\|_{(n/(n-2),\infty)} = \sup_{t>1} t^{1-2/n} f^*(t) < \infty. \quad (19)$$

Lemma 10. Consider $L_{(n/2,1)} \subset M_p(\mathbb{R}^n)$.

Proof. Let $f \in L_{(n/2,1)}$; then

$$\int_0^\infty t^{2/n-1} f^*(t) dt < \infty. \quad (20)$$

Since $|f| \chi_{B(x,r)} \leq |f|$, we have

$$(|f| \chi_{B(x,r)})^*(t) \leq f^*(t); \quad (21)$$

then,

$$\int_0^\infty t^{2/n-1} (|f| \chi_{B(x,r)})^*(t) dt \leq \int_0^\infty t^{2/n-1} f^*(t) dt < \infty. \quad (22)$$

Thus, $|f| \chi_{B(x,r)} \in L_{(n/2,1)}$.

On the other hand, let $g(x) = |x|^{-(1-n)}$; then

$$\begin{aligned} m(\{x : |g(x)| > \lambda\}) &= m(\{x : |x|^{-(1-n)} > \lambda\}), \\ m\left(\left\{x : |x| < \left(\frac{1}{\lambda}\right)^{1/(n-1)}\right\}\right) &= C_n \left(\frac{1}{\lambda}\right)^{n/(n-1)}, \end{aligned} \quad (23)$$

where $C_n = m(B(0,1))$.

Next, we set $t = C_n(1/\lambda)^{n/(n-1)}$, and then $\lambda = C_n t^{1/n-1}$. Thus, $g^*(t) = C_n t^{1/n-1}$. From this, we obtain

$$\begin{aligned} \|g\|_{(n/(n-2),\infty)} &= \left\| \frac{1}{|\cdot|^{n-1}} \right\|_{(n/(n-2),\infty)} \\ &= \sup_{t>1} C_n t^{1-2/n} t^{1/n-1} = \sup_{t>1} C_n t^{-1/n} \\ &\leq C_n < \infty, \end{aligned} \quad (24)$$

which means that $g \in L_{(n/(n-2),\infty)}$. Finally, by Fubini's theorem and Hölder's inequality, we have

$$\begin{aligned} \phi(r) &= \sup_{x \in \mathbb{R}^n} \left(\int_{B(x,r)} \frac{1}{|x-y|^{n-1}} \times \left(\int_{B(x,r)} \frac{|f(z)|}{|y-z|^{n-1}} dz \right)^{1/(p-1)} dy \right)^{p-1} \\ &\leq \sup_{x \in \mathbb{R}^n} \left(\int_{B(x,r)} \frac{1}{|x-y|^{n-1}} \times \left(\int_{B(x,2r)} \frac{|f(z)|}{|y-z|^{n-1}} dz \right)^{1/(p-1)} dy \right)^{p-1} \\ &= \sup_{x \in \mathbb{R}^n} \left(\int_{B(x,r)} \frac{1}{|x-y|^{n-1}} \times \left(\int_{\mathbb{R}^n} \frac{|f(z)| \chi_{B(0,2r)}(y-z) dz}{|y-z|^{n-1}} \right)^{1/(p-1)} dy \right)^{p-1} \end{aligned}$$

$$\begin{aligned}
&\leq \sup_{x \in \mathbb{R}^n} \left(\int_{B(x,r)} \frac{dy}{|x-y|^{n-1}} \right)^{p-1} \\
&\quad \times \|f\chi_{B(0,2r)}\|_{(n/2,1)} \left\| \frac{1}{|\cdot|^{n-1}} \right\|_{(n/(n-2),\infty)} \\
&= C_n r^{p-1} \|f\chi_{B(0,2r)}\|_{(n/2,1)} \rightarrow 0 \quad \text{as } r \rightarrow 0,
\end{aligned} \tag{25}$$

which means that $f \in M_p(\mathbb{R}^n)$ and the proof is complete. $\square$

Remark 11. (i) For $0 < r < 1$, it is not hard to check that for $1 < p \leq 2$, the expression

$$\begin{aligned}
&\|f\|_{\widetilde{M}_p(\mathbb{R}^n)} \\
&= \sup_{x \in \mathbb{R}^n} \left(\int_{B(x,1)} \frac{1}{|x-y|^{n-1}} \right. \\
&\quad \times \left. \left(\int_{B(x,1)} \frac{|f(z)|}{|z-y|^{n-1}} dz \right)^{1/(p-1)} dy \right)^{p-1}
\end{aligned} \tag{26}$$

defines a norm on $\widetilde{M}_p(\mathbb{R}^n)$.

(ii) For $p > 2$, the expression (26) satisfies the following inequality:

$$\|f + g\|_{\widetilde{M}_p(\mathbb{R}^n)} \leq 2^{p-2} (\|f\|_{\widetilde{M}_p(\mathbb{R}^n)} + \|g\|_{\widetilde{M}_p(\mathbb{R}^n)}) \tag{27}$$

for all f and g in $\widetilde{M}_p(\mathbb{R}^n)$.

If U is a neighborhood of 0 from (27), we have

$$2^{p-1}U + 2^{p-1}U \subset U; \tag{28}$$

then, $\widetilde{M}_p(\mathbb{R}^n)$ is a topological vector space.

Lemma 12. Consider $\widetilde{M}_p(\mathbb{R}^n) \subset L_{\text{loc},u}^1(\mathbb{R}^n)$ for $1 < p < n$.

Proof. Let $f \in \widetilde{M}_p(\mathbb{R}^n)$, and fix $r_0 > 0$. Then, there exists a positive constant C such that $\Phi(r_0) \leq C$. It follows that

$$\begin{aligned}
&\sup_{x \in \mathbb{R}^n} \left(\int_{B(x,r_0)} \frac{1}{|x-y|^{n-1}} \left(\int_{B(x,r_0)} \frac{|f(z)|}{|z-y|^{n-1}} dz \right)^{1/(p-1)} dy \right)^{p-1} \\
&\geq \sup_{x \in \mathbb{R}^n} \left(\int_{B(x,r_0)} \frac{dy}{r_0^{n-1}} \left(\int_{B(x,r_0)} \frac{|f(z)|}{(2r_0)^{n-1}} dz \right)^{1/(p-1)} \right)^{p-1} \\
&\geq \sup_{x \in \mathbb{R}^n} \left(\frac{1}{2r_0} \right)^{n-1} \left(\frac{m(B(x,r_0))}{r_0^{n-1}} \right)^{p-1} \int_{B(x,r_0)} |f(z)| dz.
\end{aligned} \tag{29}$$

Therefore,

$$\sup_{x \in \mathbb{R}^n} \int_{B(x,r_0)} |f(z)| dz < BC, \tag{30}$$

where

$$B = (2r_0)^{n-1} (r_0 m(B(0,1)))^{p-1}. \tag{31}$$

Finally, let $B(x,1) \subseteq \bigcup_{k=1}^n B(x_k, r_0)$; then

$$\sup_{x \in \mathbb{R}^n} \int_{B(x,1)} |f(z)| dz \leq \sum_{k=1}^n \sup_{x \in \mathbb{R}^n} \int_{B(x_k, r_0)} |f(z)| dz, \tag{32}$$

so

$$\sup_{x \in \mathbb{R}^n} \int_{B(x,1)} |f(z)| dz < \infty. \tag{33}$$

Therefore,

$$\widetilde{M}_p(\mathbb{R}^n) \subset L_{\text{loc},u}^1(\mathbb{R}^n). \tag{34}$$

$\square$

Lemma 13. For $1 < p < n$, $\widetilde{M}_p(\mathbb{R}^n)$ is a complete space.

Proof. Let $\{f_n\}_{n \in \mathbb{N}}$ be a Cauchy sequence in

$$\overline{B}(0,r) = \{f \in \widetilde{M}_p(\mathbb{R}^n) : \|f\|_{\widetilde{M}_p(\mathbb{R}^n)} \leq r\}. \tag{35}$$

By Lemma 10, $\{f_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in $L_{\text{loc},u}^1(\mathbb{R}^n)$. Since this space is complete, there exists a function $f \in L_{\text{loc},u}^1(\mathbb{R}^n)$ such that $f_n \rightarrow f$ in $L_{\text{loc},u}^1(\mathbb{R}^n)$. By Fatous's lemma, we have

$$\|f\|_{\widetilde{M}_p(\mathbb{R}^n)} \leq \liminf \|f_n\|_{\widetilde{M}_p(\mathbb{R}^n)} \leq r. \tag{36}$$

Thus, $f \in \overline{B}(0,r)$, which means that $\overline{B}(0,r)$ is complete with respect to the topology generated by $L_{\text{loc},u}^1(\mathbb{R}^n)$ -norm. By Corollary 2 of Proposition 9 in [8, Chapter III, Section 3, no. 5] we obtain the assertion. $\square$

Lemma 14. If $1 < p < n$, then $M_p(\mathbb{R}^n)$ is closed in $\widetilde{M}_p(\mathbb{R}^n)$.

Proof. Let us define the map $\varphi : \widetilde{M}_p(\mathbb{R}^n) \rightarrow [0, \infty)$ by $\varphi(f) = \lim_{r \rightarrow 0} \phi_f(r)$ (see Definition 3).

It is not hard to prove that the family $\{\varphi_r\}_{r>0}$ where $\varphi_r(f) = \phi_f(r)$ is equicontinuous and $\varphi_r \rightarrow \varphi$ pointwise as $r \rightarrow 0$. Since $M_p(\mathbb{R}^n) = \varphi^{-1}(0)$, we obtain the result. $\square$

For more details on nonlinear Kato class, we refer the readers to [9].

3. Some Useful Inequalities

For the sake of completeness and convenience of the reader, we include the proof of the next result which is due to Schechter [3].

Theorem 15. Assume that $V \in \widetilde{M}_p(\mathbb{R}^n)$. Then, for any $r > 0$ there exists a positive constant $C(n, p)$, such that

$$\int_{\mathbb{R}^n} |V(x)| |u(x)|^p dx \leq C(n, p) \Phi(2r) \int_{\mathbb{R}^n} |\nabla u(x)|^p dx, \tag{37}$$

for any $u \in C_0^\infty(\mathbb{R}^n)$ supported in $B(x_0, r)$.

Proof. For any $u \in C_0^\infty(\mathbb{R}^n)$ supported in $B(x_0, r)$, using the well-known inequality

$$|u(x)| \leq C(n, p) \int_{B(x_0, r)} \frac{|\nabla u(y)|}{|x - y|^{n-1}} dy, \quad (38)$$

Fubini's theorem, and Hölder's inequality, we obtain

$$\begin{aligned} & \int_{\mathbb{R}^n} |V(x)| |u(x)|^p dx \\ &= \int_{B(x_0, r)} |V(x)| |u(x)|^p dx \\ &\leq C(n, p) \int_{B(x_0, r)} |V(x)| |u(x)|^{p-1} \\ &\quad \times \left(\int_{B(x_0, r)} \frac{|\nabla u(y)|}{|x - y|^{n-1}} dy \right) dx \\ &\leq C(n, p) \int_{B(x_0, r)} |\nabla u(y)| \\ &\quad \times \left(\int_{B(x_0, r)} |V(x)| |u(x)|^{p-1} \frac{1}{|x - y|^{n-1}} dx \right) dy \\ &\leq C(n, p) \left(\int_{B(x_0, r)} |\nabla u(y)|^p dy \right)^{1/p} \\ &\quad \times \left[\int_{B(x_0, r)} \left(\int_{B(x_0, r)} |V(x)| |u(x)|^{p-1} \right. \right. \\ &\quad \left. \left. \times \frac{1}{|x - y|^{n-1}} dx \right)^{p/(p-1)} dy \right]. \end{aligned} \quad (39)$$

On the other hand, using Hölder's inequality one more time, we have

$$\begin{aligned} & \int_{B(x_0, r)} \left(\int_{B(x_0, r)} |V(x)| |u(x)|^{p-1} \frac{1}{|x - y|^{n-1}} dx \right)^{p/(p-1)} dy \\ &\leq \int_{B(x_0, r)} \left(\int_{B(x_0, r)} \frac{|V(z)|}{|z - y|^{n-1}} dz \right)^{1/(p-1)} \\ &\quad \times \int_{B(x_0, r)} \frac{|V(x)| |u(x)|^p}{|x - y|^{n-1}} dx dy \\ &= \int_{B(x_0, r)} |V(x)| |u(x)|^p \\ &\quad \times \int_{B(x_0, r)} \frac{1}{|x - y|^{n-1}} \left(\int_{B(x_0, r)} \frac{|V(z)|}{|z - y|^{n-1}} dz \right)^{1/(p-1)} dy dx \\ &\leq [\Phi(2r)]^{1/(p-1)} \int_{B(x_0, r)} |V(x)| |u(x)|^p dx. \end{aligned} \quad (40)$$

By (39) and (40), we obtain

$$\begin{aligned} & \int_{B(x_0, r)} |V(x)| |u(x)|^p dx \\ &\leq C(n, p) [\Phi(2r)]^{1/(p-1)} \left(\int_{B(x_0, r)} |\nabla u(y)|^p dy \right)^{1/p} \\ &\quad \times \left(\int_{B(x_0, r)} |V(x)| |u(x)|^p dx \right)^{1-1/p}. \end{aligned} \quad (41)$$

Hence

$$\begin{aligned} & \int_{B(x_0, r)} |V(x)| |u(x)|^p dx \\ &\leq C(n, p) [\Phi(2r)]^{q/p} \int_{B(x_0, r)} |\nabla u(x)|^p dx. \end{aligned} \quad (42)$$

□

The next corollary is an easy consequence of the previous theorem. It can be obtained via a standard partition of unity.

Corollary 16. Let $V \in \widetilde{M}_p(\mathbb{R}^n)$ and let Ω be a bounded subset of $\mathbb{R}^n$, $\text{supp } V \subseteq \Omega$. Then, for any $\sigma > 0$ there exists a positive constant K depending on σ , such that

$$\begin{aligned} & \int_{\Omega} |V(x)| |u(x)|^p dx \\ &\leq \sigma \int_{\Omega} |\nabla u(x)|^p dx + K(\sigma) \int_{\Omega} |u(x)|^p dx, \end{aligned} \quad (43)$$

for all $u \in C_0^\infty(\Omega)$.

Proof. Let $\sigma > 0$. Let r be a positive number that will be chosen later. Let $\{\alpha_k^p\}$, $k = 1, 2, \dots, N(r)$, be a finite partition of the unity of $\overline{\Omega}$, such that $\text{supp } V \subseteq B(x_k, r)$ with $x_k \in \overline{\Omega}$. We apply Theorem 15 to the functions α_k and we get

$$\begin{aligned} & \int_{\Omega} |V(x)| |u(x)|^p dx \\ &= \int_{\Omega} |V(x)| |u(x)|^p \sum_{k=1}^{N(r)} \alpha_k^p(x) dx \\ &= \sum_{k=1}^{N(r)} \int_{\Omega} |V(x)| |u(x) \alpha_k^p(x)|^p dx \\ &\leq \sum_{k=1}^{N(r)} C\Phi_V(2r) \left(\int_{\Omega} |\nabla u(x)|^p \alpha_k^p(x) dx \right. \\ &\quad \left. + \int_{\Omega} |\nabla u(x)|^p |u(x)|^p dx \right) \\ &\leq C\Phi_V(2r) \left(\int_{\Omega} |\nabla u(x)|^p dx + \frac{N(r)}{r^p} \int_{\Omega} |u(x)|^p dx \right). \end{aligned} \quad (44)$$

Finally, to obtain the result, it is sufficient to choose r such that $C\Phi_V(2r) = \sigma$. After that, we note that $N(r) \approx r^{-n}$ and the corollary follows. □

Lemma 17. Let B_r and B_{2r} be two concentric balls contained in Ω . Then,

$$\int_{B_r} |\nabla u(x)|^p dx \leq \frac{C}{r^p} \int_{B_{2r}} |u|^p, \quad (45)$$

where the constant C does not depend on r and $u \in C_0^\infty(B_r)$.

Proof. Take $\varphi \in C_0^\infty(\Omega)$, with $\text{supp } \varphi \subset B_{2r}$, $\varphi(x) = 1$ for $x \in B_r$ and $|\nabla \varphi| \leq C/r$ using φ^p as a test function in (6); we get

$$\int_{B_{2r}} -\text{div}(|\nabla u|^{p-2} \nabla u) \varphi^p u + \int_{B_{2r}} V |u|^{p-2} u \varphi^p u = 0. \quad (46)$$

Thus,

$$\begin{aligned} & \int_{B_{2r}} |\nabla u|^p \varphi^p \\ &= -p \int_{B_{2r}} |\nabla u|^{p-2} \varphi^{p-2} \nabla u \cdot \nabla \varphi (\varphi u) - \int_{B_{2r}} V |\varphi u|^p. \end{aligned} \quad (47)$$

Using Young's inequalities for $(p-1)/p + 1/p = 1$, we can estimate the first integral in the right-hand side of (47) by

$$(p-1) \varepsilon^{p/(p-1)} \int_{B_{2r}} |\nabla u|^p \varphi^p + \varepsilon^{-p} \int_{B_{2r}} |\nabla \varphi|^p |u|^p. \quad (48)$$

Also by result of Corollary 16, we can estimate the second integral in the right-hand side of (47) by

$$\varepsilon \int_{B_{2r}} |\nabla(\varphi u)|^p + C_\varepsilon \int_{B_{2r}} |\varphi u|^p. \quad (49)$$

Using these estimates in (47), we have

$$\begin{aligned} & \int_{B_{2r}} |\nabla u|^p \varphi^p \\ & \leq ((p-1) \varepsilon^{p/(p-1)} + \varepsilon) \int_{B_{2r}} |\nabla u|^p \varphi^p \\ & \quad + (\varepsilon^p + \varepsilon) \int_{B_{2r}} |u|^p |\nabla u|^p + C_\varepsilon \int_{B_{2r}} |\nabla u|^p |\varphi|^p. \end{aligned} \quad (50)$$

Using the fact that $|\nabla \varphi| \leq C/r$, $|\varphi| \leq C/r$, and $\varphi = 1$ in B_r , we immediately have inequality (45). $\square$

Lemma 18. Let $u \in C_0^\infty(B_r)$ where B_r is the ball of radius r in $\mathbb{R}^n$ and let $E = \{x \in B_r : u(x) = 0\}$. Then, there exists a constant β depending only on n , such that

$$\int_A |u| \leq \beta \frac{r^n}{m(E)} [m(A)]^{1/n} \int_{B_{2r}} |\nabla u|, \quad (51)$$

for all ball B_r , u as above, and all measurable sets $A \subset B_r$.

To prove this lemma see [5]. Note that $m(A)$ and $m(E)$ denote the Lebesgue measure of the sets A and E .

4. Strong Unique Continuation

In this section, we proceed to establish the strong unique continuation property of the eigenfunction for the p -Laplacian operator in the case $V \in \widetilde{M}_p(\Omega)$.

Theorem 19. Let $u \in C_0^\infty(\Omega)$ be a solution of (6). If $u = 0$ on a set E of positive measures, then u has zero of infinite order in p -mean.

Proof. We know that almost every point of E is a point of density of E . Let $x_0 \in E$ be such point. This means that

$$\lim_{r \rightarrow 0} \frac{m(E \cap B_r)}{m(B_r)} = 1, \quad (52)$$

where B_r denotes the ball of radius r centered at x_0 , and thus, given that $\varepsilon > 0$, there is an $r_0 = r_0(\varepsilon)$, such that

$$\frac{m(E^c \cap B_r)}{m(B_r)} < \varepsilon, \quad \frac{m(E \cap B_r)}{m(B_r)} > 1 - \varepsilon \quad \text{for } r \leq r_0, \quad (53)$$

where E^c denotes the complement of the set E . Taking r_0 , smaller if necessary, we can assume that $B_{r_0} \subset \Omega$. Since $u = 0$ on E , by Lemmas 18 and 11, we have

$$\begin{aligned} \int_{B_r} |u|^p &= \int_{B_r \cap E^c} |u|^p \leq \beta \frac{r^n}{m(E \cap B_r)} [m(E^c \cap B_r)]^{1/n} \\ &\quad \times \int_{B_r} |\nabla(u)|^p, \end{aligned} \quad (54)$$

$$p\beta \frac{r^n \varepsilon^{1/n}}{[m(B_r)]^{1-1/n} (1-\varepsilon)} \int_{B_r} |u|^{p-1} |\nabla(u)|.$$

By Hölder inequality

$$\int_{B_r} |u|^p \leq C \frac{\varepsilon^{1/n}}{1-\varepsilon} \left(\int_{B_r} |\nabla(u)|^p \right)^{1/p} \left(\int_{B_r} |u|^p \right)^{(p-1)/p}, \quad (55)$$

and by using the Young inequality, we get

$$\int_{B_r} |u|^p \leq C \frac{\varepsilon^{1/n}}{1-\varepsilon} r \left(r^{p-1} \int_{B_r} |\nabla(u)|^p + \frac{p-1}{r} \int_{B_r} |u|^p \right). \quad (56)$$

Finally, by Lemma 17, we have

$$\int_{B_r} |u|^p \leq C \frac{\varepsilon^{1/n}}{1-\varepsilon} \int_{B_{2r}} |u|^p, \quad (57)$$

where C is independent of ε and of r , as $r \rightarrow 0$. Now, let us introduce the following function:

$$f(r) = \int_{B_{2r}} |u|^p. \quad (58)$$

And let us fix $n \in \mathbb{N}$ and choose $\varepsilon > 0$ such that $(C\varepsilon^{1/n})(1-\varepsilon) \leq 2^{-n}$. Observe that consequently r_0 depends on n . Then, (57) can be written as

$$f(r) \leq 2^{-n} f(2r), \quad \text{for } r \leq r_0. \quad (59)$$

Iterating (59), we get

$$f(\rho) \leq 2^{-kn} f(2^k \rho), \quad \text{if } 2^{k-1} \rho \leq r_0. \quad (60)$$

Now, given that $0 < r < r_0(n)$, choose $k \in \mathbb{N}$, such that

$$2^{-k} r_0 \leq r \leq 2^{-k+1} r_0. \quad (61)$$

From (60), we obtain

$$f(r) \leq 2^{-kn} f(2^k r) \leq 2^{-kn} f(2r). \quad (62)$$

Since $2^{-k} \leq r/r_0$, we finally obtain

$$f(r) \leq \left(\frac{r}{r_0}\right)^n f(2r_0). \quad (63)$$

And thus, we have

$$\int_{B_r(x_0)} |u(y)|^p dy \leq \left(\frac{r}{r_0}\right)^n f(2r_0), \quad (64)$$

and this shows that (7) holds, which means that u has a zero of infinite order in p -mean at x_0 . $\square$

Corollary 20. Equation (6) has a strong unique continuation property.

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