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Research Article

The D-Property of Finite Unions of t-Metrizable Spaces and Certain Function Spaces

Xin Zhang and Hongfeng Guo

School of Mathematics and Quantitative Economics, Shandong University of Finance and Economics, Jinan, Shandong 250014, China

Correspondence should be addressed to Hongfeng Guo; guohongfeng@sdufe.edu.cn

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The additivity of D-property is studied on t-metrizable spaces and certain function spaces. It is shown that a space of countable tightness is a D-space provided that it is the union of finitely many t-metrizable subspaces, or function spaces $C_p(X_i)$ where each X_i is Lindelöf Σ.

1. Introduction and Definitions

The class of D-spaces was introduced by van Douwen and Pfeffer in [1]. It is well known that the extent coincides with the Lindelöf number in a D-space, every countably compact D-space is compact and every D-space of countable extent is Lindelöf. These facts make it valuable as a covering property.

A lot of work has been done these years by many topologists, especially by Arhangel'skii and Buzyakova (see [2–5]), Gruenhage (see [6]), Peng (see [7–9]), Fleissner and Stanley (see [10]), Soukup (see [11, 12]), Nyikos (see [13]), Alas et al. (see [14]), and so forth.

Among the topics for studying D-spaces, the additivity of D-property has been an important one since Arhagel'skii raised the question in [3] whether the union of two D-subspaces is a D-space. Recently, Soukup and Szeptycki constructed in [11] a T_2 non D-space which is the union of two D-subspaces. However, the answer is positive in some typical D-classes (see [2, 4, 15–17]). Then it becomes an interesting work to find important D-classes that preserve D-property under finite unions. Motivated by this point, we try to discover some more general classes and obtain that a space of countable tightness is a D-space if it is the union of finitely many t-metrizable spaces, or function spaces $C_p(X_i)$ where each X_i is Lindelöf Σ. It must be pointed out, in our work, we use the concept of nearly good relations, which was first introduced and well used by Gruenhage in [6]. To exhibit its importance, some more examples are shown in [18]. In this paper, we use it creatively to deal with the finite unions of D-spaces. We believe that more results will be obtained if we pay more attention to it.

For convenience, we show some related definitions below. All spaces we consider in this paper are assumed to be T_1 spaces.

Firstly, we define $[A]^{<\omega} = \{H \subset A : |H| < \omega\}$ for a set A, and denote by $\overline{A}$ the closure of A in the whole space and by $\mathrm{Cl}_Y A$ the closure of A in the space Y. The symbol $\mathbb{N}$ stands for the set of all positive natural numbers and $\mathbb{R}$ for the real line equipped with the usual metric.

Definition 1 (see [6]). A relation R from X to $[X]^{<\omega}$ is *nearly good* if $x \in \overline{A}$ implies xRH for some $H \in [A]^{<\omega}$.

Definition 2 (see [6]). Given a neighborhood assignment ϕ on X, a subset Z of X is *ϕ-close* if $x, x' \in Z \Rightarrow x \in \phi(x')$ (equivalently, $Z \subset \phi(x)$ for every $x \in Z$).

Definition 3 (see [19]). A topological space (X, τ) is *t-metrizable* if there exists a metrizable topology π on X with $\tau \subset \pi$ and an assignment $H \mapsto J_H$ from $[X]^{<\omega}$ to $[X]^{\leq\omega}$ such that $\overline{A}^{\tau} \subset \overline{\bigcup_{H\in[A]^{<\omega}} J_H}^{\pi}$ for every $A \subset X$.

Definition 4 (see [19]). A cover $\mathscr{L}$ of a topological space X is *thick* if it satisfies the following condition:

One can assign $\mathscr{L}(H) \in \mathscr{L}^{\leq\omega}$ to each $H \in [X]^{<\omega}$ so that $\overline{A} \subset \bigcup\{\bigcup \mathscr{L}(H) : H \in [A]^{<\omega}\}$ for every $A \subset X$.

Definition 5 (see [20]). A space X has *countable tightness* if $x \in \overline{A}$ implies that $x \in \overline{C}$ for some countable subset C of A.

Definition 6 (see [21]). The topology of $C(X, L)$ is called a *point-open topology* if the family $\{[S, V] : S \in [X]^{<\omega}, V \in \mathcal{T}_X\}$ is a subbase of the topology, where $[S, V] = \{f \in C(X, L) : f(S) \subset V\}$ and $\mathcal{T}_X$ is the topology of X. The point-open topology is denoted by $C_p(X, L)$, and when $L = \mathbb{R}$, denoted by $C_p(X)$ for short.

Definition 7 (see [22]). A *Lindelöf* Σ-space is known as a *K-countably determined* space, that is, there is a cover $\mathcal{K}$ by compact sets and a countable collection $\mathcal{F}$ such that, for any $K \in \mathcal{K}$ and $K \subset U$, where U is open in X, then $K \subset F \subset U$ for some $F \in \mathcal{F}$.

For other definitions and terminologies without showing here, please refer to [19–21].

2. Finite Unions of t-Metrizable Spaces and Function Spaces

In [6], Gruenhage introduced the concept of nearly good relations and build the following method to help discover D-classes.

Proposition 8 (see [6]). *Let ϕ be a neighborhood assignment on X. Suppose there is a nearly good relation from X to $[X]^{<\omega}$ such that for any $H \in [X]^{<\omega}$, $R^{-1}(H) \setminus \bigcup \phi(H)$ is the countable union of ϕ-close sets. Then there is a closed and discrete set D such that $\bigcup \phi(D) = X$.*

In [18], some interesting spaces are shown to be D-spaces by constructing reasonable nearly good relations. Among them, t-metrizable space is an important one. In this section, we use the method to discuss the relation between D-property and the finite unions of t-metrizable spaces. Note that it may be the first time to deal with finite unions of D-spaces in this way.

Lemma 9. *Suppose X has countable tightness and $X = \bigcup_{i=1}^{m} X_i$, where each X_i is t-metrizable. Then $Y = \bigcap_{i=1}^{m} \overline{X_i}$ is a D-space.*

Proof. For all $i \in \{1, 2, \ldots, m\}$, since X_i is t-metrizable, by [19, Theorem 3.4], let $\mathcal{L}_i = \bigcup_{n\in\mathbb{N}} \mathcal{L}_{i,n}$ be the network of X_i, where each $\mathcal{L}_{i,n}$ is a thick partition of X_i, and the assignment $\mathcal{L}_{i,n}(H) \in [\mathcal{L}_{i,n}]^{\leq\omega}$ to each $H \in [X_i]^{<\omega}$ satisfies that, for every $A \subset X_i$,

$$\mathrm{Cl}_{X_i} A \subset \bigcup \left\{ \bigcup \mathcal{L}_{i,n}(H) : H \in [A]^{<\omega} \right\}. \tag{1}$$

For all $n \in \mathbb{N}$ and $A \in [Y \cap X_i]^{<\omega}$, we define $\mathcal{F}_{i,n}(A)$ in the following way.

For each $x \in A$, and $j \neq i$, since $x \in \overline{Y \cap X_j}$ and X has countable tightness, there exists a $B_j(x) \in [Y \cap X_j]^{\leq\omega}$, such that $x \in \overline{B_j(x)}$. Now let $B_j(A) = \bigcup\{B_j(x) : x \in A\}$, and then we have that $A \subset \overline{B_j(A)}$. Let $\mathcal{F}_{i,n}(A) = \{L \cap Y : L \in \mathcal{L}_{i,n}(A)$, or $L \in \mathcal{L}_{j(n)}(C)$, where $C \in [B_j(A)]^{<\omega}, 1 \leq j \leq n, j \neq i\}$. It is trivial that $\mathcal{F}_{i,n}(A)$ is a countable family. Furthermore, it satisfies the following condition.

Claim. For any $H \subset X_i \cap Y$, we have that $\mathrm{Cl}_Y H \subset \bigcup\{\bigcup \mathcal{F}_{i,n}(A) : A \in [H]^{<\omega}\}$.

To prove the claim, let $x \in \mathrm{Cl}_Y H$. If $x \in X_i$, it follows from the thick property of $\mathcal{L}_{i,n}$ that there is an $A \in [H]^{<\omega}$ and $L \in \mathcal{L}_{i,n}(A)$ such that $x \in L$, and hence $x \in L \cap Y \in \mathcal{F}_{i,n}(A)$.

Or else, there exists $j \neq i$, such that $x \in X_j$. Since X has countable tightness, we can fix a $K \in [H]^{\leq\omega}$ such that $x \in \overline{K}$. For every $A \in [K]^{<\omega}$, we have $A \subset \overline{B_j(A)}$ by the construction of $B_j(A)$. Therefore, the following holds,

$$x \in \overline{K} = \overline{\bigcup_{A\in[K]^{<\omega}} A} \subset \overline{\bigcup_{A\in[K]^{<\omega}} \overline{B_j(A)}} = \overline{\bigcup_{A\in[K]^{<\omega}} B_j(A)}. \tag{2}$$

Let $B = \bigcup_{A\in[K]^{<\omega}} B_j(A)$. Then there exists $C \in [B]^{<\omega}$ and $L \in \mathcal{L}_{j(n)}(C)$ such that $x \in L$. Assume that $C = \{y_k : 1 \leq k \leq p\}$. For each y_k, there exists an $A_k \in [K]^{<\omega}$ such that $y_k \in B_j(A_k)$. Let $A_0 = \bigcup_{1\leq k\leq p} A_k$. Then $A_0 \in [K]^{<\omega} \subset [H]^{<\omega}$ and $C \in [B_j(A_0)]^{<\omega}$. Consequently, $x \in L \cap Y \in \mathcal{F}_{i,n}(A_0)$.

Thus, we complete the proof of Claim and proceed to prove Lemma 9.

Now, for every $A \in [Y]^{<\omega}$, let $\mathcal{F}_n(A) = \bigcup_{i=1}^{m} \mathcal{F}_{i,n}(A \cap X_i)$. Then $\mathcal{F}_n(A)$ is also a countable family.

Let ϕ be an arbitrary neighborhood assignment on Y, and define a relation R from Y to $[Y]^{<\omega}$ as follows,

$$xRH \Longleftrightarrow \exists n \in \mathbb{N}, L \cap Y \in \mathcal{F}_n(H) \left(x \in L \cap Y \subset \phi(x)\right). \tag{3}$$

To show that R is nearly good, let $A \subset Y$ and $x \in \overline{A}$. There must exist $k, i \in \{1, 2, \ldots, m\}$ such that $x \in X_k \cap \overline{A \cap X_i}$. Without loss of generality, we assume that $i = 1$ and $A \subset X_1 \cap Y$. Since $\mathcal{L}_k = \bigcup_{n\in\mathbb{N}} \mathcal{L}_{k,n}$ is a network of X_k, there is an $n \in \mathbb{N}$ and $L_x \in \mathcal{L}_{k,n}$ such that $x \in L_x \cap Y \subset \phi(x)$. The following discussion help us know that R is nearly good.

(i) If $k = 1$, by the thick property of $\mathcal{L}_{1,n}$ on X_1, we have that $\mathrm{Cl}_{X_1} A \subset \bigcup\{\bigcup \mathcal{L}_{1,n}(H) : H \in [A]^{<\omega}\}$. It follows that there exists a $J \in [A]^{<\omega}$ such that $x \in \bigcup \mathcal{L}_{1,n}(J)$. Moreover, since $\mathcal{L}_{1,n}$ is a partition of X_1 and $x \in L_x \in \mathcal{L}_{1,n}$, then $L_x \in \mathcal{L}_{1,n}(J)$, and hence $L_x \cap Y \in \mathcal{F}_{1,n}(J)$. It witnesses that xRJ.

(ii) If $k \neq 1$, by the foregoing claim, $\mathrm{Cl}_Y A \subset \bigcup\{\bigcup \mathcal{F}_{1,n}(H) : H \in [A]^{<\omega}\}$. There exists a $Q \in [A]^{<\omega}$ such that $x \in \bigcup \mathcal{F}_{1,n}(Q)$. Then, by the definition of $\mathcal{F}_{1,n}(Q)$, there is a set $C \in [B_k(Q)]^{<\omega}$ and $L \in \mathcal{L}_{k,n}(C)$ so that $x \in L \cap Y \in \mathcal{F}_{1,n}(Q)$. Since $\mathcal{L}_{k,n}(C) \in [\mathcal{L}_{k,n}]^{\leq\omega}$ and $\mathcal{L}_{k,n}$ is a partition of X_k. then this $L = L_x$, and hence $x \in L_x \cap Y \in \mathcal{F}_{1,n}(Q)$. Therefore, we have that xRQ.

By (i) and (ii), we know that R is a nearly good relation.

For every $H \in [Y]^{<\omega}$, $n \in \mathbb{N}$ and $L \cap Y \in \mathcal{F}_n(H)$, let $(L \cap Y)_C = \{x \in R^{-1}(H) : x \in L \cap Y \subset \phi(x)\}$. Then $(L \cap Y)_C$ is ϕ-close. By the definition of the relation R, it is easy to see

that $R^{-1}(H) = \bigcup_{n\in\mathbb{N}} \bigcup_{L\cap Y\in\mathcal{F}_n(H)} (L \cap Y)_C$. Since $\mathcal{F}_n(H)$ is a countable family, $R^{-1}(H)$ is a countable union of ϕ-close sets.

By Proposition 8, there exists a closed and discrete subset D of Y so that $\phi(D)$ covers Y. It follows that Y is a D-space. □

Theorem 10. *Suppose X has countable tightness and $X = \bigcup_{i=1}^{m} X_i$, where each X_i is t-metrizable. Then X is a D-space.*

Proof. By [23, Corollary 4.9], the result is true for $m = 1$. We prove inductively and assume that the result is true for $m - 1$ many t-metrizable subspaces.

Denote $Y = \bigcap_{i=1}^{m} \overline{X_i}$. It follows from Lemma 9 that Y is a D-space. On the other hand, $X \setminus Y = X \setminus \bigcap_{i=1}^{m} \overline{X_i} = \bigcup_{i=1}^{m} (X \setminus \overline{X_i})$. For every $i \in \{1, 2, \ldots, m\}$, since countable tightness and t-metrizability is hereditary, $X \setminus \overline{X_i}$ has countable tightness and it is the union of $m - 1$ many t-metrizable subspaces. By our assumption, each $X \setminus \overline{X_i}$ is a D-space.

It is not difficult to check that, as the union of m many open D-subspaces, $X \setminus Y$ is a D-space. Now we know that Y is a closed D-subspace and $X \setminus Y$ is an open D-subspace of X. Then by [2, Proposition 1.2], $X = (X \setminus Y) \cup Y$ is also a D-space. Thus, the result is also true for m many t-metrizable subspaces. □

Since all first countable spaces, Frechet-Urysohn spaces and sequential spaces have countable tightness (see [20, Theorem 1.7.13]), we have the following consequence of Theorem 10.

Corollary 11. *If a space from the following spaces is the union of finitely many t-metrizable subspaces, then it is a D-space.*

(a) First countable spaces;

(b) Frechet-Urysohn spaces;

(c) Sequential spaces.

By [19, Theorem 3.2], some classes have t-metrizability, and then we can obtain the result below as another corollary of Theorem 10.

Corollary 12. *If a space of countable tightness is the union of finitely many spaces below, then it is a D-space.*

(a) The free topological groups of a t-metrizable Tychonoff spaces;

(b) The σ-products of t-metrizable spaces;

(c) Hereditarily metaLindelöf descriptive spaces;

(d) Spaces L_w, where L is a metrizable locally convex vector space;

(e) Spaces with point-countably expandable networks.

Remark 13. The result concerning point-countably expandable networks answers [23, Problem 5.1] in a large part.

Note that Buzyakova obtained in [5] the hereditary D-property of $C_p(K)$, where K is a compact Hausdorff space. It also follows from [19, Theorem 3.2] that the function space $C_p(K)$ is t-metrizable. Therefore, we obtain the result below.

Corollary 14. *A space of countable tightness is a D-space if it is the union of finitely many function spaces $C_p(K_i)$, where each K_i is a compact Hausdorff space.*

Moreover, in [6], Gruenhage generalized Buzyakova's result to the function space $C_p(X)$ for the Lindelöf Σ-space X. Now using similar arguments as we have shown in the proofs of Lemma 9 and Theorem 10, we can generalize Corollary 14 to such function spaces. The idea of the construction of the nearly good relation in the proof below benefits from [6, Propositions 2.6 and 2.7].

Theorem 15. *If a space of countable tightness is the union of finitely many function spaces $C_p(X_i)$, where each X_i is Lindelöf Σ, then it is a D-space.*

Proof. Suppose that Y has countable tightness and $Y = \bigcup_{i=1}^{m} C_p(X_i)$, where each X_i is Lindelöf Σ. Based on the ideas shown in Lemma 9 and Theorem 10, it suffices to prove the result for $m = 2$ and $Y = \overline{C_p(X_1)} \cap \overline{C_p(X_1)}$.

For $i, j \in \{1, 2\}$ with $i \neq j$, and every $f \in C_p(X_i)$, since $f \in \overline{C_p(X_j)}$, and Y has countable tightness, we fix $D_j(f) \in [X]^{\leq\omega}$ such that $x \in \overline{D_j(f)}$. Then for every $A \subset C_p(X_i)$, let $D_j(A) = \bigcup\{D_j(f) : f \in A\}$.

Since X_i is Lindelöf Σ, there is a cover $\mathcal{K}_i$ by compact sets and a countable collection $\mathcal{F}_i$ such that, for any $K \in \mathcal{K}_i$ and $K \subset U$, where U is open in X_i, then $K \subset F \subset U$ for some $F \in \mathcal{F}_i$.

Let $\mathcal{B}$ be a countable base for the real line $\mathbb{R}$. For $S \subset X_i$ and $B \in \mathcal{B}$, let $[S, B]_i = \{f \in C_p(X_i) : f(S) \subset B\}$. For $A \subset C(X_i)$, let $\mathcal{G}_{i(A)}$ be the set of all $G = \bigcap_{k<n} [S_k, B_k]_i$ where $B_k \in \mathcal{B}$ and S_k can be written in the form $F \setminus \bigcup_{a\in A'} a^{-1}(B_a)$ for some finite $A' \subset A$, where $F \in \mathcal{F}_i$.

For every $H \in [C_p(X_i)]^{<\omega}$, let $\mathcal{G}_H = \mathcal{G}_{i(H)} \cup \bigcup\{\mathcal{G}_{j(Q)} : Q \in [D_{j(H)}]^{<\omega}\}$, and for every $H \in [Y]^{<\omega}$, let $\mathcal{G}_H = \mathcal{G}_{H\cap C_p(X_1)} \cup \mathcal{G}_{H\cap C_p(X_2)}$.

To show Y is a D-space, let ϕ be a neighborhood assignment on Y. We define R from Y to $[Y]^{<\omega}$ as follows,

$$fRH \Longleftrightarrow \exists G \in \mathcal{G}_H \left(f \in G \subset \phi(f)\right). \tag{4}$$

For each $H \in [Y]^{<\omega}$ and $G \in \mathcal{G}_H$, let $G_C = \{f \in R^{-1}(H) : f \in G \subset \phi(f)\}$. Then G_C is ϕ-close. Moreover, it is easy to check that $\mathcal{G}_{H\cap C_p(X_i)}$ is countable, and then $\mathcal{G}_H$ is countable. It follows that $R^{-1}(H) = \bigcup_{G\in\mathcal{G}_H} G_C$ is a countable union of ϕ-close sets.

To show R is nearly good, let $f \in \overline{A}$. Without loss of generality, we assume that $f \in C_p(X_1)$ and $A \subset C_p(X_2)$. Then $f \in \overline{C}$ for some countable set $C \subset A$. Therefore, $f \in \overline{C} \subset \overline{D_1(C)}$. As shown in the proof of [6, Proposition 2.7], there is a $Q \in [D_1(C)]^{<\omega}$ and $G \in \mathcal{G}_{1(Q)}$ such that $f \in G \subset \phi(f)$. There must be an $H \in [C]^{<\omega}$, such that $Q \in [D_1(H)]^{<\omega}$. And hence $G \in \mathcal{G}_H$. Thus, we have that fRH, where $H \in [C]^{<\omega} \subset [A]^{<\omega}$.

By Proposition 8, there exists a closed and discrete subset E of Y so that $\phi(E)$ covers Y. This complete the proof of the D-property of Y. □

Conflict of Interests Statement

The authors declare that there is no conflict of interests regarding the publication of this article.

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