Research Article

Nonuniform Exponential Trichotomy for Linear Discrete-Time Systems in Banach Spaces

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The aim of this paper is to give several characterizations for nonuniform exponential trichotomy properties of linear difference equations in Banach spaces. Well-known results for exponential stability and exponential dichotomy are extended to the case of nonuniform exponential trichotomy.

1. Introduction

In the mathematical literature of the last decades, the asymptotic properties of solutions of evolution equations in finite or infinite dimensional space have proved to be research area of large intensity. There were defined and developed concepts of the asymptotic behaviors, as stability, expansivity, dichotomy, and trichotomy (see [1–25] and the references therein), based on the fact that the dynamical systems which describe processes from economics, physical sciences, or engineering are extremely complex and the identification of the proper mathematical model is difficult.

As a natural generalization of exponential dichotomy, exponential trichotomy is one of the most complex asymptotic properties of dynamical systems arising from the central manifold theory. When people analyze the asymptotic behavior of dynamical systems, exponential trichotomy is a powerful tool. Starting from the idea that the center manifold of an equilibrium point of a dynamical system consists of orbits whose behavior around the equilibrium point is not controlled by either the attraction of the stable manifold or the repulsion of the unstable manifold, it is obvious that the concept of exponential dichotomy describes a rather idealistic situation when the solution is either exponentially stable on the stable subspaces or exponentially unstable on the unstable subspaces (see [4, 7, 8, 16]). Thus, with motivation from the properties arising in bifurcation theory, a new asymptotic concept called exponential trichotomy

which reflects a deeper analysis of the behavior of solutions of dynamical systems is introduced. Under this case the main idea in the study of the asymptotic behavior is to obtain, at any moment, a decomposition of the state space in three subspaces: a stable one, an instable one, and a third one, the central manifold.

The conception of trichotomy firstly arose in the works of Sacker and Sell [19] in 1976. They described trichotomy for linear differential systems by linear skew-product flows. Later, Elaydi and Hájek [6, 7] gave the notions of exponential trichotomy for differential systems and for nonlinear differential systems, respectively. The case of difference equations received a special attention in the paper of Elaydi and Janglajew [8] where the authors deduced the first input-output criteria for exponential trichotomy, on one hand, and they introduced the first nonlinear discrete concepts of exponential trichotomy, on the other hand. Despite the increasing interest on this topic, most of papers were devoted to problems regarding the robustness of the exponential trichotomy (see [1, 9, 10, 12, 15, 25]) and only in the past few years the existence criteria started to be obtained (see [2, 13, 18, 20–22]). For instance, in [9, 10] Hong and his partners studied the relationship between exponential trichotomy and the ergodic solutions of linear differential and difference equations with ergodic perturbations. In [20], the connections between the existence of exponential trichotomy of variational difference equations and the solvability of the associated variational control system were established. And in [22] B. Sasu and A.L.

Sasu obtained some nonlinear conditions for the existence of the exponential trichotomy of skew-product flows in infinite dimensional spaces.

In this paper, we introduce the concept of nonuniform exponential trichotomy for linear difference equations which is an extension of classical concept of uniform exponential trichotomy. Our main objective is to give some characterizations for nonuniform exponential trichotomy properties of linear difference equations in Banach spaces, and variants for nonuniform exponential trichotomy of some well-known results in uniform exponential stability theory (Datko [5], Przyłuski and Rolewicz [17]) and exponential dichotomy theory (Popa et al. [16]) are obtained.

2. Preliminaries

Let X be a real or complex Banach space. The norm on X and on $\mathcal{B}(X)$ the Banach algebra of all bounded linear operators acting on X will be denoted by $\|\cdot\|$. We denote $\Delta = \{(m, n) \in \mathbb{N}^2, m \geq n\}$ and $T = \{(m, p, n) \in \mathbb{N}^3, m \geq p \geq n\}$. Let $\mathcal{F}$ be the set of all nondecreasing functions $F : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ with the properties $F(0) = 0$ and $F(t) > 0$ for every $t > 0$. Let I be the identity operator on X .

In the present paper we consider linear discrete-time system of difference equations

$$x_{n+1} = A(n) x_n, \quad n \in \mathbb{N}, \quad (1)$$

where $A : \mathbb{N} \rightarrow \mathcal{B}(X)$ is a sequence in $\mathcal{B}(X)$. Then every solution $x = \{x_n\}$ of the system (1) is given by

$$x_m = U(m, n) x_n \quad (2)$$

for all $(m, n) \in \Delta$, where the mapping $U : \Delta \rightarrow \mathcal{B}(X)$ is defined by

$$U(m, n) := \begin{cases} I, & m = n \\ A(m-1) \cdots A(n), & m > n. \end{cases} \quad (3)$$

It is easy to see that $U(m, p)U(p, n) = U(m, n)$, for all $(m, p, n) \in T$.

For the particular case when (1) is autonomous, that is, $A(n) = A \in \mathcal{B}(X)$ for all $n \in \mathbb{N}$, then $U(m, n) = A^{m-n}$ for all $(m, n) \in \Delta$.

Definition 1. An application $P : \mathbb{N} \rightarrow \mathcal{B}(X)$ is said to be a projection family on X if

$$P^2(n) = P(n), \quad (4)$$

for all $n \in \mathbb{N}$.

Definition 2. Three projection families $P_1, P_2, P_3 : \mathbb{N} \rightarrow \mathcal{B}(X)$ are said to be compatible with the system (1), if

- (c1) $P_1(n) + P_2(n) + P_3(n) = I$, for all $n \in \mathbb{N}$,
- (c2) $P_i(n)P_j(n) = 0$, for all $n \in \mathbb{N}$, for all $i, j \in \{1, 2, 3\}$, $i \neq j$,
- (c3) $\|P_i(n)x + P_j(n)x\|^2 = \|P_i(n)x\|^2 + \|P_j(n)x\|^2$, for all $n \in \mathbb{N}$, all $x \in X$, and all $i, j \in \{1, 2, 3\}$, $i \neq j$,
- (c4) $A(n)P_i(n) = P_i(n+1)A(n)$, for all $n \in \mathbb{N}$ and all $i \in \{1, 2, 3\}$.

Remark 3. If $P_1, P_2, P_3 : \mathbb{N} \rightarrow \mathcal{B}(X)$ are projection families compatible with the system (1) then

$$U(m, n)P_i(n) = P_i(m)U(m, n) \quad (5)$$

for all $i \in \{1, 2, 3\}$ and all $(m, n) \in \Delta$.

Definition 4. The linear discrete-time system (1) is said to be uniformly exponentially trichotomic if there exist three projection families $\{P_i\}_{i \in \{1, 2, 3\}}$ compatible with the system (1), constants ν_1, ν_2, ν_3 , and ν_4 with the property $\nu_1 \leq \nu_2 < 0 < \nu_3 \leq \nu_4$, and $N \geq 1$ such that

$$\begin{aligned} (\text{uet}_1) \quad & \|U(m, n)P_1(n)x\| \leq N\|P_1(n)x\|e^{\nu_1(m-n)}, \\ (\text{uet}_2) \quad & \|P_2(n)x\| \leq N\|U(m, n)P_2(n)x\|e^{-\nu_4(m-n)}, \\ (\text{uet}_3) \quad & \|U(m, n)P_3(n)x\| \leq N\|P_3(n)x\|e^{\nu_3(m-n)}, \\ (\text{uet}_4) \quad & \|P_3(n)x\| \leq N\|U(m, n)P_3(n)x\|e^{-\nu_2(m-n)}, \end{aligned}$$

for all $(m, n, x) \in \Delta \times X$.

Example 5. Let $X = \mathbb{R}^3$ with the Euclidean norm. The application $A : \mathbb{N} \rightarrow \mathcal{B}(\mathbb{R}^3)$ is defined by

$$\begin{aligned} A(n)(x_1, x_2, x_3) &= \left(ae^{\lambda_n} x_1, \frac{x_2}{a}, \frac{x_3}{e} \right), \\ \forall (n, x_1, x_2, x_3) &\in \mathbb{N} \times \mathbb{R}^3, \end{aligned} \quad (6)$$

where $a \in (0, e^{-3})$ and $\lambda_n = (n+2)\cos \ln(n+2) - (n+1)\cos \ln(n+1)$.

Let us consider the projection families $P_1, P_2, P_3 : \mathbb{N} \rightarrow \mathcal{B}(\mathbb{R}^3)$ defined by

$$\begin{aligned} P_1(n)(x_1, x_2, x_3) &= (x_1, 0, 0), \\ P_2(n)(x_1, x_2, x_3) &= (0, x_2, 0), \\ P_3(n)(x_1, x_2, x_3) &= (0, 0, x_3) \end{aligned} \quad (7)$$

for all $n \in \mathbb{N}$ and all $x = (x_1, x_2, x_3) \in X$.

A simple calculus shows that

$$\begin{aligned} U(m, n)P_1(n)x &= (a^{m-n}e^{\mu_{mn}}x_1, 0, 0), \\ U(m, n)P_2(n)x &= (0, a^{-(m-n)}x_2, 0), \\ U(m, n)P_3(n)x &= (0, 0, e^{-(m-n)}x_3), \end{aligned} \quad (8)$$

where $\mu_{mn} = (m+1)\cos \ln(m+1) - (n+1)\cos \ln(n+1)$.

If we denote $(x) = x \cos \ln x$, then by Lagrange's mean value theorem it results that for every $n \in \mathbb{N}$ there exists $\tau_n \in (n+1, n+2)$ such that

$$\lambda_n = \varphi(n+2) - \varphi(n+1) = \cos \ln \tau_n - \sin \ln \tau_n \quad (9)$$

and hence $|\lambda_n| \leq \sqrt{2} \leq 2$ for every $n \in \mathbb{N}$.

This implies that

$$\sup_{n \in \mathbb{N}} \|A(n)\| \leq ae^2 + \frac{1}{a} + \frac{1}{e}, \quad |\mu_{mn}| \leq 2(m-n). \quad (10)$$

In the following we obtain the relations

$$\begin{aligned}
 e^{m-n} \|U(m, n) P_1(n) x\| &\leq e^{\varepsilon(m-n) + \mu_{mn}} a^{m-n} |x_1| \\
 &\leq |x_1| = \|P_1(n) x\|, \\
 e^{m-n} \|P_2(n) x\| &\leq e^{3(m-n)} |x_2| \leq a^{-(m-n)} |x_2| \\
 &= \|U(m, n) P_2(n) x\|, \\
 \|U(m, n) P_3(n) x\| &= e^{-(m-n)} |x_3| \\
 &\leq e^{(m-n)} |x_3| = \|P_3(n) x\| e^{(m-n)}, \\
 \|P_3(n) x\| &= |x_3| \leq \|U(m, n) P_3(n) x\| e^{(m-n)}, \quad (11)
 \end{aligned}$$

for all $(m, n, x) \in \Delta \times \mathbb{R}^3$, where $\varepsilon = \ln(ae^2)^{-1} > \ln e = 1$. Thus Definition 4 is satisfied for $N = -v_1 = -v_2 = v_3 = v_4 = 1$; hence, the system (1) is uniformly exponentially trichotomic.

Definition 6. The linear discrete-time system (1) is said to be nonuniformly exponentially trichotomic if there exist a nondecreasing sequence of real numbers $N : \mathbb{N} \rightarrow \mathbb{R}_+^*$, constants v_1, v_2, v_3 , and v_4 with the property $v_1 \leq v_2 < 0 < v_3 \leq v_4$, and three projection families $\{P_i\}_{i \in \{1,2,3\}}$ compatible with the system (1) such that

$$\begin{aligned}
 (\text{net}_1) \quad &\|U(m, n) P_1(n) x\| \leq N(n) \|P_1(n) x\| e^{v_1(m-n)}, \\
 (\text{net}_2) \quad &\|P_2(n) x\| \leq N(m) \|U(m, n) P_2(n) x\| e^{-v_4(m-n)}, \\
 (\text{net}_3) \quad &\|U(m, n) P_3(n) x\| \leq N(n) \|P_3(n) x\| e^{v_3(m-n)}, \\
 (\text{net}_4) \quad &\|P_3(n) x\| \leq N(m) \|U(m, n) P_3(n) x\| e^{-v_2(m-n)},
 \end{aligned}$$

for all $(m, n, x) \in \Delta \times X$.

Remark 7. Consider the following.

- (i) For $P_3 = 0$ in Definition 6 we obtain the property of nonuniform exponential dichotomy.
- (ii) For $P_2 = P_3 = 0$, the property of nonuniform exponential stability is obtained. It follows that a nonuniformly exponentially stable linear discrete-time system is nonuniformly exponentially dichotomic and, further, nonuniformly exponentially trichotomic.
- (iii) For $P_1 = P_3 = 0$, we obtain the property of nonuniform exponential expansivity. Also it is easy to see that the property of nonuniform exponential expansivity implies the nonuniform exponential dichotomy and, further, the nonuniform exponential trichotomy.

Remark 8. The linear discrete-time system (1) is nonuniformly exponentially trichotomic if and only if there exist a nondecreasing sequence of real numbers $N : \mathbb{N} \rightarrow \mathbb{R}_+^*$, constants v_1, v_2, v_3 , and v_4 with the property $v_1 \leq v_2 < 0 < v_3 \leq v_4$, and three projection families $\{P_i\}_{i \in \{1,2,3\}}$ compatible with the system (1) such that

$$\begin{aligned}
 (a) \quad &\|U(m, n) P_1(n) x\| \leq N(p) \|U(p, n) P_1(n) x\| e^{v_1(m-p)}, \\
 &\quad \quad \quad (12)
 \end{aligned}$$

(b)

$$\|U(p, n) P_2(n) x\| \leq N(m) \|U(m, n) P_2(n) x\| e^{-v_4(m-p)}, \quad (13)$$

(c)

$$\|U(m, n) P_3(n) x\| \leq N(p) \|U(p, n) P_3(n) x\| e^{v_3(m-p)}, \quad (14)$$

(d)

$$\|U(p, n) P_3(n) x\| \leq N(m) \|U(m, n) P_3(n) x\| e^{-v_2(m-p)}, \quad (15)$$

for all $(m, p, n, x) \in T \times X$.

Remark 9. It is obvious that if the system (1) is uniformly exponentially trichotomic then it is nonuniformly exponentially trichotomic. But the converse statement is not necessarily valid. This fact is illustrated by the following example.

Example 10. Let $X = \mathbb{R}^3$ with the Euclidean norm. The application $A : \mathbb{N} \rightarrow \mathcal{B}(\mathbb{R}^3)$ is defined by

$$\begin{aligned}
 A(n) (x_1, x_2, x_3) &= (ax_1, x_2, x_3), \\
 \forall (n, x_1, x_2, x_3) &\in \mathbb{N} \times \mathbb{R}^3, \quad (16)
 \end{aligned}$$

where $a \in (0, e^{-1}]$. We consider the projection families $P_1, P_2, P_3 : \mathbb{N} \rightarrow \mathcal{B}(\mathbb{R}^3)$,

$$\begin{aligned}
 P_1(n) (x_1, x_2, x_3) &= (x_1, 0, 0), \\
 P_2(n) (x_1, x_2, x_3) &= (0, x_2, 0), \\
 P_3(n) (x_1, x_2, x_3) &= (0, 0, x_3)
 \end{aligned} \quad (17)$$

for all $n \in \mathbb{N}$ and all $x = (x_1, x_2, x_3) \in X$, compatible with the system (1).

Then

$$\begin{aligned}
 U(m, n) P_1(n) x &= (a^{m-n} x_1, 0, 0), \\
 U(m, n) P_2(n) x &= (0, x_2, 0), \\
 U(m, n) P_3(n) x &= (0, 0, x_3).
 \end{aligned} \quad (18)$$

Further we have that

$$\begin{aligned}
 e^{m-n} \|U(m, n) P_1(n) x\| &= (ea)^{m-n} |x_1| \\
 &\leq |x_1| \leq e^n |x_1| \\
 &= N(n) \|P_1(n) x\|, \\
 e^{m-n} \|P_2(n) x\| &= e^{m-n} |x_2| \leq e^m |x_2| \\
 &= N(m) \|U(m, n) P_2(n) x\|, \\
 \frac{1}{N(m)} e^{-(m-n)} \|P_3(n) x\| &= e^{-m} \cdot e^{-(m-n)} |x_3| \\
 &\leq \|U(m, n) P_3(n) x\| \\
 &= |x_3| \leq e^n \cdot e^{m-n} \|P_3(n) x\|
 \end{aligned} \quad (19)$$

for all $(m, n, x) \in \Delta \times \mathbb{R}^3$. Thus Definition 6 is satisfied for $N(n) = e^n$ and $v_1 = v_2 = -v_3 = -v_4 = -1$; hence, the system (1) is nonuniformly exponentially trichotomic.

On the other hand, if we suppose that system (1) is uniformly exponentially trichotomic, then there exist two constants $N \geq 1$ and $v_4 > 0$ such that

$$\begin{aligned} \|x_2\| &= \|P_2(n)x\| \leq N \|U(m, n)P_2(n)x\| e^{-v_4(m-n)} \\ &= N \|x_2\| e^{-v_4(m-n)} \end{aligned} \quad (20)$$

for all $(m, n, x) \in \Delta \times \mathbb{R}^3$. This implies that $e^{v_4(m-n)} \leq N$ for all $(m, n) \in \Delta$, which is false.

3. The Main Results

Theorem 11. *The linear discrete-time system (1) is nonuniformly exponentially trichotomic if and only if there exist a function $F \in \mathcal{F}$ and three projection families $\{P_i\}_{i \in \{1,2,3\}}$ compatible with the system (1) such that the following relations hold:*

(i) *There exist a constant $d_1 > 0$ and a sequence of positive real numbers $\{\xi_n\}_{n \in \mathbb{N}}$ such that*

$$\sum_{k=n}^m F(e^{d_1(k-n)} \|U(k, n)P_1(n)x\|) \leq F(\xi_n \|P_1(n)x\|), \quad (21)$$

(ii) *There exist a constant $d_2 > 0$ and a sequence of positive real numbers $\{\eta_n\}_{n \in \mathbb{N}}$ such that*

$$\begin{aligned} \sum_{k=n}^m F(e^{-d_2(k-n)} \|U(k, n)P_2(n)x\|) \\ \leq F(\eta_n e^{-d_2(m-n)} \|U(m, n)P_2(n)x\|), \end{aligned} \quad (22)$$

(iii) *There exist a constant $d_3 > 0$ and a sequence of positive real numbers $\{\alpha_n\}_{n \in \mathbb{N}}$ such that*

$$\begin{aligned} \sum_{k=p}^m F(e^{-d_3(k-n)} \|U(k, n)P_3(n)x\|) \\ \leq F(\alpha_n e^{-d_3(p-n)} \|U(p, n)P_3(n)x\|), \end{aligned} \quad (23)$$

(iv) *There exist a constant $d_4 > 0$ and a sequence of positive real numbers $\{\beta_n\}_{n \in \mathbb{N}}$ such that*

$$\begin{aligned} \sum_{k=p}^m F(e^{d_4(k-n)} \|U(k, n)P_3(n)x\|) \\ \leq F(\beta_n e^{d_4(m-n)} \|U(m, n)P_3(n)x\|) \end{aligned} \quad (24)$$

for all $(m, p, n, x) \in T \times X$.

Proof. (a) $\Leftrightarrow$ (i) *Necessity.* We consider $F(t) = t$, $t \geq 0$. As system (1) is nonuniformly exponentially trichotomic, Remark 8 assures the existence of a constant $v_1 < 0$, a sequence of real numbers $N : \mathbb{N} \rightarrow \mathbb{R}_+^*$, and a projection family

P_1 such that (a) holds. We obtain for $d_1 = -(v_1/2) > 0$ and according to (a)

$$\begin{aligned} \sum_{k=n}^m e^{d_1(k-n)} \|U(k, n)P_1(n)x\| \\ \leq N(n) \sum_{k=n}^m e^{d_1(k-n)} e^{v_1(k-n)} \|U(n, n)P_1(n)x\| \\ = N(n) \|P_1(n)x\| \sum_{k=n}^m e^{(v_1/2)(k-n)} \\ \leq \xi_n \|P_1(n)x\|, \quad \forall (m, n, x) \in \Delta \times X, \end{aligned} \quad (25)$$

where we have denoted $\xi_n = (1/(1 - e^{-d_1}))N(n)$, $n \in \mathbb{N}$.

Sufficiency. According to the hypothesis, if we consider $k = m$ then

$$F(e^{d_1(m-n)} \|U(m, n)P_1(n)x\|) \leq F(\xi_n \|P_1(n)x\|) \quad (26)$$

for all $(m, n, x) \in \Delta \times X$. By means of the properties of function F , we have

$$\|U(m, n)P_1(n)x\| \leq \xi_n \|P_1(n)x\| e^{-d_1(m-n)}. \quad (27)$$

Thus relation (a) is obtained.

Similarly the other equivalences can also be proved, that is, (b) $\Leftrightarrow$ (ii), (c) $\Leftrightarrow$ (iii), and (d) $\Leftrightarrow$ (iv). $\square$

Remark 12. The preceding theorem is an extension for the case of nonuniform exponential trichotomy of a result due to Popa et al. in [16]. Additionally, it is variant for the case of nonuniform exponential trichotomy property of a well-known theorem due to Przyłuski and Rolewicz [17] for exponential stability. If we consider $F(t) = t^p$, $t \geq 0$, $p > 0$, then Theorem 11 can be considered a version for the case of nonuniform exponential trichotomy of some results due to Datko [5].

It is well known that the exponential dichotomy involves two commuting projection families. In order to emphasize the natural extension of the nonuniform exponential trichotomy relative to the property of dichotomy, we will present a characterization by means of two commuting projection families, introduced by the following.

Definition 13. Two projection families $R_1, R_2 : \mathbb{N} \rightarrow \mathcal{B}(X)$ are said to be compatible with the system (1), if

- (r1) $R_i(n)R_j(n) = 0$, for all $n \in \mathbb{N}$, for all $i, j \in \{1, 2\}$, $i \neq j$,
- (r2) $\|R_1(n)x + R_2(n)x\|^2 = \|R_1(n)x\|^2 + \|R_2(n)x\|^2$, for all $n \in \mathbb{N}$, all $x \in X$,
- (r3) $\|[I - R_1(n)]\|^2 = \|[I - R_1(n) - R_2(n)]x\|^2 + \|R_2(n)x\|^2$, for all $n \in \mathbb{N}$, all $x \in X$,
- (r4) $\|[I - R_2(n)]\|^2 = \|[I - R_1(n) - R_2(n)]x\|^2 + \|R_1(n)x\|^2$, for all $n \in \mathbb{N}$, all $x \in X$,
- (r5) $A(n)R_i(n) = R_i(n+1)A(n)$, for all $n \in \mathbb{N}$ and all $i \in \{1, 2\}$.

Theorem 14. *The linear discrete-time system (1) is nonuniformly exponentially trichotomic if and only if there exist a nondecreasing sequence of real numbers $N_1 : \mathbb{N} \rightarrow \mathbb{R}_+^*$, constants $\mu > \nu > 0$, and two projection families $\{R_i\}_{i \in \{1,2\}}$ compatible with the system (1) such that*

(a')

$$\|U(m, n) R_1(n) x\| \leq N_1(p) \|U(p, n) R_1(n) x\| e^{-\mu(m-p)}, \quad (28)$$

(b')

$$\|U(p, n) R_2(n) x\| \leq N_1(m) \|U(m, n) R_2(n) x\| e^{-\nu(m-p)}, \quad (29)$$

(c')

$$\begin{aligned} & \|U(m, n) [I - R_2(n)] x\| \\ & \leq N_1(p) \|U(p, n) [I - R_2(n)] x\| e^{\nu(m-p)}, \end{aligned} \quad (30)$$

(d')

$$\begin{aligned} & \|U(p, n) [I - R_1(n)] x\| \\ & \leq N_1(m) \|U(m, n) [I - R_1(n)] x\| e^{\mu(m-p)}, \end{aligned} \quad (31)$$

for all $(m, p, n, x) \in T \times X$.

Proof. Necessity. As the system (1) is nonuniformly exponentially trichotomic, according to Remark 8 there exist three projection families $\{P_i\}_{i \in \{1,2,3\}}$ compatible with the system (1), constants $\nu_1 \leq \nu_2 < 0 < \nu_3 \leq \nu_4$, and a nondecreasing sequence of real numbers $N : \mathbb{N} \rightarrow \mathbb{R}_+^*$ such that relations (a)–(d) hold.

If we denote $R_1 = P_1$ and $R_2 = P_2$ then conditions (r1)–(r5) of Definition 13 result from (c1)–(c4) of Definition 2. Thus, the projection families R_1 and R_2 are compatible with the system (1).

Let us define $\mu = -\nu_1 = -\nu_2 > 0$, $\nu = \nu_3 = \nu_4 > 0$, and $N_1(n) = N(n)$, $n \in \mathbb{N}$. It is obvious that relations (a') and (b') hold. Using conditions (c3), (c4), (b), and (d) we obtain

$$\begin{aligned} & \|U(p, n) [I - R_1(n)] x\|^2 \\ & = \|U(p, n) [P_2(n) + P_3(n)] x\|^2 \\ & = \|U(p, n) P_2(n) x\|^2 + \|U(p, n) P_3(n) x\|^2 \\ & \leq N(m)^2 \|U(m, n) P_2(n) x\|^2 e^{-2\nu_4(m-p)} \\ & \quad + N(m)^2 \|U(m, n) P_3(n) x\|^2 e^{-2\nu_2(m-p)} \\ & \leq N(m)^2 \|U(m, n) [P_2(n) + P_3(n)] x\|^2 e^{-2\nu_2(m-p)} \\ & = N_1(m)^2 \|U(m, n) [I - R_1(n)] x\|^2 e^{2\mu(m-p)} \end{aligned} \quad (32)$$

for all $(m, p, n, x) \in T \times X$. Hence, relation (d') holds.

The proof of (c') is similar.

Sufficiency. We consider the projection families

$$\begin{aligned} P_1 &= R_1, & P_2 &= R_2, \\ P_3 &= I - R_1 - R_2 = (I - R_1)(I - R_2). \end{aligned} \quad (33)$$

These are compatible with the system (1).

Let us define $\nu_1 = \nu_2 = -\mu < 0$, $\nu_3 = \nu_4 = \nu > 0$, and $N(n) = N_1(n)$, $n \in \mathbb{N}$. It is clear that (a') $\Leftrightarrow$ (a) and (b') $\Leftrightarrow$ (b). By condition (c') we obtain

$$\begin{aligned} & \|U(m, n) P_3(n) x\| \\ & = \|U(m, n) [I - R_1(n)] [I - R_2(n)] x\| \\ & \leq N_1(p) \|U(p, n) [I - R_1(n)] [I - R_2(n)] x\| e^{\nu(m-p)} \\ & = N(p) \|U(p, n) P_3(n) x\| e^{\nu_3(m-p)} \end{aligned} \quad (34)$$

for all $(m, p, n, x) \in T \times X$, and hence (c) holds.

Similarly, we can prove that (d') implies relation (d). Finally, by Remark 8 we conclude that the system (1) is nonuniformly exponentially trichotomic. $\square$

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