Research Article

New Sequence Spaces and Function Spaces on Interval $[0, 1]$

Cheng-Zhong Xu¹ and Gen-Qi Xu²

¹ Université de Lyon, LAGER, Bâtiment CPE, Université Lyon 1, 43 boulevard du 11 Novembre 1918, 69622 Villeurbanne, France

² Department of Mathematics, Tianjin University, Tianjin 300072, China

Correspondence should be addressed to Gen-Qi Xu; gqxu@tju.edu.cn

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We study the sequence spaces and the spaces of functions defined on interval $[0, 1]$ in this paper. By a new summation method of sequences, we find out some new sequence spaces that are interpolating into spaces between ℓ^p and ℓ^q and function spaces that are interpolating into the spaces between the polynomial space $P[0, 1]$ and $C^\infty[0, 1]$. We prove that these spaces of sequences and functions are Banach spaces.

1. Introduction

With development of sciences and technologies, more and more information are obtaining and need to be reserved and transmitted in the form of data sequence, such as DNA sequence, protein structure [1], brain imaging data, optic spectral analysis, text retrieval, financial data, and climate data. These data have common features: (1) there are at most finite many nonzero elements in the sequence; (2) their dimensions have not bounded from above; (3) the sample size is relatively small. In particular, some elements in the sequence repeat many times, for instance, there are only four different elements in DNA sequence: T, A, C, and G. When the data have much greater dimension, their record and reserve also become a serious problem. On the other hand, we usually use the data to obtain some information, such as the image reconstruction, sequence comparison in medicine, and plant classification in biology. From application point view, the basic requirement is that one can draw easily information from the reservoir; the is to use this data to handle some things. When the data have lower dimension and the samples have larger size, the statistics method such as the covariance matrix can give a good treatment; for instance, see [2] for the semiparameter estimation, [3] for the sparse data estimation, and [4, 5] for the threshold sparse sample covariance matrix method. However, when the data have higher dimensional

and the sample size is smaller, the statistics method shall lead to great errors. So, we need new methods to treat them.

Let us consider a simple example from a classification problem. Set S as a set of some class samples and a as a given data. Is a close to someone of S or a new class? A simpler approach is to consider problem $\inf_{s \in S} \|a - s\|_p$, where p denotes the norm in ℓ^p space. In most cases, there is at least one $s_0 \in S$ such that $\|a - s_0\|_p = \inf_{s \in S} \|a - s\|_p$. We denote by $F(a)$ the feasible set. Can we say that a is close to some $s_0 \in F(a)$? To see disadvantage, we divide sequence $s \in S$ into three segments (s_1, s_2, s_3) ; the first segment s_1 is composed of the first n_1 elements, the second segment s_2 is made of the next n_2 elements, and the third is composed of the others. Similarly, we also divide a into corresponding three parts (a_1, a_2, a_3) . Now, we reconsider

$$\inf_{s_1} \|a_1 - s_1\|_p, \quad \inf_{s_2} \|a_2 - s_2\|_p, \quad \inf_{s_3} \|a_3 - s_3\|_p. \quad (1)$$

Perhaps we would find that $F(a_1) \cap F(a_2) \cap F(a_3) = \emptyset$. Can one say that a is a new class? From the above example, we see that we need a new definition of the norm to fit application.

Motivated by these questions, we revisit the sequence spaces and function spaces defined on $[0, 1]$ in this paper. We have observed recent studies on the sequence spaces, for instants, [6–8] for different requirements. Here, the sequence spaces we work on are different from the existing spaces, this

is because the spaces are aimed to solve our problem. Now, let us introduce our idea and the resulted sequence space and function spaces.

Let $a = (a_1, a_2, a_3, \dots, a_n, \dots)$ be a DNA sequence. Obviously, there are at most finite many nonzero terms and $a_n \in \{A, C, T, G\}$. To shorten the representation, we can embed a into a polynomial. In this way, we can write a as

$$a(x) = Ap_1(x) + Tp_2(x) + Cp_3(x) + Gp_4(x). \quad (2)$$

For a different DNA sequence, we have different polynomial $p_j(x)$. Obviously, it is a simpler reserve form.

How do we extract original sequence from the polynomial? By the classical mathematics, we know that

$$a_k = \frac{1}{n!} \frac{d^{(k)} a(x)}{dx^k} \Big|_{x=0}, \quad k = 0, 1, 2, \dots, \quad (3)$$

so we recover the sequence.

To extend this form into a sequence of infinite many nonzero terms, we usually take $x \in [0, 1]$; $a(x)$ is called the generation function in the classical queuing theory. Note that the generation function is not a continuous function defined on $[0, 1]$. So, the differential operation is not fitting to such a function, although the formal differential is always feasible. To find out a feasible form of $a(x)$, let us consider the operations of integral and derivative. We denote by L the integral operation. Operating for the constant 1 leads to

$$\begin{aligned} L^1(1)(x) &= \int_0^x 1 dx = x, \\ L^2(1)(x) &= \int_0^x L^1(1) x dx = \frac{x^2}{2}, \\ L^3(1)(x) &= \int_0^x L^2(1)(x) dx = \frac{x^3}{3!}. \end{aligned} \quad (4)$$

Generally, we have

$$L^n(1)(x) = \frac{x^n}{n!}, \quad \forall n \in \mathbb{N}. \quad (5)$$

For any polynomial of n -order, $p_n(x)$, it can be written as

$$\begin{aligned} p_n(x) &= a_0 1 + a_1 x + a_2 \frac{x^2}{2!} + \dots + a_n \frac{x^n}{n!} \\ &= \left[\sum_{k=0}^n a_k L^k \right] (1)(x). \end{aligned} \quad (6)$$

Next, let us consider the differential operation $D(f) = f'(x)$. Taking differential for function $x^n/n!$ leads to

$$D\left(\frac{x^n}{n!}\right) = \frac{x^{n-1}}{(n-1)!}, \quad D^2\left(\frac{x^n}{n!}\right) = \frac{x^{n-2}}{(n-2)!}. \quad (7)$$

In general, for any $1 \leq k \leq n$, it holds that

$$D^k\left(\frac{x^n}{n!}\right) = \frac{x^{n-k}}{(n-k)!}. \quad (8)$$

Obviously, using D , we get once again the coefficients in (6):

$$\begin{aligned} a_0 &= p_n(0), \quad a_1 = Dp_n(0), \dots, \\ a_k &= D^k p_n(0), \dots, \quad a_n = D^n p_n(0). \end{aligned} \quad (9)$$

Therefore, the coefficient sequence is given by

$$(a_0, a_1, \dots, a_n) = (D^0, D^1, \dots, D^n) p_n|_{x=0}. \quad (10)$$

Clearly, we should take functions $x^n/n!$ as the basis functions.

Moreover, we note that in the polynomial space over $[0, 1]$, denoted by $P[0, 1]$, if the norm is defined as

$$\|p\|_\phi = \sup_{n \geq 0} \{ \|D^n p\|_\infty \}; \quad (11)$$

where $\|f\|_\infty = \max_{0 \leq x \leq 1} |f(x)|$, then it is a normed linear space. In this space, the integral and differential operations are bounded linear operators. To extend to an infinite sequence, we should choose such a function space in which the integral and differential operations are bounded linear operators. What is such a function space?

Consider a subset of $C^\infty[0, 1]$ defined as

$$C_M^\infty[0, 1] = \left\{ f \in C^\infty[0, 1] \mid \sup_{n \geq 0} \|D^n f\|_\infty < \infty \right\}. \quad (12)$$

The set is in fact a linear space. We define a norm on it by

$$\|f\|_\phi = \sup_{n \geq 0} \|D^n f\|_\infty, \quad (13)$$

then it becomes a Banach space. Now, for the function spaces over interval $[0, 1]$, there are the following inclusion relations

$$\begin{aligned} P[0, 1] &\subset C_M^\infty[0, 1] \subset C^\infty[0, 1] \subset C^k[0, 1] \subset C[0, 1] \\ &\subset L^\infty[0, 1] \subset L^p[0, 1] \subset L^1[0, 1]. \end{aligned} \quad (14)$$

But the completion of $(P[0, 1], \|\cdot\|_\phi)$ is not the space $(C_M^\infty[0, 1], \|\cdot\|_\phi)$. Let $C_{\phi,0}[0, 1]$ be the completion space of $(P[0, 1], \|\cdot\|_\phi)$. Clearly,

$$P[0, 1] \subset C_{\phi,0}[0, 1] \subset C_M^\infty[0, 1]. \quad (15)$$

And the integral and differential operations are bounded operators.

For space $C_{\phi,0}[0, 1]$, we have the following representation theorem.

Theorem 1. *The set $C_{\phi,0}[0, 1]$ has the following representation:*

$$C_{\phi,0}[0, 1] = \left\{ f(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!} \mid \lim_{n \rightarrow \infty} a_n = 0 \right\}. \quad (16)$$

Moreover, the space $C_{\phi,0}[0, 1]$ is isomorphic to the sequence space c_0 , and the isomorphism mapping is given by

$$T\{a_n\} = f(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}, \quad \forall \{a_n\} \in c_0. \quad (17)$$

Motivated by above result, for $p \geq 1$, we define a new norm on the space $P[0, 1]$ by

$$\|f\|_{\phi,p} = \left(\sum_k \|D^k f\|_{\infty}^p \right)^{1/p}. \quad (18)$$

The completion of space $P[0, 1]$ under this norm is denoted by $C_{\phi,p}$. Similarly, we have the following result.

Theorem 2. *The space $C_{\phi,p}[0, 1]$ has a representation:*

$$C_{\phi,p}[0, 1] = \left\{ f(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!} \mid \sum_{n=0}^{\infty} |a_n|^p < \infty \right\}. \quad (19)$$

Furthermore, $C_{\phi,p}[0, 1]$ is isomorphic to the space ℓ^p , and the isomorphism operator is given by

$$T\{a_n\} = f(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}, \quad \forall \{a_n\} \in \ell^p. \quad (20)$$

Theorem 3. *The space $C_{\phi,\infty}[0, 1]$ has a representation:*

$$C_{\phi,\infty}[0, 1] = \left\{ f(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!} \mid \sup_{n \geq 0} |a_n| < \infty \right\}. \quad (21)$$

Moreover, $C_{\phi,\infty}[0, 1] = C_M^{\infty}[0, 1]$ is isomorphic to ℓ^{∞} , and the isomorphism operator is given by

$$T\{a_n\} = f(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}, \quad \forall \{a_n\} \in \ell^{\infty}. \quad (22)$$

By now, we have gotten a series spaces in which both the differential operator and integral operator are bounded linear operators. Obviously, these sets have the following inclusion relations:

$$\begin{aligned} P[0, 1] &\subset C_{\phi,1}[0, 1] \subset C_{\phi,p}[0, 1] \subset C_{\phi,0}[0, 1] \\ &\subset C_{\phi,\infty}[0, 1] = C_M^{\infty}[0, 1], \quad 1 \leq p < \infty. \end{aligned} \quad (23)$$

We observe that for $p \geq 1$, in definition of these spaces, the term

$$\sum_{n=0}^{\infty} \|D^n f\|_{\infty}^p < \infty, \quad (24)$$

means that the terms $\sum_{k=1}^n \|D^{n+k} f\|_{\infty}^p$ are small as n is large enough.

To insert a new space between $C_{\phi,p}[0, 1]$ and $C_{\phi,0}[0, 1]$, let us consider the sequence formed by $f \in C_{\phi,0}[0, 1]$,

$$f(x), f'(x), f''(x), \dots, f^{(n)}(x), \dots; \quad (25)$$

the norm $\|f\|_{\phi} = \sup_{n \geq 0} \|D^n f\|_{\infty}$ is maximum of the function sequence under space $C[0, 1]$. To define a new normed space, in Section 2, we shall study new summation approach for a sequence. Based on such a new summation approach, we shall define some new sequence spaces. In Section 3, we shall define some function spaces and discuss their completeness. Furthermore, we compare these spaces and give some inclusion relations.

2. New Summation Method

2.1. Summation of Absolutely Dominant Operator. Let $\{a_n\}$ be a sequence (real or complex number) and satisfy $\lim_{n \rightarrow \infty} a_n = 0$. We define an operator σ by

$$\sigma\{a_n\} = \{a_{\sigma(k)}; k \geq 0\}, \quad (26)$$

where $|a_{\sigma(k)}| \geq |a_{\sigma(k+1)}|$. It is called the absolutely dominant queuing operator.

Removing the first n terms, the remainder sequence $\{a_{n+k}; k \geq 0\}$ has a new queuing:

$$\sigma\{a_{n+k}\} = \{a_{n+\sigma(k)}; k \geq 0\}. \quad (27)$$

Using the queuing operator, for a sequence $\{a_n\}$ of zero limit (this is only used to ensure that we can take the maximal value for such a sequence), we define a positive number by

$$s_0 = \sup_{n \geq 0} |a_n| = |a_{\sigma(0)}|. \quad (28)$$

By removing the first term a_0 from $\{a_n\}_{n=0}^{\infty}$, we define the second number s_1 , that is, the absolute summation of the first two terms in $\sigma\{a_{n+1}\}$, that is,

$$s_1 = \sum_2 \sigma\{a_{1+n}, n \geq 0\} = |a_{1+\sigma(0)}| + |\sigma_{1+\sigma(1)}|, \quad (29)$$

where the number in subscription of summation is the number of the terms σ denotes the absolutely dominant queuing operator.

After removing the first two terms a_0 and a_1 from $\{a_n\}_{n=0}^{\infty}$, we define the third positive number s_2 , that is, the absolute summation of the first three terms in $\sigma\{a_{2+n}\}$, that is,

$$s_2 = \sum_3 \sigma\{a_{2+n}, n \geq 0\} = |a_{2+\sigma(0)}| + |a_{2+\sigma(1)}| + |a_{2+\sigma(2)}|. \quad (30)$$

Generally, we define positive number s_k as

$$\begin{aligned} s_k &= \sum_{k+1} \sigma\{a_{k+n}, n \geq 0\} = |a_{k+\sigma(0)}| + |a_{k+\sigma(1)}| \\ &\quad + |a_{k+\sigma(2)}| + \dots + |a_{k+\sigma(k)}|. \end{aligned} \quad (31)$$

According to this rule, we get a new nonnegative sequence $\{s_k; k \geq 0\}$.

Example 4. Let $\{a_k\} = \{1, 2, 3, 4, 5, 6\}$. Then, we have

$$\begin{aligned} s_0 &= 6, & s_1 &= 6 + 5, \\ s_2 &= 6 + 5 + 4, & s_3 &= 6 + 5 + 4, \\ s_4 &= 6 + 5, & s_5 &= 6. \end{aligned} \quad (32)$$

2.2. Relationship between Summation and Order of Sequence. To explain the thing we concerned about, let us see an example.

Example 5. Let scalar group be $\{b_k\} = \{6, 5, 4, 3, 2, 1\}$. Then, we have

$$\begin{aligned}\hat{s}_0 &= 6, & \hat{s}_1 &= 5 + 4, \\ \hat{s}_2 &= 4 + 3 + 2, & \hat{s}_3 &= 3 + 2 + 1, \\ \hat{s}_4 &= 2 + 1, & \hat{s}_5 &= 1.\end{aligned}\quad (33)$$

Comparing this with Example 4, we see that the summation of a sequence has a relationship with its order.

From Examples 4 and 5, we see that the sequences $\{\hat{s}_n\}$ and $\{s_n\}$ generated by a new summation method have relation of order of a sequence. In the sequel, we mainly discuss the infinite sequence. If a sequence has only finite many terms, we shall complement zero after the last term so that it becomes an infinite sequence.

2.3. Distribution of s -Sequence. In this subsection, we shall consider the distribution of the sequence $\{s_k, k \geq 0\}$. We discuss it according to the different cases.

(1) Let $\{a_k\}$ be a positive and increasing sequence, that is,

$$0 < a_0 < a_1 < a_2 < a_3 < \cdots < a_{N-1} < a_N. \quad (34)$$

According to the absolutely dominant summation, we have

$$\begin{aligned}s_0 &= a_N, \\ s_1 &= a_N + a_{N-1}, \\ s_2 &= a_N + a_{N-1} + a_{N-2}, \\ &\dots\end{aligned}\quad (35)$$

$$s_k = \sum_{j=0}^k a_{N-j}, \quad k \leq \frac{N}{2},$$

when $N = 2m + 1$, the sequence has even terms $2(m + 1)$; then

$$\begin{aligned}s_{m-1} &= \sum_{j=0}^{m-1} a_{N-j}, \\ s_m &= \sum_{j=0}^m a_{N-j}, \\ s_{m+1} &= \sum_{j=0}^m a_{N-j}, \\ s_{m+2} &= \sum_{j=0}^{m-1} a_{N-j}, \dots, \\ s_{m+k} &= \sum_{j=0}^{m+1-k} a_{N-j} \\ &\dots \\ s_{N-1} &= a_N + a_{N-1}, \\ s_N &= a_N.\end{aligned}\quad (36)$$

In this case, the s -sequence has a U -type distribution shown as follows:

$$\begin{array}{ccccccc}s_0 & & s_1 & & s_2 & \cdots & s_{m-1} & s_m & s_{m+1} & \cdots & s_{N-2} & s_{N-1} & s_N.\end{array}\quad (37)$$

If $N = 2m$, the sequence has odd terms $(2m + 1)$; then

$$\begin{aligned}s_{m-1} &= \sum_{j=0}^{m-1} a_{N-j}, \\ s_m &= \sum_{j=0}^m a_{N-j}, \\ s_{m+1} &= \sum_{j=0}^{m-1} a_{N-j}, \dots, \\ s_{m+k} &= \sum_{j=0}^{m-k} a_{N-j}, \dots \\ s_{N-1} &= a_N + a_{N-1}, \\ s_N &= a_N.\end{aligned}\quad (38)$$

In this case, the s -sequence has V -type distribution as follows:

$$\begin{array}{ccccccc}s_0 & & s_1 & & s_2 & \cdots & s_{m-1} & s_m & s_{m+1} & \cdots & s_{N-2} & s_{N-1} & s_N.\end{array}\quad (39)$$

If $\{a_k\}$ is an increasing sequence, then $\{s_k\}$ has a symmetrical form; the s -data at the medial term is its maximal value.

(2) $\{a_n\}$ is a positive and decreasing sequence

$$a_N > a_{N-1} > a_{N-2} > \cdots > a_{N/2} > a_{(N/2)-1} > \cdots > a_1 > a_0. \quad (40)$$

According to the absolutely dominant summation, we have

$$\begin{aligned}s_0 &= a_N, \\ s_1 &= a_{N-1} + a_{N-2}, \\ s_2 &= a_{N-2} + a_{N-3} + a_{N-4}, \\ s_k &= \sum_{j=0}^k a_{N-k-j}, \quad 0 \leq k \leq \frac{N}{2}.\end{aligned}\quad (41)$$

If $N = 2m + 1$, the sequence has an even term $2(m + 1)$; then

$$\begin{aligned} s_{m-1} &= \sum_{j=0}^{m-1} a_{N-(m-1)-j}, \\ s_m &= \sum_{j=0}^m a_{N-m-j}, \\ s_{m+1} &= \sum_{j=0}^m a_{N-j} \\ s_{m+2} &= \sum_{j=0}^{m-1} a_{N-j}, \dots, \\ s_{N-k} &= \sum_{j=0}^k a_j, \quad s_N = a_0. \end{aligned} \quad (42)$$

If $N = 2m$, the sequence has odd terms $(2m + 1)$, then

$$\begin{aligned} s_{m-1} &= \sum_{j=0}^{m-1} a_{N-j}, \\ s_m &= \sum_{j=0}^m a_j, \\ s_{m+1} &= \sum_{j=0}^{m-1} a_j, \dots, \\ s_{N-k} &= \sum_{j=0}^k a_j, \\ s_N &= a_0. \end{aligned} \quad (43)$$

So, s -sequence has distribution as

$$\begin{array}{ccccccc} & & s_k & & & & \\ & \dots & & s_{k+1} & \dots & & \\ s_0 & s_1 & & & & s_m & \dots \\ & & & & & & s_N \end{array} \quad (44)$$

In this case, the s -sequence has a character that the initial original data is the absolute largest; at the first several steps, s -sequence arrives at its maximum value; after then, s -sequence decreases until it arrives at its minimum value. The final value is minimal.

(3) Let $\{a_n\}$ be an arbitrary sequence.

In this case, it is difficult to give a general distribution. Although so, we have the following relation

$$s_0 = \sup_{k \geq 0} |a_k| = |a_{\sigma(0)}|, \quad s_k^m \leq s_k \leq s_k^M, \quad \forall k, \quad (45)$$

where s_k^m denotes the value in decreasing queuing, s_k^M denotes the value in increasing queuing.

From above, we see that s -sequence undergoes a great contortion due to different queuing order of a sequence. Denote by $J(a_n)$ the maximum change of s -sequence for $\{a_n\}$ under different queuing order; then

$$J(a_n) = \frac{\max \{s_k^M\}}{\max \{s_k^m\}}. \quad (46)$$

For the sequence $\{a_n\} = \{1, 2, 3, 4, 5, 6\}$, it is easy to see that $J(a_n) = 15/9$.

2.4. Generalized p -Summation. From the absolutely dominant summation, we get a new sequence $\{s_k\}$. Now, we define a new p -summation for $\{a_n\}$ sequence:

$$s_{0,p} = s_0 = \sup_{n \geq 0} |a_n| = |a_{\sigma(0)}|. \quad (47)$$

Removing the first term a_0 , we define the second value $s_{1,p}$ by

$$s_{1,p} = \sum_{2,p} \sigma \{a_{1+n}, n \geq 0\} = (|a_{1+\sigma(0)}|^p + |a_{1+\sigma(1)}|^p)^{1/p}. \quad (48)$$

Again, removing the second term a_1 , we define the summation of the first three absolute maximum value as $s_{2,p}$, that is,

$$\begin{aligned} s_{2,p} &= \sum_{3,p} \sigma \{a_{2+n}, n \geq 0\} \\ &= (|a_{2+\sigma(0)}|^p + |a_{2+\sigma(1)}|^p + |a_{2+\sigma(2)}|^p)^{1/p}. \end{aligned} \quad (49)$$

In general, we define

$$\begin{aligned} s_{k,p} &= \sum_{k+1,p} \sigma \{a_{k+n}, n \geq 0\} \\ &= (|a_{k+\sigma(0)}|^p + |a_{k+\sigma(1)}|^p + |a_{k+\sigma(2)}|^p \\ &\quad + \dots + |a_{k+\sigma(k)}|^p)^{1/p}. \end{aligned} \quad (50)$$

According to this definition, we get a new nonnegative sequence $\{s_{k,p}; k \geq 0\}$.

Obviously, when $p = 1$, we recover the above summation sequence $s_k = s_{k,1}$, for all $k \geq 0$. For the sequence $\{a_n\}$, we define a positive number $\|\{a_n\}\|_{G,p}$ by

$$\|\{a_n\}\|_{G,p} = \sup_{k \geq 0} s_{k,p}, \quad (51)$$

where G denotes the generalized summation and p denotes the p -norm in finite-dimensional space.

In particular, when $\{a_n\} \in \ell^p$, $s_{k,p} \rightarrow 0$ as $k \rightarrow \infty$, and hence $\|\{a_n\}\|_{G,p} < (\sum_{n \geq 0} |a_n|^p)^{1/p} < \infty$.

Now, we define the sequence spaces by

$$\begin{aligned} c_{G,p,M} &= \left\{ \{a_n\} \mid \sup_{k \geq 0} s_{k,p} < \infty \right\}, \\ c_{G,p} &= \left\{ \{a_n\} \mid \lim_{k \rightarrow \infty} s_{k,p} = 0 \right\}. \end{aligned} \quad (52)$$

We can prove the following result.

Theorem 6. $(C_{G,p,M}, \|\cdot\|_{G,p})$ and $(C_{G,p}, \|\cdot\|_{G,p})$ are Banach spaces.

3. $C_{G,p}[0, 1]$ -Type Spaces

3.1. *Definition of Spaces $C_{G,p,M}[0, 1]$.* Let $p \geq 1$. For each $f \in C_{\phi,0}[0, 1]$, we define positive number:

$$\begin{aligned} s_{k,p}(f) &= \sum_{n,p} \sigma \{D^{k+n} f, n \geq 0\} \\ &= \left(\|D^{k+\sigma(0)}(f)\|_{\infty}^p + \|D^{k+\sigma(1)}(f)\|_{\infty}^p \right. \\ &\quad \left. + \|D^{k+\sigma(2)}(f)\|_{\infty}^p + \dots + \|D^{k+\sigma(k)}(f)\|_{\infty}^p \right)^{1/p}. \end{aligned} \quad (53)$$

Example 7. To show the computation method, let us consider the case that $p = 1$ and discuss the function $e_n(x) = x^n/n!$.

By computing various derivative functions

$$\begin{aligned} \|D^0 e_n\|_{\infty} &= \frac{1}{n!}, \\ \|D^1 e_n\|_{\infty} &= \frac{1}{(n-1)!}, \\ \|D^2 e_n\|_{\infty} &= \frac{1}{(n-2)!}, \\ &\dots \\ \|D^k e_n\|_{\infty} &= \frac{1}{(n-k)!}, \\ &\dots \\ \|D^{n-2} e_n\|_{\infty} &= \frac{1}{2!}, \\ \|D^{n-1} e_n\|_{\infty} &= 1, \\ \|D^n e_n\|_{\infty} &= 1, \\ \|D^{n+1} e_n\|_{\infty} &= 0, \\ \|D^{n+2} e_n\|_{\infty} &= 0, \\ &\dots \end{aligned} \quad (54)$$

We calculate $s_k(e_n)$ as follows:

$$\begin{aligned} s_0(e_n) &= \sup_{k \geq 0} \|D^k e_n\|_{\infty} = 1, \\ s_1(e_n) &= \sum_2 \sigma \{D^{1+k} e_n, k \geq 0\} = 1 + 1, \end{aligned}$$

$$s_2(e_n) = \sum_3 \sigma \{D^{2+k} e_n, k \geq 0\} = 1 + 1 + \frac{1}{2!},$$

....

$$\begin{aligned} s_j(e_n) &= \sum_k \sigma \{D^{j+k} e_n, k \geq 0\} \\ &= 1 + 1 + \frac{1}{2!} + \dots + \frac{1}{j!}, \quad j \leq \frac{n}{2}, \end{aligned}$$

$$s_{(n/2)}(e_n) = \sum_{n/2} \sigma \{D^{(n/2)+k} e_n, k \geq 0\} = \sum_{k=0}^{n/2} \frac{1}{k!},$$

$$s_{(n/2)+1} = \sum_{k=0}^{n/2} \frac{1}{k!},$$

$$s_{(n/2)+2} = \sum_{k=0}^{(n/2)-1} \frac{1}{k!},$$

$$s_{n-2} = 1 + 1 + \frac{1}{2!},$$

$$s_{n-1} = 1 + 1,$$

$$s_n = 1,$$

$$s_{n+j} = 0, \quad j \geq 1.$$

(55)

According to the definition of $s_{k,p}(f)$, it has the following properties:

(1) for any $k \geq 0$,

$$s_{0,p}(f) = \sup_{n \geq 0} \|D^n f\|_{\infty}, \quad (56)$$

$$s_{k,p}(f) \geq \|D^k f\|_{\infty} \geq 0;$$

(2) for each $\alpha \in \mathbb{C}$,

$$s_{k,p}(\alpha f) = |\alpha| s_{k,p}(f), \quad \forall f \in C_{\phi,0}[0, 1]; \quad (57)$$

(3) for any $g, f \in C_{\phi,0}[0, 1]$,

$$s_{k,p}(f + g) \leq s_{k,p}(f) + s_{k,p}(g). \quad (58)$$

This can be obtained from the Minkowski inequality.

These properties show that $s_{0,p}(f)$ satisfies the norm axiom, and hence it is a norm on space $C_{\phi,0}[0, 1]$. But for $k \geq 1$, $s_{k,p}(f)$ only is a seminorm (or prenorm), it is continuous in the topology of $C_{\phi,0}[0, 1]$.

For each $f \in C_{\phi,0}[0, 1]$, we define the following:

$$\|f\|_{G,p} = \sup_{n \geq 0} s_{n,p}(f). \quad (59)$$

We define the following functions spaces:

$$C_{G,p,M}[0, 1] = \{f \in C_{\phi,0}[0, 1] \mid \|f\|_{G,p} < \infty\}, \quad (60)$$

$$C_{G,p}[0, 1] = \left\{f \in C_{\phi,0}[0, 1] \mid \lim_{k \rightarrow \infty} s_{k,p}(f) = 0\right\}. \quad (61)$$

Theorem 8. For $f \in C_{\phi,0}[0,1]$, the scalar $\|f\|_{G,p}$ is defined as above. Then, $\|f\|_{G,p}$ is a norm on $C_{G,p,M}[0,1]$, and hence $(C_{G,p,M}[0,1], \|\cdot\|_{G,p})$ is a normed linear space. Moreover, $(C_{G,p,M}[0,1], \|\cdot\|_{G,p})$ is a Banach space, which is isomorphic to the sequence space $c_{G,p,M}$.

Proof. Here, we only prove the second assertion.

Let $f_n \in C_{G,p,M}[0,1]$ be a Cauchy sequence, that is, for any $\varepsilon > 0$, there exists $N(\varepsilon)$ such that

$$\|f_n - f_m\|_{G,p} < \varepsilon, \quad \forall n, m > N(\varepsilon). \quad (62)$$

For any $k \geq 0$, it holds that

$$\|D^k(f_n - f_m)\|_{\infty} \leq s_{k,p}(f_n - f_m) \leq \|f_n - f_m\|_{G,p}. \quad (63)$$

This shows that $\{f_n\}$ also is a Cauchy sequence in $C_{\phi,0}[0,1]$. By the completeness of space $C_{\phi,0}[0,1]$, there is a $f_0 \in C_{\phi,0}[0,1]$ such that $\|f_n - f_0\|_{\phi} \rightarrow 0$.

For each $k \geq 0$, by the continuity of $s_{k,p}(f)$, we can get

$$\|D^k(f_n - f_0)\|_{\infty} \leq s_{k,p}(f_n - f_0) \leq \varepsilon. \quad (64)$$

Thus, we find that

$$\|f_n - f_0\|_{G,p} = \sup_{k \geq 0} s_{k,p}(f_n - f_0) \leq \varepsilon, \quad \forall n > N(\varepsilon). \quad (65)$$

This means that the sequence converges to f_0 in the sense of norm on $C_{G,p,M}[0,1]$. Since $s_{k,p}(f)$ is a seminorm, it holds that

$$\begin{aligned} s_{k,p}(f_0) &\leq s_{k,p}(f_n - f_0) + s_{k,p}(f_n) \\ &< \varepsilon + s_{k,p}(f_n), \quad \text{for } n > N(\varepsilon). \end{aligned} \quad (66)$$

Therefore, $f_0 \in C_{G,p,M}[0,1]$, which implies that $C_{\phi,p,M}[0,1]$ is a Banach space.

For each $f \in C_{G,p,M}[0,1]$,

$$f(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!},$$

$$|a_k| = |D^k f(0)| \leq \|D^k f\|_{\infty} \leq \sum_{n=0}^{\infty} |a_{n+k}| \frac{1}{n!} \leq e \sup_{n \geq 0} |a_{k+n}|. \quad (67)$$

Clearly,

$$\begin{aligned} |a_{k+\sigma(0)}| &\leq \|D^{k+\sigma(0)} f\|_{\infty} \leq e |a_{k+\sigma(0)}|, \\ |a_{k+\sigma(j)}| &\leq \|D^{k+\sigma(j)} f\|_{\infty} \leq e |a_{k+\sigma(j)}|. \end{aligned} \quad (68)$$

Thus, we have

$$s_{k,p} \leq s_{k,p}(f) \leq e s_{k,p}. \quad (69)$$

This implies that

$$\|\{a_n\}\|_{G,p} \leq \|f\|_{G,p} \leq e \|\{a_n\}\|_{G,p}. \quad (70)$$

Therefore, the mapping

$$T(f) = \{f^{(n)}(0)\}_{n=0}^{\infty}, \quad (71)$$

is a bounded invertible linear operator from $C_{G,p,M}[0,1]$ to $c_{G,p,M}$. $\square$

Theorem 9. Let $C_{G,p}[0,1]$ be defined as (61). Then, $C_{G,p}[0,1]$ is the completion of space $(P[0,1], \|\cdot\|_{G,p})$, and

$$C_{\phi,p}[0,1] \subset C_{G,p}[0,1] \subset C_{\phi,0}[0,1]. \quad (72)$$

Moreover, space $C_{G,p}[0,1]$ is isomorphic to $c_{G,p}$.

The proof of Theorem 9 is similar to that of Theorem 8; we omit the details.

3.2. Comparison of Spaces. So far, we have introduced some Banach spaces. The question is whether there is an inclusion relation $C_{\phi,q}[0,1] \subset C_{G,p}[0,1]$ for $q > p$. In general, the answer is negative.

Example 10. Let $q > p \geq 1$. Then, $C_{\phi,q}[0,1] \not\subset C_{G,p,M}[0,1]$

Take $p' \in (p, q)$. Define a function by

$$f(x) = \sum_{k=1}^{\infty} \left(\frac{1}{k}\right)^{1/p'} \frac{x^k}{k!}. \quad (73)$$

Obviously, $f \in C_{\phi,q}[0,1]$, but $f \notin C_{\phi,p}[0,1]$. Furthermore, we also have $f \notin C_{G,p,M}[0,1]$.

In fact, for any $n \in \mathbb{N}$,

$$D^n f(0) = \left(\frac{1}{n}\right)^{1/p'}. \quad (74)$$

Clearly,

$$\sum_{n=1}^{\infty} |D^n f(0)|^q = \sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^{q/p'} < \infty. \quad (75)$$

Due to

$$D^n f(x) = \sum_{k=1}^{\infty} \left(\frac{1}{k}\right)^{1/p'} \frac{x^{k-n}}{(k-n)!} = \sum_{r=0}^{\infty} \left(\frac{1}{n+r}\right)^{1/p'} \frac{x^r}{r!}, \quad (76)$$

$$\|D^n f\|_{\infty} \leq \sum_{k=0}^{\infty} \frac{1}{(n+k)^{1/p'}} \frac{1}{k!}, \quad \forall n \geq 1,$$

it holds that

$$\begin{aligned} \sum_{n=1}^{\infty} |D^n f(0)|^q &\leq \sum_{n=1}^{\infty} \|D^n f\|_{\infty}^q \leq \sum_{n=1}^{\infty} \left(\sum_{k=0}^{\infty} \left(\frac{1}{n+k}\right)^{1/p'} \frac{1}{k!} \right)^q \\ &< \infty. \end{aligned} \quad (77)$$

For any $n \in \mathbb{N}$,

$$\sum_{k=1}^n |D^{n+k} f(0)|^p = \sum_{k=1}^n \left(\frac{1}{n+k} \right)^{p/p'} = \frac{1}{n^{p/p'}} \sum_{k=1}^n \frac{1}{(1+(k/n))^{p/p'}}, \quad (78)$$

while

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \frac{1}{(1+(k/n))^{p/p'}} = \int_0^1 \frac{dx}{(1+x)^{p/p'}}, \quad (79)$$

this shows that $\sup_{n \geq 1} \sum_{k=1}^n |D^{n+k} f(0)|^p = \infty$, and hence $\sup_{n \geq 0} s_{n,p}(f) = \infty$. So, $f \notin C_{G,p,M}[0, 1]$.

Example 11. It holds that $C_{\phi,p}[0, 1] \neq C_{G,p}[0, 1]$.

Define a function as

$$f(x) = \sum_{n=1}^{\infty} \left(\frac{1}{(1+n) \ln(1+n)} \right)^{1/p} \frac{x^n}{n!}. \quad (80)$$

Obviously,

$$\begin{aligned} D^k f(0) &= \left(\frac{1}{(1+k) \ln(1+k)} \right)^{1/p}, \\ \sum_{k=1}^n |D^k f(0)|^p &= \sum_{k=1}^n \frac{1}{(1+k) \ln(1+k)} \\ \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{(1+k) \ln(1+k)} &= \infty. \end{aligned} \quad (81)$$

So, $f \notin C_{\phi,p}[0, 1]$.

Furthermore,

$$\begin{aligned} |D^n f(0)| &\leq \|D^n f\|_{\infty} \\ &\leq \sum_{k=1}^{\infty} \left(\frac{1}{(1+n+k) \ln(1+n+k)} \right)^{1/p} \frac{1}{k!} \\ &\leq e |D^n f(0)|, \\ \sum_{k=1}^n |D^{n+k} f(0)|^p &= \sum_{k=1}^n \frac{1}{(n+k) \ln(n+k)} \\ &= \frac{1}{n \ln n} \sum_{k=1}^n \frac{1}{(1+(k/n)) (1+(\ln(1+(k/n))/\ln n))}. \end{aligned} \quad (82)$$

Notice the identity

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \frac{1}{(1+(k/n)) (1+(\ln(1+(k/n))/\ln n))} = \int_0^1 \frac{dx}{1+x}, \quad (83)$$

and so $f \in C_{G,p}[0, 1]$.

Example 12. It holds that $C_{G,p}[0, 1] \subset C_{G,p,M}[0, 1]$.

Let us consider the following function:

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{(1+n)^{1/p}} \frac{x^n}{n!}. \quad (84)$$

Since

$$D^n f(0) = \frac{1}{(1+n)^{1/p}}, \quad (85)$$

$$|D^n f(0)| \leq \|D^n f\|_{\infty} \leq e |D^n f(0)|,$$

we have

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{(1+k+n)} = \int_0^1 \frac{dx}{1+x}, \quad (86)$$

this means that $f \in C_{G,p,M}[0, 1]$, but $f \notin C_{G,p}[0, 1]$.

A new question is whether there is a inclusion relation between $C_{G,p,M}[0, 1]$ and $C_{G,q}[0, 1]$ for any $q > p$? The following result gives a positive answer.

Theorem 13. For any $q > p \geq 1$, the inclusion $C_{G,p,M}[0, 1] \subset C_{G,q}[0, 1]$ holds true.

Proof. Set $f \in C_{G,p,M}[0, 1]$. Then,

$$f(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}, \quad (87)$$

and for any $n \in \mathbb{N}$,

$$\begin{aligned} |D^n f(0)| &\leq \|D^n f\|_{\infty} \leq \sum_{k=0}^{\infty} |a_{n+k}| \frac{1}{k!} \leq e \sup_{k \geq 0} |a_{n+k}| \\ &= e |a_{n+\sigma(0)}|. \end{aligned} \quad (88)$$

From above, we get that

$$|D^{n+\sigma(0)} f(0)| \leq \|D^{n+\sigma(0)} f\|_{\infty} \leq e |a_{n+\sigma(0)}|. \quad (89)$$

Similarly, for $k \in \{1, 2, \dots, n\}$,

$$|D^{n+\sigma(k)} f(0)| \leq \|D^{n+\sigma(k)} f\|_{\infty} \leq e |a_{n+\sigma(k)}|. \quad (90)$$

Since $f \in C_{G,p,M}[0, 1]$, it holds that

$$\sum_{k=1}^n \|D^{(n+\sigma(k))} f\|_{\infty}^p \leq \|f\|_{G,p}^p, \quad \forall n \in \mathbb{N}. \quad (91)$$

Let $q > p \geq 1$. Set $q = p + \delta$, then

$$\begin{aligned} &\sum_{k=1}^n \|D^{(n+\sigma(k))} f\|_{\infty}^q \\ &= \sum_{k=1}^n \|D^{(n+\sigma(k))} f\|_{\infty}^{\delta} \|D^{(n+\sigma(k))} f\|_{\infty}^p \\ &\leq \|D^{n+\sigma(0)} f\|_{\infty}^{\delta} \|f\|_{G,p}^p, \quad \forall n \in \mathbb{N}. \end{aligned} \quad (92)$$

Since

$$\lim_{n \rightarrow \infty} \|D^n f\|_{\infty} = 0, \quad (93)$$

it holds that

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \|D^{(n+\sigma(k))} f\|_{\infty}^q = 0. \quad (94)$$

This gives that $f \in C_{G,q}[0, 1]$. $\square$

3.3. $C_{\phi,g,1}[0, 1]$ -Type Space. In the previous discussion, we insert the some spaces between $C_{\phi,p}[0, 1]$ and $C_{\phi,q}[0, 1]$. The questions is can one insert a Banach space between $P[0, 1]$ and $C_{\phi,1}[0, 1]$?

For each $f \in C_{\phi,1}[0, 1]$, whose norm is

$$\|f\|_{\phi,1} = \sum_{n=0}^{\infty} \|D^n f\|_{\infty}, \quad (95)$$

the corresponding s -sequence is

$$s_k(f) = \sum_k \sigma \{D^{k+n} f, n \geq 0\} = \sum_{j=0}^k \|D^{k+\sigma(j)} f\|_{\infty}, \quad \forall k \geq 0. \quad (96)$$

Define a positive number

$$\|f\|_{g,1} = \sum_{k=0}^{\infty} s_k(f), \quad (97)$$

where $(g, 1)$ denotes the ℓ^1 sum after the generalized 1-summation.

Define the function space by

$$C_{\phi,g,1}[0, 1] = \{f \in C_{\phi,1}[0, 1] \mid \|f\|_{g,1} < \infty\}. \quad (98)$$

Then, we have the following result.

Theorem 14. Let $\|f\|_{g,1}$ be defined as before. Then, for any $f \in C_{\phi,g,1}[0, 1]$,

(1) the following inequality holds

$$\|f\|_{g,1} \geq \|f\|_{\phi,1}, \quad \forall f \in C_{\phi,g,1}[0, 1]; \quad (99)$$

(2) $\|f\|_{g,1}$ is a norm on $C_{\phi,g,1}[0, 1]$. Further, $(C_{\phi,g,1}[0, 1], \|\cdot\|_{g,1})$ is a Banach space;

(3) $P[0, 1] \subset C_{\phi,g,1}[0, 1]$ is a dense subspace.

Proof. For any $f \in C_{\phi,1}[0, 1]$, we have

$$s_0(f) \geq \|f\|_{\infty}, \quad s_k(f) \geq \|D^k f\|_{\infty}, \quad \forall k \geq 1. \quad (100)$$

This leads to

$$\sum_{k=0}^n s_k(f) \geq \sum_{k=0}^n \|D^k f\|_{\infty}. \quad (101)$$

Obviously, when $f \in C_{\phi,g,1}[0, 1]$, it holds that $\|f\|_{\phi,1} \leq \|f\|_{g,1}$.

Since $s_k(f)$ is a seminorm, the above relation shows that $\|f\|_{g,1}$ is a norm. In what follows, we shall prove that $C_{\phi,g,1}[0, 1]$ is a Banach space.

Let $f_n \in C_{\phi,g,1}[0, 1]$ be a Cauchy sequence, that is,

$$\lim_{m,n \rightarrow \infty} \|f_n - f_m\|_{g,1} = 0. \quad (102)$$

Since

$$\|f_n - f_m\|_{\phi,1} \leq \|f_n - f_m\|_{g,1}, \quad (103)$$

this implies that $\{f_n\} \subset C_{\phi,1}[0, 1]$ also is a Cauchy sequence, so there exists a $f_0 \in C_{\phi,1}[0, 1]$ such that

$$\lim_{m \rightarrow \infty} \|f_m - f_0\|_{\phi,1} = 0. \quad (104)$$

For any $k \in \mathbb{N}$, when N is large enough, we have

$$\sum_{j=0}^k s_j(f_n - f_m) \leq \|f_n - f_m\|_{g,1} < \varepsilon, \quad \forall n, m > N. \quad (105)$$

Using the continuity of $s_k(f)$ with respect to $\|f\|_{\phi,1}$, and taking $m \rightarrow \infty$, we get that

$$\sum_{j=0}^k s_j(f_n - f_0) \leq \varepsilon, \quad \forall n \geq N. \quad (106)$$

Again taking $k \rightarrow \infty$, we get that

$$\|f_n - f_0\|_{g,1} \leq \varepsilon, \quad \forall n > N. \quad (107)$$

So, $f_0 \in C_{\phi,g,1}[0, 1]$, and

$$\lim_{n \rightarrow \infty} \|f_n - f_0\|_{g,1} = 0. \quad (108)$$

The completeness of the space follows.

Let $p(x)$ be a n -order polynomial. Then,

$$s_0(p) \geq \sup_{k \geq 0} |p(x)|, \quad s_k(p) \leq \|p\|_{\phi,1}, \quad (109)$$

$$s_n(p) = \|D^n p\|_{\infty}, \quad s_{n+k}(p) = 0, \quad k \geq 1.$$

Therefore,

$$\|p\|_{g,1} = \sum_{k=0}^n s_k(p) < \infty. \quad (110)$$

For any $f \in C_{\phi,g,1}[0, 1]$,

$$f(x) = \sum_{j=0}^{\infty} a_j \frac{x^j}{j!}, \quad (111)$$

for $\varepsilon > 0$, there exists a N such that

$$\frac{e}{2} \sup_{j \geq n+1} |a_{j+k}| + \sum_{k=n}^{\infty} s_k(f) < \varepsilon, \quad \forall n > N. \quad (112)$$

Taking a polynomial

$$p_n(x) = \sum_{j=0}^n a_j \frac{x^j}{j!} \quad (113)$$

and noticing

$$\begin{aligned} \|(f - p_n)^{(k)}\|_{\infty} &\leq \sum_{j=n}^{\infty} |a_j| \frac{1}{(j-k)!} \\ &\leq \sup_{j \geq n} |a_j| \sum_{j=n}^{\infty} \frac{1}{(j-k)!}, \quad k \leq n, \end{aligned} \quad (114)$$

and $(f - p_n)^{(k)} = f^{(k)}$ for $k > n$, we have

$$\|f - p_n\|_{g,1} \leq \frac{e}{2} \sup_{j \geq n+1} |a_{j+k}| + \sum_{k=n}^{\infty} s_k(f) < \varepsilon. \quad (115)$$

This means that $P[0, 1]$ is dense in $C_{\phi,g,1}[0, 1]$. $\square$

3.4. Some Open Questions. From the previous discussion we see that

$$\begin{aligned} C_{\phi,1}[0, 1] &\subset C_{\phi,p}[0, 1] \subset C_{\phi,q}[0, 1] \\ &\subset C_{\phi,M}[0, 1], \quad \forall 1 < p < q, \\ C_{\phi,p}[0, 1] &\subset C_{G,p}[0, 1] \subset C_{G,p,M}[0, 1] \subset C_{G,q}[0, 1] \\ &\subset C_{G,q,M}[0, 1], \quad \forall 1 < p < q. \end{aligned} \quad (116)$$

Note that the first relation

$$C_{\phi,q}[0, 1] \subset C_{G,q}[0, 1] \subset C_{G,q,M}[0, 1], \quad (117)$$

and the relation $C_{\phi,q}[0, 1] \not\subset C_{G,p,M}[0, 1]$ (see, Example 10). These spaces are based on the new summation approach.

Now, we propose some open questions in mathematics aspect.

Question 1. Does the inclusion $C_{G,p,M}[0, 1] \subset C_{\phi,q}[0, 1]$ hold? yes or no.

Question 2. What is the dual space of $C_{G,p}[0, 1]$? Is it a reflexive space?

Question 3. Can one find out a Banach space $(\tilde{P}[0, 1], \|\cdot\|_*)$, that is, the least in the following sense: $\tilde{P}[0, 1]$ is the completion of $P[0, 1]$ under the norm $\|\cdot\|_*$ such that, for every Banach space $\mathbb{X}$ of functions, $\tilde{P}[0, 1]$ embeds densely and continuously into $\mathbb{X}$?

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