Research Article

New Proof for Balian-Low Theorem of Nonlinear Gabor System

D. H. Yuan,¹ S. Z. Yang,² X. W. Zheng,² and Y. F. Shen²

¹ Department of Mathematics, Hanshan Normal University, Chaozhou, Guangdong 521041, China

² Department of Mathematics, Shantou University, Shantou, Guangdong 515063, China

Correspondence should be addressed to D. H. Yuan; dhyuan@hstc.edu.cn

Received 27 July 2013; Revised 24 September 2013; Accepted 25 September 2013

Academic Editor: T. S. S. R. K. Rao

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The main purpose of this paper is to give a new proof of the Balian-Low theorem for Gabor system $\{e^{im\theta(2\pi t)}g(t-n), m, n \in \mathbb{Z}\}$, which is proposed by Fu et al. and associated with nonlinear Fourier atoms. To this end, we establish the relationships between spaces $L^2(\mathbb{R}, d\theta)$ and $L^2(\mathbb{R})$. We also introduce the concept of frame associated with nonlinear Fourier atoms for $L^2(\mathbb{R}, d\theta)$ and obtain many subsidiary results for this kind of (Gabor) frames.

1. Introduction

Note that the classical Fourier atoms $e^{2\pi i n t}$ cannot expose the time-varying property of nonstationary signals [1]. Recently, a kind of specific nonlinear phase function $\theta_a(2\pi t)$ is introduced in [2–6]. Note that, for different a , the shapes of $\cos\theta_a(2\pi t)$ (also those of $\sin\theta_a(2\pi t)$) are different. It is observed that the closer a gets to 1, the sharper the graph of $\cos\theta_a(2\pi t)$ is. This means that the nontrivial Harmonic waves $e^{im\theta_a(2\pi t)}$ can represent a conformal rescaling of classic Fourier atoms. Thus, the nontrivial Harmonic waves are expected to be better suitable and adaptable, along with different choices of a , to capture nonstationary features of band-limited signals. In fact, Ren et al. [7] obtained some new phenomena on the Shannon sampling theorem by dealing with sampling points which are nonequally distributed.

Motivated by these points, Fu et al. [8] considered a newly Gabor system $\{e^{im\theta(2\pi t)}g(t-n), m, n \in \mathbb{Z}\}$ generated by a function g , where $\theta(t)$ satisfies certain assumptions. Note that they were not restricted to the conformal phase functions $\theta(t)$ in their discussion. This freedom allows us to choose phase functions adequate to the necessary nonuniform sampling of the signal [7]. Using the Zak transform technique, they established the Balian-Low theorem for this newly Gabor system.

We point out that the Gabor system $\{e^{im\theta(2\pi t)}g(t-n), m, n \in \mathbb{Z}\}$ proposed by [8] can be related to already existing

cases. A particular case of this kind of Gabor system is the nonlinear Fourier atoms $e^{im\theta_a(2\pi t)}$ which was discussed in [2–6]. Using the nonlinear Fourier atoms in [2–6], we have that the frequency modulation $e^{im\theta_a(2\pi t)}$ represents a conformal dilation of the classical modulation $e^{im2\pi t}$ on the unit circle. Taking the proposed Gabor systems with different parameters a , we can obtain a dictionary of Gabor frames with different dilation parameters in the modulation part. A simple change of variables can establish a clear relation between this system and the system generated by the affine Weyl-Heisenberg group with dilation on the window function [9, 10].

Basing on these points, we can say that establishing relationships between frames for $L^2(\mathbb{R}, d\theta)$ and $L^2(\mathbb{R})$ is an interesting issue. In this paper, our main purpose is to give a different proof for the Balian-Low theorem proposed in [8]. For this purpose, we firstly establish the relationships between spaces $L^2(\mathbb{R}, d\theta)$ and $L^2(\mathbb{R})$. Basing on this relationship, we obtain many properties for general frame system $\{e^{im\theta(2\pi t)}g_n(t), m, n \in \mathbb{Z}\}$ and its special case $\{e^{im\theta(2\pi t)}g(t-n), m, n \in \mathbb{Z}\}$, where θ is a nonlinear function. With these results for general Gabor system $\{e^{im\theta(2\pi t)}g(t-n), m, n \in \mathbb{Z}\}$, we give a new and simple proof for the Balian-Low theorem proposed in [8].

The rest of the paper is organized as follows. Section 2 is devoted to giving some notations and lemmas. In Section 3, we establish the relationship between spaces $L^2(\mathbb{R}, d\theta)$ and $L^2(\mathbb{R})$; we also depict some properties of general frame

$\{e^{im\theta(2\pi t)}g_n(t), m, n \in \mathbb{Z}\}$ for $L^2(\mathbb{R}, d\theta)$. In Section 4, we establish the relationship between Gabor frame $\{e^{im\theta(2\pi t)}g(t-n), m, n \in \mathbb{Z}\}$ for $L^2(\mathbb{R}, d\theta)$ and classical one $\{e^{im\pi t}g^\theta(t-n), m, n \in \mathbb{Z}\}$ for $L^2(\mathbb{R})$ under some assumptions on θ ; further, a new and simple proof is presented for the Balian-Low theorem which was proposed by Fu et al. [8].

2. Notations

In this section, we present some notations and lemmas, which will be needed in the rest of the paper.

For an arbitrary measure θ in $\mathbb{R}$, consider the space $L^2(\mathbb{R}, d\theta)$ of square integrable functions in $\mathbb{R}$ with respect to θ and the finite norm:

$$\|f\|_\theta = \left(\frac{1}{\theta(2\pi) - \theta(0)} \int_{-\infty}^{\infty} |f(x)|^2 d\theta(2\pi x) \right)^{1/2} \quad (1)$$

induced by the inner product

$$\langle f, g \rangle_\theta := \frac{1}{\theta(2\pi) - \theta(0)} \int_{-\infty}^{\infty} f(x) \overline{g(x)} d\theta(2\pi x). \quad (2)$$

To obtain the Balian-Low theorem for Gabor system $\{e^{im\theta(2\pi t)}g(t-n), m, n \in \mathbb{Z}\}$, Fu et al. introduced some assumptions including the Assumptions 2.1 and 2.2 in [8] for a nonlinear phase function θ . Combining these two assumptions together, we obtain the following Assumption 1.

Assumption 1. Let function $\theta : \mathbb{R} \rightarrow \mathbb{R}$ be a measure on $\mathbb{R}$ and satisfy

$$\theta(x + 2k\pi) = \theta(x) + 2k\pi, \quad (3)$$

for any $x \in \mathbb{R}$ and $k \in \mathbb{Z}$. Further, $\theta'(x) > 0$ for all $x \in \mathbb{R}$.

Note that $\theta'(x) > 0$ for all $x \in \mathbb{R}$; one obtains that the inverse of θ (denote by θ^{-1}) exists. Moreover, it is obvious to check that θ satisfies

$$\theta^{-1}(x + 2k\pi) = \theta^{-1}(x) + 2k\pi, \quad (4)$$

for any $x \in \mathbb{R}$ and $k \in \mathbb{Z}$. In fact, we obtain from (3) that

$$\theta^{-1}(\theta(x) + 2k\pi) = \theta^{-1}(\theta(x)) + 2k\pi, \quad (5)$$

or

$$x + 2k\pi = \theta^{-1}(\theta(x) + 2k\pi). \quad (6)$$

Replacing $\theta(x)$ and x by t and $\theta^{-1}(t)$ in (6), respectively, we obtain (4).

For a function f defined in $\mathbb{R}$, denote by

$$f^\theta(t) := f\left(\frac{1}{2\pi}\theta^{-1}(2\pi t)\right) \quad (7)$$

through the rest of paper.

For $a, b \in \mathbb{R}$, consider the translation operator $(T_a g)(x) = g(x - a)$ and the modulation operator $(E_b^\theta g)(x) = e^{ib\theta(2\pi x)}g(x)$, both acting on $L^2(\mathbb{R}, d\theta)$. In [8], Fu et al. proposed a general Gabor frame for $L^2(\mathbb{R}, d\theta)$. We say that the system $\{E_m^\theta T_n g, m, n \in \mathbb{Z}\}$ is a general Gabor frame for $L^2(\mathbb{R}, d\theta)$ if there exist two constants $A, B > 0$ such that

$$A\|f\|_\theta^2 \leq \sum_{m, n \in \mathbb{Z}} |\langle f, E_m^\theta T_n g \rangle_\theta|^2 \leq B\|f\|_\theta^2 \quad (8)$$

holds for all $f \in L^2(\mathbb{R}, d\theta)$. To further study the general Gabor frame as defined in [8], we introduce a general frame concept as follows.

Definition 2. Let $g_n \in L^2(\mathbb{R}, d\theta)$, $n \in \mathbb{Z}$. One says that the system $\{E_m^\theta g_n, m, n \in \mathbb{Z}\}$ is a general frame for $L^2(\mathbb{R}, d\theta)$ if there exist two constants $A, B > 0$ such that

$$A\|f\|_\theta^2 \leq \sum_{m, n \in \mathbb{Z}} |\langle f, E_m^\theta g_n \rangle_\theta|^2 \leq B\|f\|_\theta^2 \quad (9)$$

holds for all $f \in L^2(\mathbb{R}, d\theta)$; moreover, one says that the frame $\{E_m^\theta g_n, m, n \in \mathbb{Z}\}$ is tight if $A = B$; in particular, the frame $\{E_m^\theta g_n, m, n \in \mathbb{Z}\}$ is Parseval if $A = B = 1$.

Given a frame $\{E_m^\theta g_n, m, n \in \mathbb{Z}\}$ for $L^2(\mathbb{R}, d\theta)$, a dual frame is a frame $\{E_m^\theta h_n, m, n \in \mathbb{Z}\}$ of $L^2(\mathbb{R}, d\theta)$ which satisfies the reconstruction property

$$f = \sum_{n, m \in \mathbb{Z}} \langle f, E_m^\theta g_n \rangle_\theta E_m^\theta h_n, \quad \forall f \in L^2(\mathbb{R}, d\theta), \quad (10)$$

and we say that the systems $\{E_m^\theta g_n, m, n \in \mathbb{Z}\}$ and $\{E_m^\theta h_n, m, n \in \mathbb{Z}\}$ constitute a pair of dual frames for $L^2(\mathbb{R}, d\theta)$, where the convergence is in the L^2 sense. Note that if $\theta(2\pi x) = 2\pi x$ for all $x \in \mathbb{R}$, then the frame $\{E_m^\theta g_n, m, n \in \mathbb{Z}\}$ for $L^2(\mathbb{R}, d\theta)$ constitutes a frame for $L^2(\mathbb{R})$.

For fixed $f, g \in L^2(\mathbb{R}, d\theta)$ and $b > 0$, we introduce the θ -bracket product as follows:

$$[f, g]_b^\theta(x) := \sum_{k \in \mathbb{Z}} f^\theta\left(x + \frac{k}{b}\right) \overline{g^\theta\left(x + \frac{k}{b}\right)}, \quad \text{a.e. } x \in \mathbb{R}. \quad (11)$$

If $\theta(2\pi x) = 2\pi x$ for $x \in \mathbb{R}$, then $[f, g]_b^\theta$ is bracket product (denote by $[f, g]_b$) introduced by Ron and Shen in [11]. Thus,

$$[f, g]_b^\theta = [f^\theta, g^\theta]_b. \quad (12)$$

Note that $[f, g]_b^\theta$ is a 1-periodic function.

With the classical bracket product, Christensen and Sun [12] proved the following Lemma 3, which is [13, Lemma 2.3].

Lemma 3. Let $g_n, h_n \in L^2(\mathbb{R})$, $n \in \mathbb{Z}$, and $b > 0$. Let the systems $\{E_m b g_n, m, n \in \mathbb{Z}\}$ and $\{E_m b h_n, m, n \in \mathbb{Z}\}$ be Bessel sequences in $L^2(\mathbb{R})$. Define

$$Sf = \sum_{m, n \in \mathbb{Z}} \langle f, E_m b g_n \rangle E_m b h_n, \quad \forall f \in L^2(\mathbb{R}). \quad (13)$$

Then, the following holds:

$$\begin{aligned} (Sf)(x) &= \frac{1}{b} \sum_{n \in \mathbb{Z}} [f, g_n]_b(x) h_n(x) \\ &= \frac{1}{b} \sum_{n \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} f\left(x + \frac{k}{b}\right) \overline{g_n\left(x + \frac{k}{b}\right)} h_n(x), \quad (14) \\ &\quad \forall f \in L^2(\mathbb{R}), \end{aligned}$$

where the convergence is in the L^2 sense. Moreover, $\{E_{mb}g_n, m, n \in \mathbb{Z}\}$ and $\{E_{mb}h_n, m, n \in \mathbb{Z}\}$ are a pair of dual frames for $L^2(\mathbb{R})$ if and only if

$$\sum_{k \in \mathbb{Z}} g_n\left(x + \frac{k}{b}\right) \overline{h_n\left(x + \frac{k}{b}\right)} = b\delta_{n,0}, \quad a.e. \ x \in \mathbb{R}. \quad (15)$$

The following lemma follows from general properties of shift-invariant frames; see [11, Corollary 1.6.2]. Alternatively, it can be proved similarly to [14, Theorem 8.4.4].

Lemma 4. Let $g_n \in L^2(\mathbb{R})$, $n \in \mathbb{Z}$, $b > 0$, and

$$B := \frac{1}{b} \sup_{x \in \mathbb{R}} \sum_{k \in \mathbb{Z}} \left| \sum_{n \in \mathbb{Z}} g_n(x) \overline{g_n\left(x - \frac{k}{b}\right)} \right| < \infty. \quad (16)$$

Then, $\{E_{mb}g_n, m, n \in \mathbb{Z}\}$ is a Bessel sequence with upper frame bound B . If also

$$A := \frac{1}{b} \inf_{x \in \mathbb{R}} \left(\sum_{n \in \mathbb{Z}} |g_n(x)|^2 - \sum_{k \neq 0} \left| \sum_{n \in \mathbb{Z}} g_n(x) \overline{g_n\left(x - \frac{k}{b}\right)} \right| \right) > 0, \quad (17)$$

then $\{E_{mb}g_n, m, n \in \mathbb{Z}\}$ constitutes a frame $L^2(\mathbb{R})$ with bounds A and B .

3. Frame for $L^2(\mathbb{R}, d\theta)$

In this section, we discuss frames for $L^2(\mathbb{R}, d\theta)$. Here, we will establish the relationship between frames for $L^2(\mathbb{R}, d\theta)$ and $L^2(\mathbb{R})$. We will also give necessary conditions for frames and characterize a pair of dual frames in $L^2(\mathbb{R}, d\theta)$. Above all, we establish the relationship between $L^2(\mathbb{R}, d\theta)$ and $L^2(\mathbb{R})$ as follows.

Theorem 5. Let f, g be functions defined on $\mathbb{R}$. Then, $\langle f, g \rangle_\theta = \langle f^\theta, g^\theta \rangle$; in particular, $\|f\|_\theta = \|f^\theta\|$, which means that $f \in L^2(\mathbb{R}, d\theta)$ if and only if $f^\theta \in L^2(\mathbb{R})$.

Proof. Denote $t := (1/2\pi)\theta(2\pi x)$. Then,

$$\begin{aligned} \int_{-\infty}^{\infty} f(x) \overline{g(x)} d\theta(2\pi x) &= \int_{-\infty}^{\infty} f\left(\frac{1}{2\pi}\theta^{-1}(2\pi t)\right) \\ &\quad \times \overline{g\left(\frac{1}{2\pi}\theta^{-1}(2\pi t)\right)} d(2\pi t) \quad (18) \\ &= 2\pi \int_{-\infty}^{\infty} f^\theta(t) \overline{g^\theta(t)} dt. \end{aligned}$$

This means that

$$\langle f, g \rangle_\theta = \langle f^\theta, g^\theta \rangle. \quad (19)$$

Thus, we can obtain the desired result. $\square$

Theorem 6. Let $g_n, n \in \mathbb{Z}$ be functions defined on $\mathbb{R}$. Then, the system $\{E_m^\theta g_n, m, n \in \mathbb{Z}\}$ constitutes a frame for $L^2(\mathbb{R}, d\theta)$ if and only if the system $\{E_m g_n, m, n \in \mathbb{Z}\}$ constitutes a frame for $L^2(\mathbb{R})$ and these two systems have the same bounds.

Proof. From Theorem 5, one obtains that $g_n \in L^2(\mathbb{R}, d\theta)$ if and only if $g_n^\theta \in L^2(\mathbb{R})$ for $n \in \mathbb{Z}$. Note that

$$\begin{aligned} \langle f, E_m^\theta g_n \rangle_\theta &= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) \overline{e^{im\theta(2\pi x)} g_n(x)} d\theta(2\pi x) \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} f\left(\frac{1}{2\pi}\theta^{-1}(2\pi x)\right) \\ &\quad \times \overline{e^{i2\pi m x} g_n\left(\frac{1}{2\pi}\theta^{-1}(2\pi x)\right)} d(2\pi x) \\ &= \int_{-\infty}^{\infty} f^\theta(x) \overline{e^{i2\pi m x} g_n^\theta(x)} dx = \langle f^\theta, E_m g_n^\theta \rangle. \quad (20) \end{aligned}$$

Then,

$$A\|f\|_\theta^2 \leq \sum_{m, n \in \mathbb{Z}} |\langle f, E_m^\theta g_n \rangle_\theta|^2 \leq B\|f\|_\theta^2, \quad \forall f \in L^2(\mathbb{R}, d\theta) \quad (21)$$

is equivalent to

$$A\|f^\theta\|^2 \leq \sum_{m, n \in \mathbb{Z}} |\langle f^\theta, E_m g_n^\theta \rangle|^2 \leq B\|f^\theta\|^2, \quad \forall f^\theta \in L^2(\mathbb{R}). \quad (22)$$

Now, we can obtain the desired results. $\square$

Theorem 7. Let $g_n, n \in \mathbb{Z}$ be functions defined on $\mathbb{R}$. Then, the systems $\{E_m^\theta g_n, m, n \in \mathbb{Z}\}$ and $\{E_m^\theta h_n, m, n \in \mathbb{Z}\}$ constitute a pair of dual frames for $L^2(\mathbb{R}, d\theta)$ if and only if the systems $\{E_m g_n, m, n \in \mathbb{Z}\}$ and $\{E_m h_n, m, n \in \mathbb{Z}\}$ constitute a pair of dual frames for $L^2(\mathbb{R})$.

Proof. “if” part. If the systems $\{E_m^\theta g_n, m, n \in \mathbb{Z}\}$ and $\{E_m^\theta h_n, m, n \in \mathbb{Z}\}$ constitute a pair of dual frames for $L^2(\mathbb{R}, d\theta)$. Then, by Theorem 6, these two systems $\{E_m^\theta g_n, m, n \in \mathbb{Z}\}$ and $\{E_m^\theta h_n, m, n \in \mathbb{Z}\}$ are frames for $L^2(\mathbb{R}, d\theta)$. Moreover, we obtain from (20) that

$$f(x) = \sum_{m, n \in \mathbb{Z}} \langle f^\theta, E_m g_n^\theta \rangle E_m h_n(x), \quad x \in \mathbb{R}, \quad (23)$$

for any $f \in L^2(\mathbb{R}, d\theta)$, where the convergence is in the L^2 sense. Replacing x by $(1/2\pi)\theta^{-1}(2\pi x)$ in the above equation, we obtain

$$f^\theta(x) = \sum_{m, n \in \mathbb{Z}} \langle f^\theta, E_m g_n^\theta \rangle E_m h_n^\theta(x), \quad x \in \mathbb{R}, \quad (24)$$

for any $f^\theta \in L^2(\mathbb{R})$, where the convergence is in the L^2 sense. Therefore, the systems $\{E_m g_n^\theta, m, n \in \mathbb{Z}\}$ and $\{E_m h_n^\theta, m, n \in \mathbb{Z}\}$ constitute a pair of dual frames for $L^2(\mathbb{R})$.

The proof of “only if” part is similar to the “if” part, and we omit it. $\square$

With the θ -bracket product proposed in the above section, we can prove the following theorem.

Theorem 8. Let $g_n, h_n \in L^2(\mathbb{R}, d\theta)$ for $n \in \mathbb{Z}$. Let $\{E_m g_n, m, n \in \mathbb{Z}\}$ and $\{E_m h_n, m, n \in \mathbb{Z}\}$ be Bessel sequences in $L^2(\mathbb{R}, d\theta)$. Define

$$Sf = \sum_{m, n \in \mathbb{Z}} \langle f, E_m g_n \rangle_\theta E_m h_n, \quad (25)$$

for $f \in L^2(\mathbb{R}, d\theta)$. Then,

$$\begin{aligned} (Sf^\theta)(x) &= \sum_{n \in \mathbb{Z}} [f, g_n]^\theta(x) h_n^\theta(x) \\ &\times \sum_{m \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} f^\theta(x+k) \overline{g_n^\theta(x+k)} h_n^\theta(x) \end{aligned} \quad (26)$$

holds for $f \in L^2(\mathbb{R}, d\theta)$, where the convergence is in the L^2 sense. Moreover, the systems $\{E_m g_n, m, n \in \mathbb{Z}\}$ and $\{E_m h_n, m, n \in \mathbb{Z}\}$ constitute a pair of dual frames for $L^2(\mathbb{R}, d\theta)$ if and only if

$$\sum_{k \in \mathbb{Z}} g_n^\theta(x+k) \overline{h_n^\theta(x+k)} = \delta_{n,0}, \quad a.e. \ x \in \mathbb{R}. \quad (27)$$

Proof. For fixed $f \in L^2(\mathbb{R}, d\theta)$, one obtains from (20) that

$$\begin{aligned} Sf(x) &= \sum_{m, n \in \mathbb{Z}} \langle f, E_m g_n \rangle_\theta E_m h_n(x) \\ &= \sum_{m, n \in \mathbb{Z}} \langle f^\theta, E_m g_n^\theta \rangle E_m h_n(x), \end{aligned} \quad (28)$$

where the convergence is in the L^2 sense. Replacing x by $(1/2\pi)\theta^{-1}(2\pi x)$ in the above equation, we obtain

$$Sf^\theta(x) = \sum_{m, n \in \mathbb{Z}} \langle f^\theta, E_m g_n^\theta \rangle E_m h_n^\theta(x), \quad \forall f^\theta \in L^2(\mathbb{R}). \quad (29)$$

Note that the systems $\{E_m g_n, m, n \in \mathbb{Z}\}$ and $\{E_m h_n, m, n \in \mathbb{Z}\}$ are Bessel sequences in $L^2(\mathbb{R}, d\theta)$. We can deduce that the systems $\{E_m g_n^\theta, m, n \in \mathbb{Z}\}$ and $\{E_m h_n^\theta, m, n \in \mathbb{Z}\}$ are Bessel sequences in $L^2(\mathbb{R})$. Therefore, one obtains (26) from (14).

From Theorem 7, we know that the systems $\{E_m g_n, m, n \in \mathbb{Z}\}$ and $\{E_m h_n, m, n \in \mathbb{Z}\}$ constitute a pair of dual frames for $L^2(\mathbb{R}, d\theta)$ if and only if the systems $\{E_m g_n^\theta, m, n \in \mathbb{Z}\}$ and $\{E_m h_n^\theta, m, n \in \mathbb{Z}\}$ constitute a pair of dual frames for $L^2(\mathbb{R})$. Thus, by (15) in Lemma 3, one obtains the desired result. $\square$

Theorem 9. Let $g_n \in L^2(\mathbb{R}, d\theta)$, $n \in \mathbb{Z}$, and suppose that

$$B := \sup_{x \in \mathbb{R}} \sum_{k \in \mathbb{Z}} \left| \sum_{n \in \mathbb{Z}} g_n^\theta(x) \overline{g_n^\theta(x+k)} \right| < \infty. \quad (30)$$

Then, the system $\{E_m g_n, m, n \in \mathbb{Z}\}$ is a Bessel sequence with upper frame bound B for $L^2(\mathbb{R}, d\theta)$. If also

$$A := \inf_{x \in \mathbb{R}} \left(\sum_{n \in \mathbb{Z}} |g_n^\theta(x)|^2 - \sum_{k \neq 0} \left| \sum_{n \in \mathbb{Z}} g_n^\theta(x) \overline{g_n^\theta(x+k)} \right| \right) > 0, \quad (31)$$

then the system $\{E_m g_n, m, n \in \mathbb{Z}\}$ constitutes a frame for $L^2(\mathbb{R}, d\theta)$ with bounds A and B .

Proof. Since $g_n \in L^2(\mathbb{R}, d\theta)$, $n \in \mathbb{Z}$, then $g_n^\theta \in L^2(\mathbb{R})$, $n \in \mathbb{Z}$. If $0 < A, B < \infty$, then by Lemma 4, the system $\{E_m g_n, m, n \in \mathbb{Z}\}$ constitutes a frame for $L^2(\mathbb{R})$ with frame bounds A and B . Therefore, by Theorem 6, one obtains that the system $\{E_m g_n, m, n \in \mathbb{Z}\}$ constitutes a frame for $L^2(\mathbb{R}, d\theta)$ with the same frame bounds A and B . $\square$

4. Gabor Frame for $L^2(\mathbb{R}, d\theta)$

In this section, Gabor frames for $L^2(\mathbb{R}, d\theta)$ are discussed. We establish the relationship between the generalized Gabor frame $\{e^{im\theta(2\pi t)} g(t-n), m, n \in \mathbb{Z}\}$ for $L^2(\mathbb{R}, d\theta)$ and the classical one $\{e^{i2m\pi t} g^\theta(t-n), m, n \in \mathbb{Z}\}$ for $L^2(\mathbb{R})$; further, we prove the Balian-Low theorem for Gabor system $\{e^{im\theta(2\pi t)} g(t-n), m, n \in \mathbb{Z}\}$ proposed by Fu et al. in [8] from a different viewpoint.

Theorem 10. Let $g \in L^2(\mathbb{R}, d\theta)$. Then,

$$(T_n g)^\theta(x) = T_n g^\theta(x). \quad (32)$$

Proof. Since

$$\theta^{-1}(x+2k\pi) = \theta^{-1}(x) + 2k\pi, \quad \forall x \in \mathbb{R}, k \in \mathbb{Z}, \quad (33)$$

then

$$\begin{aligned} \theta^{-1}(2\pi(x+k)) &= \theta^{-1}(2\pi x + 2k\pi) \\ &= \theta^{-1}(2\pi x) + 2k\pi, \quad \forall x \in \mathbb{R}, k \in \mathbb{Z}. \end{aligned} \quad (34)$$

Hence,

$$\begin{aligned} (T_n g)^\theta(x) &= g\left(\frac{1}{2\pi}\theta^{-1}(2\pi x) - n\right) \\ &= g\left(\frac{1}{2\pi}\theta^{-1}(2\pi(x-n))\right) = T_n g^\theta(x). \end{aligned} \quad (35)$$

We complete the proof. $\square$

Theorem 11. Let g be a function defined on $\mathbb{R}$. Then, the general Gabor system $\{E_m^\theta T_n g, m, n \in \mathbb{Z}\}$ constitutes a frame for $L^2(\mathbb{R}, d\theta)$ if and only if the classical Gabor system $\{E_m T_n g^\theta, m, n \in \mathbb{Z}\}$ constitutes a frame for $L^2(\mathbb{R})$ with the same bounds.

Proof. Define $g_n(x) := T_n g(x)$, $n \in \mathbb{Z}$. From Theorem 6, one obtains that the general Gabor system $\{E_m^\theta T_n g, m, n \in \mathbb{Z}\}$ constitutes a frame for $L^2(\mathbb{R}, d\theta)$ if and only if the system $\{E_m(T_n g)^\theta, m, n \in \mathbb{Z}\}$ constitutes a frame for $L^2(\mathbb{R})$ with the same bounds. So, one obtains the desired result from Theorem 10. $\square$

Combining Theorems 8 and 10 together, we obtain the following Theorem 12.

Theorem 12. Consider $g, h \in L^2(\mathbb{R}, d\theta)$. Let the systems $\{E_m^\theta T_n g, m, n \in \mathbb{Z}\}$ and $\{E_m^\theta T_n h, m, n \in \mathbb{Z}\}$ be Bessel sequences in $L^2(\mathbb{R}, d\theta)$. Define

$$Sf = \sum_{m, n \in \mathbb{Z}} \langle f, E_m^\theta T_n g \rangle E_m^\theta T_n h, \quad \forall f \in L^2(\mathbb{R}, d\theta). \quad (36)$$

Then, for any $f \in L^2(\mathbb{R}, d\theta)$,

$$\begin{aligned} (Sf^\theta)(x) &= \sum_{n \in \mathbb{Z}} [f^\theta, T_n g^\theta](x) T_n h^\theta(x) \\ &= \sum_{n \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} f^\theta(x+k) \overline{T_n g^\theta(x+k)} T_n h^\theta(x), \end{aligned} \quad (37)$$

where the convergence is in the L^2 sense. Moreover, the systems $\{E_m^\theta T_n g, m, n \in \mathbb{Z}\}$ and $\{E_m^\theta T_n h, m, n \in \mathbb{Z}\}$ constitute a pair of dual frames for $L^2(\mathbb{R}, d\theta)$ if and only if

$$\sum_{k \in \mathbb{Z}} T_n g^\theta(x+k) \overline{T_n h^\theta(x+k)} = \delta_{n,0}, \quad \text{a.e. } x \in \mathbb{R}. \quad (38)$$

Proof. Replacing g_n and h_n by $T_n g$ and $T_n h$ in (26) and (27), respectively, we have

$$\begin{aligned} (Sf^\theta)(x) &= \sum_{n \in \mathbb{Z}} [f, T_n g]^\theta(x) (T_n h)^\theta(x) \\ &= \sum_{n \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} f^\theta(x+k) \overline{(T_n g)^\theta(x+k)} (T_n h)^\theta(x), \end{aligned} \quad (39)$$

$$\sum_{k \in \mathbb{Z}} (T_n g)^\theta(x+k) \overline{(T_n h)^\theta(x+k)} = \delta_{n,0}, \quad \text{a.e. } x \in \mathbb{R}. \quad (40)$$

Equations (37) and (38) follow from (39) and (40), respectively. Here, we used the facts that $(T_n g)^\theta(x) = T_n g^\theta(x)$ and $(T_n h)^\theta(x) = T_n h^\theta(x)$. $\square$

By Theorems 9 and 10, we obtain Theorem 13.

Theorem 13. Consider $g \in L^2(\mathbb{R}, d\theta)$, $n \in \mathbb{Z}$, and suppose that

$$B := \sup_{x \in [0,1]} \sum_{k \in \mathbb{Z}} \left| \sum_{n \in \mathbb{Z}} T_n g^\theta(x) \overline{T_n g^\theta(x+k)} \right| < \infty. \quad (41)$$

Then, the system $\{E_m^\theta T_n g, m, n \in \mathbb{Z}\}$ is a Bessel sequence for $L^2(\mathbb{R}, d\theta)$ with upper frame bound B . If also

$$A := \inf_{x \in [0,1]} \left(\sum_{n \in \mathbb{Z}} |T_n g^\theta(x)|^2 - \sum_{k \neq 0} \left| \sum_{n \in \mathbb{Z}} T_n g^\theta(x) \overline{T_n g^\theta(x+k)} \right| \right) > 0, \quad (42)$$

then the system $\{E_m^\theta T_n g, m, n \in \mathbb{Z}\}$ constitutes a frame for $L^2(\mathbb{R}, d\theta)$ with bounds A and B .

Proof. Since $T_n g^\theta(x) = (T_n g)^\theta(x)$ and $T_n h^\theta(x) = (T_n h)^\theta(x)$, then

$$\begin{aligned} B &= \sup_{x \in [0,1]} \sum_{k \in \mathbb{Z}} \left| \sum_{n \in \mathbb{Z}} (T_n g)^\theta(x) \overline{(T_n g)^\theta(x+k)} \right|, \\ A &= \inf_{x \in [0,1]} \left(\sum_{n \in \mathbb{Z}} |(T_n g)^\theta(x)|^2 - \sum_{k \neq 0} \left| \sum_{n \in \mathbb{Z}} (T_n g)^\theta(x) \overline{(T_n g)^\theta(x+k)} \right| \right). \end{aligned} \quad (43)$$

Define

$$\begin{aligned} H_1(x) &:= \sum_{k \in \mathbb{Z}} \left| \sum_{n \in \mathbb{Z}} (T_n g)^\theta(x) \overline{(T_n g)^\theta(x+k)} \right|, \\ H_2(x) &:= \sum_{n \in \mathbb{Z}} |(T_n g)^\theta(x)|^2 \\ &\quad - \sum_{k \neq 0} \left| \sum_{n \in \mathbb{Z}} (T_n g)^\theta(x) \overline{(T_n g)^\theta(x+k)} \right|, \end{aligned} \quad (44)$$

then H_1 and H_2 are 1-periodic functions. Thus,

$$\begin{aligned} B &= \sup_{x \in \mathbb{R}} \sum_{k \in \mathbb{Z}} \left| \sum_{n \in \mathbb{Z}} (T_n g)^\theta(x) \overline{(T_n g)^\theta(x+k)} \right|, \\ A &= \inf_{x \in \mathbb{R}} \left(\sum_{n \in \mathbb{Z}} |(T_n g)^\theta(x)|^2 - \sum_{k \neq 0} \left| \sum_{n \in \mathbb{Z}} (T_n g)^\theta(x) \overline{(T_n g)^\theta(x+k)} \right| \right). \end{aligned} \quad (45)$$

By Theorem 9, one obtains the results. $\square$

Theorem 14. Let $g \in L^2(\mathbb{R}, d\theta)$. Assume that the system $\{E_m^\theta T_n g, m, n \in \mathbb{Z}\}$ constitutes a generalized Gabor frame for $L^2(\mathbb{R}, d\theta)$ with bounds A and B . Then,

$$A \leq \sum_{n \in \mathbb{Z}} |g^\theta(x-n)|^2 \leq B, \quad \text{a.e. } x \in \mathbb{R}. \quad (46)$$

Proof. If the system $\{E_m^\theta T_n g, m, n \in \mathbb{Z}\}$ constitutes a generalized Gabor frame for $L^2(\mathbb{R}, d\theta)$ with bounds A and B . Then, by Theorem 7, $\{E_m(T_n g)^\theta, m, n \in \mathbb{Z}\}$ constitutes a frame for $L^2(\mathbb{R})$ with the same bounds A and B . Note that $(T_n g)^\theta = T_n g^\theta$. We can say that the system $\{E_m T_n g^\theta, m, n \in \mathbb{Z}\}$ constitutes a Gabor frame for $L^2(\mathbb{R})$ with the same bounds A and B . Thus, one obtains from [14, Proposition 8.3.2] the desired result. $\square$

Theorem 15. Let $g \in L^2(\mathbb{R}, d\theta)$. Suppose that the system $\{E_m^\theta T_n g, m, n \in \mathbb{Z}\}$ constitutes a general Gabor frame for $L^2(\mathbb{R}, d\theta)$. If the derivative θ' of function θ is continuous on $\mathbb{R}$, then either

$$\int_{-\infty}^{\infty} x^2 |g(x)|^2 d\theta(2\pi x) = \infty \quad (47)$$

or

$$\int_{-\infty}^{\infty} \xi^2 |\widehat{g}(\xi)|^2 d\theta(2\pi\xi) = \infty. \quad (48)$$

Proof. Since $\{E_m^\theta T_n g, m, n \in \mathbb{Z}\}$ constitutes a general Gabor frame for $L^2(\mathbb{R}, d\theta)$, then, by Theorem 11, the system $\{E_m T_n g^\theta, m, n \in \mathbb{Z}\}$ constitutes a classical Gabor frame for $L^2(\mathbb{R})$. Therefore, by the classical Balian-Low theorem, we have either

$$\int_{-\infty}^{\infty} x^2 |g^\theta(x)|^2 dx = \infty \quad (49)$$

or

$$\int_{-\infty}^{\infty} \xi^2 |\widehat{g^\theta}(\xi)|^2 d\xi = \infty. \quad (50)$$

That is either

$$\int_{-\infty}^{\infty} \theta^2(2\pi x) |g(x)|^2 d\theta(2\pi x) = \infty \quad (51)$$

or

$$\int_{-\infty}^{\infty} \theta^2(2\pi\xi) |\widehat{g}(\xi)|^2 d\theta(2\pi\xi) = \infty. \quad (52)$$

We need to prove that (51) implies (47) or (52) implies (48). Next, we only prove that (51) implies (47) (the case (52) implies (48) can be obtained similarly). Without loss of generality, let

$$\int_0^{\infty} \theta^2(2\pi x) |g(x)|^2 d\theta(2\pi x) = \infty. \quad (53)$$

Since the derivative θ' of function θ is continuous on $\mathbb{R}$ and θ satisfies Assumption 1, then

$$0 < \min_{x \in [0, 2\pi]} \theta'(x) \leq \theta'(t) \leq \max_{x \in [0, 2\pi]} \theta'(x), \quad t \in \mathbb{R}. \quad (54)$$

By the Lagrange mean-valued theorem, there exists $\zeta \in [0, x]$ such that

$$\theta(x) = \theta(0) + \theta'(\zeta)x. \quad (55)$$

Therefore, for any fixed $x_0 > 0$, there exists a constant $C > 0$ such that

$$|\theta(x)| \leq Cx, \quad \forall x \geq x_0. \quad (56)$$

Note that

$$\int_0^{x_0} \theta^2(2\pi x) |g(x)|^2 d\theta(2\pi x) < \infty. \quad (57)$$

Therefore,

$$\begin{aligned} & \int_{x_0}^{\infty} C^2 x^2 |g(x)|^2 d\theta(2\pi x) \\ & \geq \int_{x_0}^{\infty} \theta^2(2\pi x) |g(x)|^2 d\theta(2\pi x) = \infty. \end{aligned} \quad (58)$$

That is,

$$\int_{x_0}^{\infty} x^2 |g(x)|^2 d\theta(2\pi x) = \infty, \quad (59)$$

or

$$\int_{-\infty}^{\infty} x^2 |g(x)|^2 d\theta(2\pi x) = \infty. \quad (60)$$

$\square$

In the proof of Theorem 15, the main technique is the inequality

$$|\theta(x)| \leq Cx, \quad \forall x \geq x_0, \quad (61)$$

for some positive constant C . Note that

$$\theta(x + 2k\pi) = \theta(x) + 2k\pi \quad (62)$$

is equivalent to

$$\theta(x) = x + \beta(x), \quad (63)$$

where β is a 2π -periodic function. We obtain the following Balian-Low theorem which weakens the conditions imposed on θ in Theorem 15.

Theorem 16. Let $g \in L^2(\mathbb{R}, d\theta)$. Suppose that the system $\{E_m^\theta T_n g, m, n \in \mathbb{Z}\}$ constitutes a general Gabor frame for $L^2(\mathbb{R}, d\theta)$. Let β be a 2π -periodic function such that

$$\begin{aligned} \theta(x) &= x + \beta(x), \quad \forall x \in \mathbb{R}, \\ |\beta(x)| &\leq Cx, \quad \forall x \geq x_0, \end{aligned} \quad (64)$$

where C is a positive constant. Then, one and only one of the inequalities (47) and (48) holds.

In applications of frames, it is inconvenient that the frame decomposition, as stated in [15, Theorem 5.1.7], requires the inverse of a frame operator. As we have seen in the discussion of general frame theory, one way of avoiding the problem is to consider tight frames. Hence, we give characterization for tight Gabor frames in $L^2(\mathbb{R}, d\theta)$.

Theorem 17. Let $g \in L^2(\mathbb{R}, d\theta)$. Then, the system $\{E_m^\theta T_n g, m, n \in \mathbb{Z}\}$ constitutes a tight frame for $L^2(\mathbb{R}, d\theta)$ with $A = 1$ if and only if

$$\begin{aligned} \sum_{n \in \mathbb{Z}} |g^\theta(x - n)|^2 &= 1, \\ \sum_{n \in \mathbb{Z}} g^\theta(x - n) \overline{g^\theta(x - n - k)} &= 0, \quad \text{for } k \neq 0 \end{aligned} \quad (65)$$

holds a.e. in $\mathbb{R}$.

Proof. By Theorem 11, one obtains that the system $\{E_m^\theta T_n g, m, n \in \mathbb{Z}\}$ constitutes a tight frame for $L^2(\mathbb{R}, d\theta)$ with $A = 1$ if and only if $\{E_m^\theta T_n g^\theta, m, n \in \mathbb{Z}\}$ constitutes a tight frame for $L^2(\mathbb{R})$ with $A = 1$. From [14, Theorem 9.5.2], one obtains the desired result. $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The authors are grateful to the referees for their valuable suggestions that helped to improve the paper in its present form. This research is supported by the National Natural Science Foundation of China (Grant no. 11071152), the Natural Science Foundation of Guangdong Province (Grant nos. S2013010013101 and S2011010004511), and the Foundation of Hanshan Normal University (Grant nos. QD20131101 and LQ200905). This work was also partially supported by the Opening Project of Guangdong Province Key Laboratory of Computational Science at the Sun Yat-sen University (Grant no. 201206012).

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