Research Article

Semicommutators and Zero Product of Block Toeplitz Operators with Harmonic Symbols

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We completely characterize the finite rank semicommutators, commutators, and zero product of block Toeplitz operators T_F and T_G with $F, G \in h^\infty \otimes M_{n \times n}$ on the vector valued Bergman space $L_a^2(\mathbb{D}, \mathbb{C}^n)$.

1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$, and let dA be the normalized Lebesgue area measure on $\mathbb{D}$. Denote by $L^\infty(\mathbb{D}, dA)$ and $L^2(\mathbb{D}, dA)$ the space of essential bounded measurable functions and the space of square integral functions with respect to dA , respectively. The Bergman space L_a^2 consists of all analytic functions in $L^2(\mathbb{D}, dA)$. Let $L^2(\mathbb{D}, \mathbb{C}^n) = L^2(\mathbb{D}, dA) \otimes \mathbb{C}^n$ and $L_a^2(\mathbb{D}, \mathbb{C}^n) = L_a^2 \otimes \mathbb{C}^n$, where $\otimes$ means the Hilbert space tensor product. Let $M_{n \times n}$ be the set of all $n \times n$ complex matrixes in the complex plane $\mathbb{C}$, and let $L^\infty(\mathbb{D}, dA) \otimes M_{n \times n}$ be the space of matrix valued essential bounded Lebesgue measurable functions on $\mathbb{D}$.

Given a matrix valued function $\Phi(z) = [\varphi_{ij}(z)]_{n \times n} \in L^\infty(\mathbb{D}, dA) \otimes M_{n \times n}$, the Toeplitz operator T_Φ and the Hankel operator H_Φ on $L_a^2(\mathbb{D}, \mathbb{C}^n)$ with symbol Φ are defined by $T_\Phi h = P(\Phi h)$ and $H_\Phi h = (I - P)(\Phi h)$, respectively, where P is the orthogonal projection from $L^2(\mathbb{D}, \mathbb{C}^n)$ onto $L_a^2(\mathbb{D}, \mathbb{C}^n)$. Then the operators T_Φ and H_Φ have the following matrix representations:

$$T_\Phi = \begin{bmatrix} T_{\varphi_{11}} & \cdots & T_{\varphi_{1n}} \\ \vdots & & \vdots \\ T_{\varphi_{n1}} & \cdots & T_{\varphi_{nn}} \end{bmatrix}, \quad H_\Phi = \begin{bmatrix} H_{\varphi_{11}} & \cdots & H_{\varphi_{1n}} \\ \vdots & & \vdots \\ H_{\varphi_{n1}} & \cdots & H_{\varphi_{nn}} \end{bmatrix}. \quad (1)$$

The problem of the finite rank semicommutator of two Toeplitz operators on the Hardy space has been completely solved in [1, 2]. The analogous problems on the Bergman

space have been characterized in [3–5]. In the case of vector valued Hardy space, Gu and Zheng [6] have studied the problems of compact semicommutators and commutators of two block Toeplitz operators.

In this paper, we study the related problems of the two block Toeplitz operators T_F and T_G with $F, G \in h^\infty \otimes M_{n \times n}$ on the vector valued Bergman space $L_a^2(\mathbb{D}, \mathbb{C}^n)$, where h^∞ denotes the space of all bounded harmonic functions on $\mathbb{D}$. Our main idea here is to reduce the problems on $L_a^2(\mathbb{D}, \mathbb{C}^n)$ to the problems of the finite sum of some Toeplitz operator products or some Hankel operator products on the Bergman space L_a^2 . In the following, we recall two useful theorems. For every $f \in h^\infty$, denote by f_+ (or $f_- = f - f_+$) the analytic part (or the coanalytic part) of f .

Theorem 1 (see [7]). Suppose that f_j and g_j are bounded harmonic functions on $\mathbb{D}$ for $j = 1, \dots, k$. Then the following are equivalent:

- (1) $\sum_{j=1}^k H_{g_j}^* H_{f_j}$ has finite rank,
- (2) $\sum_{j=1}^k H_{g_j}^* H_{f_j} = 0$,
- (3) $\sigma(f_1, \dots, f_k; g_1, \dots, g_k) \equiv 0$,

where $\sigma(f_1, \dots, f_k; g_1, \dots, g_k) = \tilde{\Delta}[\sum_{i=1}^k (f_i)_-(g_i)_+] = (1 - |z|^2)^2 \sum_{i=1}^k (f_i)'_-(g_i)'_+$.

Theorem 2 (see [8]). Let $u_1, \dots, u_n, v_1, \dots, v_n$ be bounded harmonic functions on $\mathbb{D}$, and $x_1, \dots, x_n, y_1, \dots, y_n \in L_a^2$. Then $\sum_{j=1}^n T_{u_j} T_{v_j} = \sum_{j=1}^n x_j \otimes y_j$ if and only if the following two conditions hold:

- (a) $\sum_{j=1}^n u_j v_j = (1 - |z|^2)^2 \sum_{j=1}^n x_j \overline{y_j}$;
- (b) $\sum_{j=1}^n [\overline{P u_j} - u_j(0)][P v_j - v_j(0)] = 0$.

2. Semicommutators of Block Toeplitz Operators

In this section, we discuss the finite rank semicommutators and commutators of the block Toeplitz operators T_F, T_G on $L_a^2(\mathbb{D}, \mathbb{C}^n)$ with $F, G \in h^\infty \otimes M_{n \times n}$. In the following part of this paper, E_j denote the matrix unit with (j, j) th entry equal to one and all others equal to zero.

Theorem 3. Let $F(z) = F_+(z) + F_-(z)$, and let $G(z) = G_+(z) + G_-(z) \in h^\infty \otimes M_{n \times n}$ such that $F_+(0) = G_-(0) = 0_{n \times n}$. Then the following statements are equivalent:

- (1) $T_{FG} - T_F T_G$ has finite rank;
- (2) $T_{FG} = T_F T_G$;
- (3) for each $1 \leq j \leq n$, there exist a matrix $M_j \in M_{n \times n}$ and a permutation matrix Q_j , such that $M_j Q_j G_- E_j = 0_{n \times n}$ and $F_+ Q_j (I + M_j) = 0_{n \times n}$.

Proof. (1) $\Rightarrow$ (2) Note that $T_{FG} - T_F T_G = H_{F^*}^* H_G$. If $H_{F^*}^* H_G$ has a finite rank, then $\sum_{p=1}^n H_{f_{ip}}^* H_{g_{pj}}$ has a finite rank for $1 \leq i, j \leq n$. By Theorem 1, we have that $\sum_{p=1}^n H_{f_{ip}}^* H_{g_{pj}} = 0$ for $1 \leq i, j \leq n$. Hence, $T_{FG} = T_F T_G$.

(2) $\Rightarrow$ (3) If $T_{FG} - T_F T_G = H_{F^*}^* H_G = 0$, then $[H_{F^*}^* H_G]_{i,j} = \sum_{p=1}^n H_{f_{ip}}^* H_{g_{pj}} = 0$, for $1 \leq i, j \leq n$. By Theorem 1, we get $(1 - |z|^2)^2 \sum_{p=1}^n [(f_{ip})'_+(z)(g_{pj})'_-(w)] = 0$ on $\mathbb{D}$. Therefore, $\sum_{p=1}^n [(f_{ip})'_+(z)(g_{pj})'_-(w)] = 0$ on $\mathbb{D}$ for $1 \leq i, j \leq n$.

It is known that if $H(z, w)$ is holomorphic in $\mathbb{D} \times \mathbb{D}$ and $H(z, \bar{z}) = 0$, then $H(z, w) \equiv 0$ for $(z, w) \in \mathbb{D} \times \mathbb{D}$ in several complex variables. So we can “complexificate” the above identity and have $\sum_{p=1}^n [(f_{ip})'_+(z)(g_{pj})'_-(w)] = 0$ for $1 \leq i, j \leq n, (z, w) \in \mathbb{D} \times \mathbb{D}$. Therefore,

$$\begin{aligned} 0 &= \sum_{p=1}^n \left[(f_{ip})'_+(u) (g_{pj})'_-(v) \right] \\ &= \int_0^z \sum_{p=1}^n \left[(f_{ip})'_+(u) (g_{pj})'_-(v) \right] du \\ &= \sum_{p=1}^n \left[(f_{ip})'_+(z) (g_{pj})'_-(v) \right] \\ &= \int_0^w \sum_{p=1}^n \left[(f_{ip})'_+(z) (g_{pj})'_-(v) \right] dv \\ &= \sum_{p=1}^n \left[(f_{ip})'_+(z) (g_{pj})'_-(w) \right], \end{aligned} \quad (2)$$

which means that $F_+(z)G_-(w) = 0_{n \times n}$ on $\mathbb{D} \times \mathbb{D}$.

Note that $\det F_+(z)$ is the analytic function of z and $\det G_-(w)$ is the coanalytic function of w . If $\det F_+(z)$ is not identically zero, then $G_- = 0_{n \times n}$; that is, G is analytic. If $\det G_-(w)$ is not identically zero, then $F_+ = 0_{n \times n}$; that is, F is coanalytic.

In the following, we deal with the case that $\det F_+ \equiv 0$ and $\det G_- \equiv 0$. Suppose $G_- E_j = (g_{\sigma(1)}^j, \dots, g_{\sigma(k_0)}^j)^T$ for $1 \leq j \leq n$. Since $\det G_- \equiv 0$, $G_- E_j$ is linearly dependent. Let Q_j be a permutation matrix such that

$$Q_j G_- E_j = \left(g_{\sigma(1)}^j, \dots, g_{\sigma(k_0)}^j, \dots, g_{\sigma(n)}^j \right)^T, \quad (3)$$

where $\{g_{\sigma(1)}^j, \dots, g_{\sigma(k_0)}^j\}$ is maximally linearly independent of $\{g_{\sigma(1)}^j, \dots, g_{\sigma(n)}^j\}$. And

$$\begin{aligned} g_{\sigma(k_0+1)}^j &= a_1^1 g_{\sigma(1)}^j + \dots + a_{k_0}^1 g_{\sigma(k_0)}^j, \\ &\vdots \\ g_{\sigma(n)}^j &= a_1^{n-k_0} g_{\sigma(1)}^j + \dots + a_{k_0}^{n-k_0} g_{\sigma(k_0)}^j, \end{aligned} \quad (4)$$

with the matrix representation

$$\begin{bmatrix} 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \dots & 0 & 0 & \dots & 0 \\ a_1^1 & \dots & a_{k_0}^1 & -1 & \dots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ a_1^{n-k_0} & \dots & a_{k_0}^{n-k_0} & 0 & \dots & -1 \end{bmatrix} \begin{bmatrix} g_{\sigma(1)}^j \\ \vdots \\ g_{\sigma(k_0)}^j \\ \vdots \\ g_{\sigma(n-1)}^j \\ g_{\sigma(n)}^j \end{bmatrix} = 0_{n \times n}, \quad (5)$$

that is, $M_j Q_j G_- E_j = 0_{n \times n}$.

Since $F_+(z)[G_-(w)E_j] = 0$, we have $[F_+(z)Q_j][Q_j G_- \times(w)E_j] = 0_{n \times n}$. Denote $F_+ = [f_{ij}]_{n \times n}$. It follows that

$$f_{k\sigma(1)} g_{\sigma(1)}^j + \dots + f_{k\sigma(k_0)} g_{\sigma(k_0)}^j + \dots + f_{k\sigma(n)} g_{\sigma(n)}^j = 0 \quad (6)$$

for $1 \leq k \leq n$. Note that $g_{\sigma(k_0+l)}^j = a_1^l g_{\sigma(1)}^j + \dots + a_{k_0}^l g_{\sigma(k_0)}^j$, $1 \leq l \leq n - k_0$. Therefore

$$\begin{aligned} &f_{k\sigma(1)} g_{\sigma(1)}^j + \dots + f_{k\sigma(k_0)} g_{\sigma(k_0)}^j \\ &+ \sum_{l=1}^{n-k_0} \left(a_1^l g_{\sigma(1)}^j + \dots + a_{k_0}^l g_{\sigma(k_0)}^j \right) f_{k\sigma(k_0+l)} = 0, \end{aligned} \quad (7)$$

and hence, $(f_{k\sigma(1)} + \sum_{l=1}^{n-k_0} a_1^l f_{k\sigma(k_0+l)})g_{\sigma(1)}^j + (f_{k\sigma(2)} + \sum_{l=1}^{n-k_0} a_2^l f_{k\sigma(k_0+l)})g_{\sigma(2)}^j + \dots + (f_{k\sigma(k_0)} + \sum_{l=1}^{n-k_0} a_{k_0}^l f_{k\sigma(k_0+l)})g_{\sigma(k_0)}^j = 0$. Since $\{g_{\sigma(1)}^j, \dots, g_{\sigma(k_0)}^j\}$ is linearly independent, we have

$$\begin{aligned} f_{k\sigma(1)} + \sum_{l=1}^{n-k_0} a_1^l f_{k\sigma(k_0+l)} &= 0, \\ &\vdots \\ f_{k\sigma(k_0)} + \sum_{l=1}^{n-k_0} a_{k_0}^l f_{k\sigma(k_0+l)} &= 0, \end{aligned} \quad (8)$$

with the matrix representation

$$\begin{aligned} &\begin{bmatrix} f_{1\sigma(1)} & \cdots & f_{1\sigma(k_0)} & \cdots & f_{1\sigma(n)} \\ f_{2\sigma(1)} & \cdots & f_{2\sigma(k_0)} & \cdots & f_{2\sigma(n)} \\ \vdots & & \vdots & & \vdots \\ \vdots & & \vdots & & \vdots \\ f_{n-1\sigma(1)} & \cdots & f_{n-1\sigma(k_0)} & \cdots & f_{n-1\sigma(n)} \\ f_{n\sigma(1)} & \cdots & f_{n\sigma(k_0)} & \cdots & f_{n\sigma(n)} \end{bmatrix} \\ &\times \begin{bmatrix} 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 1 & 0 & \cdots & 0 \\ a_1^1 & \cdots & a_{k_0}^1 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ a_1^{n-k_0} & \cdots & a_{k_0}^{n-k_0} & 0 & \cdots & 0 \end{bmatrix} = 0_{n \times n}, \end{aligned} \quad (9)$$

that is $F_+ Q_j (I + M_j) = 0_{n \times n}$.

In the following, we prove (3) $\Rightarrow$ (1).

Note that

$$\begin{aligned} T_F T_G - T_{FG} &= T_{(F_+ + F_-)} T_{(G_+ + G_-)} - T_{(F_+ + F_-)(G_+ + G_-)} \\ &= T_{(F_+ G_+ + F_- G_- + F_- G_+)} + T_{F_+ G_-} \\ &\quad - T_{(F_+ G_+ + F_- G_- + F_- G_+)} - T_{F_+ G_-} \\ &= T_{F_+} T_{G_-} - T_{F_+ G_-}. \end{aligned} \quad (10)$$

By the hypothesis, we have

$$\begin{aligned} T_{F_+} T_{G_- E_j} &= T_{F_+ Q_j (I + M_j - M_j) Q_j} T_{G_- E_j} \\ &= T_{F_+ Q_j (I + M_j) Q_j} T_{G_- E_j} - T_{F_+ Q_j M_j Q_j} T_{G_- E_j} \\ &= T_{F_+ Q_j (I + M_j) Q_j} T_{G_- E_j} - T_{F_+ Q_j} T_{M_j Q_j G_- E_j} = 0_{n \times n}, \\ F_+ G_- E_j &= F_+ Q_j [I + M_j - M_j] Q_j G_- E_j \\ &= [F_+ Q_j (I + M_j)] Q_j G_- E_j \\ &\quad - F_+ Q_j [M_j Q_j G_- E_j] = 0_{n \times n}. \end{aligned} \quad (11)$$

Hence, we obtain the desired result. $\square$

Remark 4. If the assumption $F_+(0) = G_-(0) = 0_{n \times n}$ is removed and F_+, G_- are replaced by F'_+, G'_- in statement (3), respectively, then the above theorem still holds.

Corollary 5. Let $F(z) = F_+(z) + F_-(z)$, and let $G(z) = G_+(z) + G_-(z) \in h^\infty \otimes M_{n \times n}$ such that $F_+(0) = G_-(0) = 0_{n \times n}$. Suppose that $|\det F_+| + |\det G_-|$ is not identically zero. Then the following are equivalent:

- (1) $T_{FG} - T_F T_G$ has finite rank;
- (2) $T_{FG} = T_F T_G$;
- (3) if $\det F_+$ is not identically zero, then G is analytic; if $\det G_-$ is not identically zero, then F is coanalytic.

Proof. The proof is obvious. $\square$

Let

$$\Phi = \begin{bmatrix} F & -G \\ 0 & 0 \end{bmatrix}, \quad \Psi = \begin{bmatrix} G & 0 \\ F & 0 \end{bmatrix}. \quad (12)$$

As in [6], the commutator $T_F T_G - T_G T_F$ can be reduced to the semicommutator $T_{\Phi\Psi} - T_\Phi T_\Psi$.

It is easy to check that

$$\begin{aligned} T_{\Phi\Psi} - T_\Phi T_\Psi &= \begin{bmatrix} T_{FG-GF} & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} T_F & -T_G \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T_G & 0 \\ T_F & 0 \end{bmatrix} \\ &= \begin{bmatrix} T_{FG-GF} & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} T_F T_G - T_G T_F & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned} \quad (13)$$

If $T_F T_G = T_G T_F$, Theorem 2 implies that $FG = GF$. Therefore, $T_\Phi T_\Psi = T_\Psi \Phi$. Combining Theorem 3, we get the following explicit theorem.

Theorem 6. Suppose that $F = F_+ + F_- \in h^\infty \otimes M_{n \times n}$ and $G = G_+ + G_- \in h^\infty \otimes M_{n \times n}$. Then $T_F T_G = T_G T_F$ if and only if F and G satisfy the following two conditions:

- (1) $FG = GF$;
- (2) for each $1 \leq j \leq n$, there exist permutation matrixes $\widetilde{Q}_j$ and $\mathbb{M}_j \in M_{2n \times 2n}$ such that $\mathbb{M}_j \widetilde{Q}_j \begin{bmatrix} G'_+ \\ F'_- \end{bmatrix} E_j = 0_{n \times n}$ and $[F'_+ - G'_+] \widetilde{Q}_j (I + \mathbb{M}_j) = 0_{n \times n}$.

3. Zero Product of Two Block Toeplitz Operators

In this section, we discuss the zero product of two block Toeplitz operators T_F and T_G with $F, G \in h^\infty \otimes M_{n \times n}$ on $L_a^2(\mathbb{D}, \mathbb{C}^n)$. $\square$

Theorem 7. Let $F(z) = F_+(z) + F_-(z)$ and $G(z) = G_+(z) + G_-(z) \in h^\infty \otimes M_{n \times n}$ such that $F_+(0) = G_-(0) = F_-(0) = G_+(0) = 0_{n \times n}$. Then $T_F T_G = 0_{n \times n}$ if and only if

- (1) $FG = 0_{n \times n}$;
- (2) for each $1 \leq j \leq n$, there exist a matrix $N_j \in M_{n \times n}$ and a permutation matrix Q_j^1 , such that $N_j Q_j^1 G_- E_j = 0_{n \times n}$ and $F_+ Q_j^1 (I + N_j) = 0_{n \times n}$.

Proof. By Theorem 2, it is easy to know that $\sum_{p=1}^n f_{ip}(z) g_{pj}(z) = 0$ for $1 \leq i, j \leq n$; that is, $FG = 0_{n \times n}$. Since $T_F T_G = T_F T_G - T_{FG} = 0_{n \times n}$, we obtain the desired result by Theorem 3. $\square$

Corollary 8. Let $F(z) = F_+(z) + F_-(z)$ and let $G(z) = G_+(z) + G_-(z) \in h^\infty \otimes M_{n \times n}$ such that $F_+(0) = G_-(0) = F_-(0) = G_+(0) = 0_{n \times n}$.

- (1) If neither $\det G_+$ nor $\det G_-$ is identically zero, then $T_F T_G = 0$ if and only if $F = 0_{n \times n}$.
- (2) If neither $\det F_+$ nor $\det F_-$ is identically zero, then $T_F T_G = 0$ if and only if $G = 0_{n \times n}$.

Proof. (1) We only need to prove the necessity. By the proof of Theorem 3, we have $F_+(z)G_-(w) = 0_{n \times n}$ on $\mathbb{D} \times \mathbb{D}$. By the theorem given above, we know that $FG = [F_+ + F_-][G_+ + G_-] = 0_{n \times n}$. Taking Laplace transform, it follows that $F'_+ G'_+ = 0_{n \times n}$. As in Theorem 3, we can prove that $F_-(w)G_+(z) = 0_{n \times n}$ on $\mathbb{D} \times \mathbb{D}$. “Complexificate” the identity $FG = 0_{n \times n}$, and then $[F_+(z) + F_-(w)][G_+(z) + G_-(w)] = 0_{n \times n}$. Hence, $F_+(z)G_+(z) = -F_-(w)G_-(w)$ on $\mathbb{D} \times \mathbb{D}$. Since $F_+(0) = G_-(0) = 0_{n \times n}$, we have $F_+ G_+ = F_- G_- = 0_{n \times n}$.

It is clear that $\det G_+(z)$ is the analytic function of z and $\det G_-(w)$ is the coanalytic function of w . If $\det G_+(z)$ is not identically zero, then $F_- \equiv 0_{n \times n}$ and $F_+ \equiv 0_{n \times n}$. Therefore, $F = 0_{n \times n}$. If $\det G_-(w)$ is not identically zero, then $F_- \equiv 0_{n \times n}$ and $F_+ \equiv 0_{n \times n}$. Therefore, $F = 0_{n \times n}$.

- (2) The proof is similar to the proof of (1). $\square$

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