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Research Article

A Rotation of Admixable Operators on Abstract Wiener Space with Applications

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We investigate certain rotation properties of the abstract Wiener measure. To determine our rotation property for the Wiener measure, we introduce the concept of an admixable operator via an algebraic structure on abstract Wiener space. As for applications, we define the analytic Fourier-Feynman transform and the convolution product associated with the admixable operators and proceed to establish the relationships between this transform and the corresponding convolution product.

1. Introduction and Preliminaries

Let $(C_0[0,T], \mathscr{M}, m_w)$ denote one-parameter Wiener space. Bearman's rotation theorem [1] for Wiener measure has played an important role in various research areas in mathematics and physics involving Wiener integration theory. Bearman's theorem was further developed by Cameron and Storvick [2] and by Johnson and Skoug [3] in their studies of Wiener integral equations. Recently, in [4], using results in [5], Chang et al. obtained results involving a very general multiple Fourier-Feynman transform on Wiener space.

Let H be a real separable infinite-dimensional Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $|\cdot| = \sqrt{\langle\cdot,\cdot\rangle}$. Let $\|\cdot\|_0$ be a measurable norm on H with respect to the Gaussian cylinder set measure ν_0 on H. Let B denote the completion of H with respect to $\|\cdot\|_0$. Let i denote the natural injection from H to B. The adjoint operator i^* of i is one to one and maps B^* continuously onto a dense subset of H^*, where B^* and H^* are topological duals of B and H, respectively. By identifying H^* with H and B^* with i^*B^*, we have a triple $B^* \subset H^* \approx H \subset B$ and $\langle x, y\rangle = (x, y)$ for all x in H and y in B^*, where $(\cdot,\cdot)$ denotes the natural dual pairing between B and B^*. By the well-known result of Gross [6], $\nu_0 \circ i^{-1}$ provides a unique countably additive extension, ν, to the Borel σ-algebra $\mathscr{B}(B)$ of B. ν is a probability measure on the Borel σ-algebra $\mathscr{B}(B)$ of B which satisfies

$$\int_B \exp\{i(y,x)\}\, d\nu(x) = \exp\left\{-\frac{1}{2}|y|^2\right\} \quad \text{for every } y \in B^*. \tag{1}$$

The triple (B, H, ν) is called an abstract Wiener space. For more details, see [6–9].

Let $\{e_j\}_{j=1}^{\infty}$ be a complete orthonormal set in H such that e_j's are in B^*. For each $h \in H$ and $x \in B$, we define the stochastic inner product $(h,x)^\sim$ by

$$(h,x)^\sim = \begin{cases} \lim\limits_{n\to\infty} \sum\limits_{j=1}^{n} \langle h, e_j\rangle (e_j, x), & \text{if the limit exists,} \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

For every $h\,(\neq 0)$ in H, $(h,x)^\sim$ exists for ν-a.e. $x \in B$, and it is a Gaussian random variable on B with mean zero and variance $|h|^2$; that is, (1) holds with $y \in B^*$ replaced with $h \in H$. In fact, the stochastic inner product $(h,x)^\sim$ is essentially independent of the choice of the complete orthonormal set used in its definition. Also, if both h and x are in H, then Parseval's

identity gives $(h,x)^{\sim} = \langle h,x\rangle$. Furthermore, $(h,\lambda x)^{\sim} = (\lambda h,x)^{\sim} = \lambda(h,x)^{\sim}$ for any $\lambda \in \mathbb{R}$, $h \in H$, and $x \in B$. We also see that, if $\{h_1,\ldots,h_n\}$ is an orthonormal set in H, then the random variables $(h_j,x)^{\sim}$'s are independent.

Let $\mathscr{W}(B)$ be the class of ν-measurable subsets of B. A subset E of B is said to be scale-invariant measurable [3, 7] provided ρE is $\mathscr{W}(B)$-measurable for every $\rho > 0$, and a scale-invariant measurable subset N of B is said to be scale-invariant null provided $\nu(\rho N) = 0$ for every $\rho > 0$. A property that holds except on a scale-invariant null set is said to hold scale invariant almost everywhere (s-a.e.). A functional F on B is said to be scale-invariant measurable provided F is defined on a scale-invariant measurable set and $F(\rho\cdot)$ is $\mathscr{W}(B)$-measurable for every $\rho > 0$. If two functionals F and G on B are equal s-a.e., that is, for each $\rho > 0$, $\nu(\{x \in B : F(\rho x) \neq G(\rho x)\}) = 0$, then we write $F \approx G$.

Next, we introduce the concept of an *admixable operator* on B.

Definition 1. Let $\odot$ be an operation between H and B^* which satisfies the conditions:

(1) $B^* \times B^* \ni (g_1,g_2) \mapsto g_1 \odot g_2 = g_2 \odot g_1 \in B^*$.

(2) $H \times B^* \ni (h,g) \mapsto h \odot g = g \odot h \in H$.

(3) If $h \odot g = 0$ for $h \in H$ and $g \in B^*$, then $h = 0$ or $g = 0$.

(4) For every $h \in H$ and every $g_1, g_2 \in B^*$,

$$(h \odot g_1) \odot g_2 = h \odot (g_1 \odot g_2). \tag{3}$$

(5) For every $h_1, h_2 \in H$ and every $g \in B^*$,

$$(h_1 + h_2) \odot g = h_1 \odot g + h_2 \odot g. \tag{4}$$

(6) For every $g_1, g_2 \in B^*$, there exists $g_3 \in B^*$ such that

$$g_1^{\odot 2} + g_2^{\odot 2} = g_3^{\odot 2}, \tag{5}$$

where $g^{\odot 2} = g \odot g$. In this case, we write $g_3 = \sqrt[\odot]{g_1^{\odot 2} + g_2^{\odot 2}}$.

(7) For every $h_1, h_2 \in H$ and every $g \in B^*$,

$$\langle h_1, h_2 \odot g\rangle = \langle h_1 \odot g, h_2\rangle. \tag{6}$$

Given $g \in B^*$, let $A_g : B \to B$ be a linear operator associated with g. The operator A_g is said to be $g^{\odot}$-admixable provided $(h, A_g x)^{\sim} = (h \odot g, x)^{\sim}$ for all $h \in H$.

For a finite subset $\mathscr{V} = \{v_1,\ldots,v_m\}$ of H, let $X_{\mathscr{V}} : B \to \mathbb{R}^m$ be the random vector given by

$$X_{\mathscr{V}}(x) = \left((v_1,x)^{\sim},\ldots,(v_m,x)^{\sim}\right) \equiv (\vec{v},x)^{\sim}. \tag{7}$$

A functional F is called a cylinder-type functional on B if there exists a linearly independent subset $\mathscr{V} = \{v_1,\ldots,v_m\}$ of H such that

$$F(x) = \psi\left(X_{\mathscr{V}}(x)\right), \quad x \in B, \tag{8}$$

where ψ is a complex-valued Lebesgue measurable function on $\mathbb{R}^m$. It is easy to show that, for the given cylinder-type functional F of the form (8), there exists an orthogonal subset $\mathscr{H} = \{h_1,\ldots,h_n\}$ of H such that F is expressed as

$$F(x) \equiv f\left(X_{\mathscr{H}}(x)\right) = f\left((h_1,x)^{\sim},\ldots,(h_n,x)^{\sim}\right), \quad x \in B, \tag{9}$$

where f is a complex-valued Lebesgue measurable function on $\mathbb{R}^n$. Thus, we lose no generality in assuming that every cylinder-type functional on B is of the form (9).

Lemma 2 (Chung, [7]). *Let (B,H,ν) be an abstract Wiener space, and let $\mathscr{H} = \{h_1,\ldots,h_n\}$ be an orthogonal set in H. Let f be a complex-valued function defined on $\mathbb{R}^n$. Then, for the cylinder-type functional F given by* (9) *on B,*

(i) *F is $\mathscr{B}(B)$-measurable if and only if f is Borel measurable on $\mathbb{R}^n$,*

(ii) *F is $\mathscr{W}(B)$-measurable if and only if f is Lebesgue measurable on $\mathbb{R}^n$.*

For $g \in B^*$, let A_g be the $g^{\odot}$-admixable operator on B. In this case, for any orthogonal subset $\mathscr{H}$ of H,

$$\begin{aligned} X_{\mathscr{H}}\left(A_g x\right) &= \left(\left(h_1, A_g x\right)^{\sim},\ldots,\left(h_n, A_g x\right)^{\sim}\right) \\ &= \left((h_1 \odot g, x)^{\sim},\ldots,(h_n \odot g, x)^{\sim}\right). \end{aligned} \tag{10}$$

The seminal results by Bearman in [1] are summarized as follows (see [2]): if $F(\sqrt{a^2+b^2}x)$ is Wiener integrable on $C_0[0,T]$ for $a,b \in \mathbb{R}$, then $F(ax_1 + bx_2)$ is integrable on $(C_0[0,T])^2$ and

$$\begin{aligned} &\int_{(C_0[0,T])^2} F\left(ax_1 + bx_2\right) d\left(m_w \times m_w\right)(x_1,x_2) \\ &\quad = \int_{C_0[0,T]} F\left(\sqrt{a^2+b^2}x\right) dm_w(x). \end{aligned} \tag{11}$$

The main purpose of this paper is to establish a rotation property for the abstract Wiener integral,

$$\begin{aligned} &\int_{B^2} F\left(A_{g_1}x_1 + A_{g_2}x_2\right) d(\nu\times\nu)(x_1,x_2) \\ &\quad = \int_B F\left(A_g x\right) d\nu(x), \end{aligned} \tag{12}$$

where F is given by (9) and A_g is determined by A_{g_1} and A_{g_2}.

2. A Typical Example of an Abstract Wiener Space

The classical Wiener space $C_0[0,T]$, which is one of the most important examples of abstract Wiener spaces (see [9]), is a triple (B,H,ν), where

(i) $B = C_0[0,T]$ is a Banach space consisting of real-valued continuous functions $x(t)$ with $x(0) = 0$ defined on the compact interval $[0,T]$ endowed with uniform norm $\|x\| = \sup_{t\in[0,T]}|x(t)|$,

(ii) $H = C_0'[0,T]$ is a real separable infinite dimensional Hilbert space consisting of absolutely continuous functions $h(t)$ with $h(0) = 0$, such that $Dh \equiv dh/dt \in L_2[0,T]$ endowed with the inner product

$$\langle h_1, h_2 \rangle = \int_0^T Dh_1(s)\, Dh_2(s)\, ds, \tag{13}$$

(iii) $\nu = m_w$ is the Wiener measure on the Borel σ-algebra $\mathscr{B}(C_0[0,T])$ of $C_0[0,T]$ with

$$m_w(\{x : x(t) \le a\}) = \int_{-\infty}^{a} \exp\left\{-\frac{u^2}{2t}\right\} du. \tag{14}$$

Let I be the unitary operator from $L_2[0,T]$, onto $C_0'[0,T]$ given by $I\nu(t) = \int_0^t \nu(s)ds$ for $\nu \in L_2[0,T]$ and let

$$\begin{aligned} C_0^*[0,T] = \{ & I\nu : \nu \text{ is continuous except for a finite} \\ & \text{number of finite jump discontinuities} \\ & \text{and is of bounded variation on } [0,T]\}. \end{aligned} \tag{15}$$

For any $h \in C_0'[0,T]$ and $g \in C_0^*[0,T]$, let the operation $\odot$ between $C_0'[0,T]$ and $C_0^*[0,T]$ be defined by

$$h \odot g = I(DhDg), \tag{16}$$

where $DhDg$ denotes the pointwise multiplication of the functions Dh and Dg.

It is readily seen that $\{x(t) : (x,t) \in C_0[0,T] \times [0,T]\}$ is a standard Wiener process on the probability space $(C_0[0,T], \mathscr{B}(C_0[0,T]), m_w)$. In this case, we know that, for each $h \in C_0'[0,T]$ and s-a.e. $x \in C_0[0,T]$,

$$(h,x)^{\sim} = \int_0^T Dh(t)\, \tilde{d}x(t), \tag{17}$$

where $\int_0^T Dh(t)\tilde{d}x(t)$ denotes the Paley-Wiener-Zygmund stochastic integral [10–12].

Remark 3. Let $\mathscr{M}$ be the set of all m_w-measurable subsets of $C_0[0,T]$. Then, $(C_0[0,T], \mathscr{M}, m_w)$ is a complete measure space. It is well known that $\mathscr{M}$ coincides with $\sigma(\mathscr{B}(C_0[0,T]))$, the completion of $\mathscr{B}(C_0[0,T])$.

Let $g \in C_0'[0,T]$ with $|g| = \sqrt{\langle g, g\rangle} > 0$. Then, the stochastic integral

$$\mathscr{Z}_g(x,t) = \int_0^t Dg(s)\, \tilde{d}x(s), \quad t \in [0,T], \tag{18}$$

which was introduced by Park and Skoug in [13], is a Gaussian process with mean zero and covariance function

$$\int_{C_0[0,T]} \mathscr{Z}_g(x,s)\, \mathscr{Z}_g(x,t)\, dm_w(x) = \int_0^{\min\{s,t\}} (Dg(u))^2 du. \tag{19}$$

In addition, $\mathscr{Z}_g(x,t)$ is stochastically continuous in t on $[0,T]$. For more detailed studies of this process, see [14–17]. Furthermore, if g is an element of $C_0^*[0,T]$, then, for all $x \in C_0[0,T]$, $\mathscr{Z}_g(x,t)$ is continuous in t, and so is $\mathscr{Z}_g(x,\cdot)$ in $C_0[0,T]$.

From [14, Lemma 1], we note that, for each $h \in C_0'[0,T]$ and each $g \in C_0'[0,T]$ with $Dg \in L_\infty[0,T]$,

$$\int_0^T Dh(t)\, \tilde{d}\mathscr{Z}_g(x,t) = \int_0^T Dh(t)\, Dg(t)\, \tilde{d}x(t) \tag{20}$$

for s-a.e. $x \in C_0[0,T]$.

Given $g \in C_0^*[0,T]$, define an operator $A_g : C_0[0,T] \to C_0[0,T]$ by

$$A_g x = \mathscr{Z}_g(x,\cdot). \tag{21}$$

Then, for all $h \in H$,

$$\begin{aligned} (h, A_g x)^{\sim} &= \int_0^T Dh(t)\, \tilde{d}\mathscr{Z}_g(x,t) \\ &= \int_0^T Dh(t)\, Dg(t)\, \tilde{d}x(t) = (h \odot g, x)^{\sim}. \end{aligned} \tag{22}$$

Thus, A_g is $g^{\odot}$-admixable in view of Definition 1.

3. A Rotation of Admixable Operators

In this section, we establish a rotation property for the abstract Wiener integral involving admixable operators. We first introduce an integration formula which plays a key role.

Lemma 4. *Let $\mathscr{A} = \{\alpha_1, \ldots, \alpha_n\}$ be an orthogonal set in H and let $X_{\mathscr{A}}$ be given by (7). Let $f : \mathbb{R}^n \to \mathbb{C}$ be a Lebesgue measurable function. Then*

$$\begin{aligned} &\int_B f(X_{\mathscr{A}}(x))\, d\nu(x) \\ &\quad = \int_B f((\alpha_1, x)^{\sim}, \ldots, (\alpha_n, x)^{\sim})\, d\nu(x) \\ &\quad \overset{*}{=} \left(\prod_{j=1}^n 2\pi |\alpha_j|^2\right)^{-1/2} \\ &\qquad \times \int_{\mathbb{R}^n} f(u_1, \ldots, u_n) \\ &\qquad\qquad \times \exp\left\{-\sum_{j=1}^n \frac{u_j^2}{2|\alpha_j|^2}\right\} du_1 \cdots du_n, \end{aligned} \tag{23}$$

where by $\overset{*}{=}$ one means that if either side exists, both sides exist and equality holds.

The following integration formula is also used several times in this paper:

$$\int_{\mathbb{R}} \exp\{-au^2 + bu\}\, du = \sqrt{\frac{\pi}{a}} \exp\left\{\frac{b^2}{4a}\right\} \tag{24}$$

for complex numbers a and b with $\mathrm{Re}(a) > 0$.

Remark 5. Let α^1 and α^2 be elements in H with $|\alpha^1| = |\alpha^2| \equiv \sigma^2 > 0$. Then, the random variables $X_1(x) = (\alpha^1, x)^{\sim}$ and $X_2(x) = (\alpha^2, x)^{\sim}$ will have the same distribution $N(0, \sigma^2)$.

Let $\mathscr{A}_1 = \{\alpha_1^1, \ldots, \alpha_n^1\}$ and $\mathscr{A}_2 = \{\alpha_1^2, \ldots, \alpha_n^2\}$ be orthogonal sets in H with $|\alpha_j^1| = |\alpha_j^2|$ for each $j \in \{1, \ldots, n\}$. Using the aforementioned facts and applying (23), we see that, for any Lebesgue measurable function f on $\mathbb{R}^n$,

$$\int_B f\left(X_{\mathscr{A}_1}(x)\right) d\nu(x) \stackrel{*}{=} \int_B f\left(X_{\mathscr{A}_2}(x)\right) d\nu(x). \tag{25}$$

To simplify the expressions in our results, we use the following notations:

$$\begin{gathered} f(\vec{u}) \equiv f(u_1, \ldots, u_n), \\ f\left(\vec{u} + X_{\mathscr{A}}(x)\right) \equiv f\left(u_1 + (\alpha_1, x)^{\sim}, \ldots, u_n + (\alpha_n, x)^{\sim}\right) \end{gathered} \tag{26}$$

for $\vec{u} = (u_1, \ldots, u_n) \in \mathbb{R}^n$ and $X_{\mathscr{A}}(x)$ given by (7).

Proposition 6. *Let $\mathscr{A}_1 = \{\alpha_1^1, \ldots, \alpha_n^1\}$ and $\mathscr{A}_2 = \{\alpha_1^2, \ldots, \alpha_\nu^2\}$ be orthogonal sets in H. Then, for any Lebesgue measurable function f on $\mathbb{R}^n$,*

$$\begin{aligned} &\int_{B^2} f\left(X_{\mathscr{A}_1}(x_1) + X_{\mathscr{A}_2}(x_2)\right) d(\nu \times \nu)(x_1, x_2) \\ &\quad \stackrel{*}{=} \int_B f\left(X_{\mathscr{W}_{(2)}}(x)\right) d\nu(x), \end{aligned} \tag{27}$$

where $\mathscr{W}_{(2)} = \{w_1, \ldots, w_n\}$ is an orthogonal set in H which satisfies the condition

$$\left|w_j\right|^2 = \left|\alpha_j^1\right|^2 + \left|\alpha_j^2\right|^2 \tag{28}$$

for each $j \in \{1, \ldots, n\}$. Also, both of the expressions in (27) *are given by the last expression in* (29).

Proof. First, using (23), the Fubini theorem, and (24), we have that

$$\begin{aligned} &\int_{B^2} f\left(X_{\mathscr{A}_1}(x_1) + X_{\mathscr{A}_2}(x_2)\right) d(\nu \times \nu)(x_1, x_2) \\ &= \int_B \left[\int_B f\left(X_{\mathscr{A}_1}(x_1) + X_{\mathscr{A}_2}(x_2)\right) d\nu(x_1)\right] d\nu(x_2) \\ &= \left(\prod_{j=1}^n 2\pi\left|\alpha_j^1\right|^2\right)^{-1/2} \\ &\quad \times \int_{\mathbb{R}^n} \left[\int_B f\left(\vec{u} + X_{\mathscr{A}_2}(x_2)\right) d\nu(x_2)\right] \\ &\qquad \times \exp\left\{-\sum_{j=1}^n \frac{u_j^2}{2\left|\alpha_j^1\right|^2}\right\} d\vec{u} \\ &= \left(\prod_{j=1}^n 2\pi\left|\alpha_j^1\right|^2\right)^{-1/2} \left(\prod_{j=1}^n 2\pi\left|\alpha_j^2\right|^2\right)^{-1/2} \\ &\quad \times \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} f(\vec{u} + \vec{v}) \right. \\ &\qquad \left. \times \exp\left\{-\sum_{j=1}^n \frac{v_j^2}{2\left|\alpha_j^2\right|^2}\right\} d\vec{v}\right] \\ &\qquad \times \exp\left\{-\sum_{j=1}^n \frac{u_j^2}{2\left|\alpha_j^1\right|^2}\right\} d\vec{u} \\ &= \left(\prod_{j=1}^n 2\pi\left|\alpha_j^1\right|^2\right)^{-1/2} \left(\prod_{j=1}^n 2\pi\left|\alpha_j^2\right|^2\right)^{-1/2} \\ &\quad \times \iint_{\mathbb{R}^n} f(\vec{r}) \exp\left\{-\sum_{j=1}^n \frac{\left(r_j - u_j\right)^2}{2\left|\alpha_j^2\right|^2}\right\} \\ &\qquad \times \exp\left\{-\sum_{j=1}^n \frac{u_j^2}{2\left|\alpha_j^1\right|^2}\right\} d\vec{u} d\vec{r} \\ &= \left(\prod_{j=1}^n 2\pi\left|\alpha_j^1\right|^2\right)^{-1/2} \left(\prod_{j=1}^n 2\pi\left|\alpha_j^2\right|^2\right)^{-1/2} \\ &\quad \times \int_{\mathbb{R}^n} f(\vec{r}) \left[\int_{\mathbb{R}^n} \exp\left\{-\sum_{j=1}^n \frac{u_j^2}{2\left|\alpha_j^1\right|^2} \right.\right. \\ &\qquad \left.\left. -\sum_{j=1}^n \frac{\left(r_j - u_j\right)^2}{2\left|\alpha_j^2\right|^2}\right\} d\vec{u}\right] d\vec{r} \\ &= \left(\prod_{j=1}^n 2\pi\left(\left|\alpha_j^1\right|^2 + \left|\alpha_j^2\right|^2\right)\right)^{-1/2} \\ &\quad \times \int_{\mathbb{R}^n} f(\vec{r}) \exp\left\{-\sum_{j=1}^n \frac{r_j^2}{2\left(\left|\alpha_j^1\right|^2 + \left|\alpha_j^2\right|^2\right)}\right\} d\vec{r}. \end{aligned} \tag{29}$$

Next, let $\{e_1, \ldots, e_n\}$ be any orthonormal set in H. For each $j \in \{1, \ldots, n\}$, let $w_j = (|\alpha_j^1|^2 + |\alpha_j^2|^2)^{1/2} e_j$. Then $\mathscr{W}_{(2)} = \{w_1, \ldots, w_n\}$ is an orthogonal set in H and satisfies (28) above. In this case, using (23), the Fubini theorem, and (24), we see that $\int_B f(X_{\mathscr{W}_{(2)}}(x)) d\nu(x)$ is given by the last expression of (29). In view of Remark 5, we obtain the desired result. □

The following corollary follows by the use of mathematical induction.

Corollary 7. *For each $k \in \{1,\dots,m\}$, let $\mathscr{A}_k = \{\alpha_1^k,\dots,\alpha_n^k\}$ be an orthogonal set in H. Then for any Lebesgue measurable function f on $\mathbb{R}^n$,*

$$\int_{B^m} f\left(X_{\mathscr{A}_1}(x_1)+\cdots+X_{\mathscr{A}_m}(x_m)\right)d\nu^m(x_1,\dots,x_m) \overset{*}{=} \int_B f\left(X_{\mathscr{W}_{(m)}}(x)\right)d\nu(x), \tag{30}$$

where $\mathscr{W}_{(m)} = \{w_1,\dots,w_n\}$ is an orthogonal set in H which satisfies the condition

$$|w_j|^2 = \sum_{k=1}^m |\alpha_j^k|^2 \tag{31}$$

for each $j \in \{1,\dots,n\}$. Also, both of the expressions in (30) *are given by the expression*

$$\left(\prod_{j=1}^n 2\pi\left(\sum_{k=1}^m |\alpha_j^k|^2\right)\right)^{-1/2} \times \int_{\mathbb{R}^n} f(\vec{r})\exp\left\{-\sum_{j=1}^n \frac{r_j^2}{2\left(\sum_{k=1}^m |\alpha_j^k|^2\right)}\right\} d\vec{r}. \tag{32}$$

The next corollary follows directly from Proposition 6.

Corollary 8. *Let $\mathscr{A}_1$ and $\mathscr{A}_2$ be as in Proposition 6. Assume that $\mathscr{A}_1 \cup \mathscr{A}_2$ is an orthogonal set in H. Then, for any Borel measurable function f on $\mathbb{R}^n$,* (27) *is valid. In this case, $\mathscr{W}_{(2)}$ is given by the set $\{\alpha_1^1+\alpha_1^2,\dots,\alpha_n^1+\alpha_n^2\}$.*

Throughout the rest of this section, for convenience, we use the following notation: for a finite sequence $\mathscr{G} = \{g_1,\dots,g_m\}$ in B^*, let

$$s(\mathscr{G}) \equiv s(g_1,\dots,g_m) = \sqrt[\odot]{\sum_{k=1}^m g_k^{\odot 2}}. \tag{33}$$

For our rotation property presented, namely, Theorem 11, we will consider the pair of finite subsets $\mathscr{H} = \{h_1,\dots,h_n\}$ of $H-\{0\}$ and $\mathscr{G} = \{g_1,\dots,g_m\}$ of $B^*-\{0\}$ (this allows $|\mathscr{G}| \le m$) such that

(c1) $\{h_1,\dots,h_n\}$ is orthogonal in H,

(c2) $\{h_1 \odot g_k,\dots,h_n \odot g_k\}$ is orthogonal in H for all $k \in \{1,\dots,m\}$.

Let us return to the classical Wiener space $(C_0[0,T], C_0'[0,T], m_w)$. See Section 2. For each $l \in \{1,2,\dots\}$, let $\beta_l(t) = \sin(((l-1/2)\pi/T)t)$ on $[0,T]$. Then, $\Theta \equiv \{\beta_l\}_{l=1}^\infty$ is an orthogonal sequence in $C_0'[0,T]$. Additionally, $\widetilde{\Theta} = \{(\sqrt{2T}/(l-1/2)\pi)\beta_l\}_{l=1}^\infty$ is a complete orthonormal set in $C_0'[0,T]$. In fact, $\beta_l \in C_0^*[0,T]$ for all $l \in \{1,2,\dots\}$. We can take different pair $(\mathscr{H},\mathscr{G})$ from Θ, which satisfies conditions (c1) and (c2) above. For instance, let $\mathscr{H} = \{\beta_{l_1},\dots,\beta_{l_n}\}$ and let $\mathscr{G} = \{\beta_{k_1},\dots,\beta_{k_m}\}$, where $l_1 < l_2 < \cdots < l_n < k_1 \le k_2 \le \cdots \le k_m$.

Remark 9. Let $\mathscr{H} = \{h_1,\dots,h_n\}$ and $\mathscr{G} = \{g_1,\dots,g_m\}$ satisfy the conditions (c1) and (c2). Then, the stochastic inner products

$$\left(h_j, A_{s(\mathscr{G})}x\right)^\sim = \left(h_j \odot s(\mathscr{G}), x\right)^\sim, \quad j = 1,\dots,n, \tag{34}$$

form a set of independent Gaussian random variables on B with mean 0 and variance

$$\begin{aligned} |h_j \odot s(\mathscr{G})|^2 &= \left\langle h_j \odot s(\mathscr{G}), h_j \odot s(\mathscr{G})\right\rangle \\ &= \left\langle h_j \odot s(\mathscr{G}) \odot s(\mathscr{G}), h_j\right\rangle \\ &= \left\langle h_j \odot \left(\sum_{k=1}^m g_k^{\odot 2}\right), h_j\right\rangle \\ &= \left\langle \sum_{k=1}^m \left(h_j \odot g_k^{\odot 2}\right), h_j\right\rangle \\ &= \sum_{k=1}^m \left\langle h_j \odot g_k^{\odot 2}, h_j\right\rangle \\ &= \sum_{k=1}^m \left\langle h_j \odot g_k, h_j \odot g_k\right\rangle \\ &= \sum_{k=1}^m |h_j \odot g_k|^2, \end{aligned} \tag{35}$$

where $s(\mathscr{G}) \equiv s(g_1,\dots,g_m)$ is given by (33).

Remark 10. Given an orthogonal subset $\mathscr{H}$, let $B_{\mathscr{H}}^*$ be the space of every element g in $B^*-\{0\}$ such that $\{h \odot g : h \in \mathscr{H}\}$ is orthogonal in H. Then, for any finite subset $\mathscr{G}$ of $B_{\mathscr{H}}^*$, the pair $(\mathscr{H},\mathscr{G})$ satisfies conditions (c1) and (c2) above.

Given an orthogonal set $\mathscr{H}$ in H, let $\mathfrak{G}_{\mathscr{H}}^{(1)}$ be the class of all cylinder-type functionals, F, given by (9) for ν-a.e. $x \in B$, where the corresponding function f of F satisfies the condition

$$\int_{\mathbb{R}^n} |f(\vec{v})|\exp\left\{-\sum_{j=1}^n \frac{v_j^2}{2|h_j \odot g|^2}\right\} d\vec{v} < +\infty \tag{36}$$

for all $g \in B_{\mathscr{H}}^*$.

We are now ready to present our rotation property of abstract Wiener measure associated with admixable operators.

Theorem 11. *Let $\mathscr{H} = \{h_1,\dots,h_n\}$ be an orthogonal set in H and let $F \in \mathfrak{G}_{\mathscr{H}}^{(1)}$ be given by* (9)*. Let $\mathscr{G} = \{g_1,\dots,g_m\}$ be a finite subset of $B_{\mathscr{H}}^*$, and, for each $k \in \{1,\dots,m\}$, let A_{g_k} be the $g_k^{\odot}$-admixable operator on B. Then,*

$$\int_{B^m} F\left(\sum_{k=1}^m A_{g_k}x_k\right)d\nu^m(x_1,\dots,x_m) = \int_B F\left(A_{s(\mathscr{G})}x\right)d\nu(x), \tag{37}$$

where $s(\mathscr{G})$ is given by (33).

Proof. For each $j \in \{1,\dots,n\}$ and each $k \in \{1,\dots,m\}$, let $\alpha_j^k = h_j \odot g_k$. For each $k \in \{1,\dots,m\}$, $\mathscr{A}_k = \{\alpha_1^k,\dots,\alpha_n^k\}$ is orthogonal in H by condition (c2). Hence, for each $k \in \{1,\dots,m\}$, the stochastic inner products

$$\left(h_j, A_{g_k}x\right)^{\sim} = \left(\alpha_j^k, x\right)^{\sim}, \quad j = 1,\dots,n, \tag{38}$$

form a set of independent Gaussian random variables with mean 0 and variance $|\alpha_j^k|^2$.

We observe that

$$\begin{aligned} X_{\mathscr{H}}&\left(\sum_{k=1}^m A_{g_k}x_k\right) \\ &= \left(\left(h_1, \sum_{k=1}^m A_{g_k}x_k\right)^{\sim},\dots,\left(h_n, \sum_{k=1}^m A_{g_k}x_k\right)^{\sim}\right) \\ &= \left(\sum_{k=1}^m \left(h_1, A_{g_k}x_k\right)^{\sim},\dots,\sum_{k=1}^m \left(h_n, A_{g_k}x_k\right)^{\sim}\right) \\ &= \sum_{k=1}^m \left(\left(h_1, A_{g_k}x_k\right)^{\sim},\dots,\left(h_n, A_{g_k}x_k\right)^{\sim}\right) \\ &= \sum_{k=1}^m \left(\left(\alpha_1^k, x_k\right)^{\sim},\dots,\left(\alpha_n^k, x_k\right)^{\sim}\right) \\ &= \sum_{k=1}^m X_{\mathscr{A}_k}(x_k). \end{aligned} \tag{39}$$

Thus, using (9), (39), and (30), we obtain

$$\begin{aligned} \int_{B^m} &F\left(\sum_{k=1}^m A_{g_k}x_k\right) d\nu^m(x_1,\dots,x_m) \\ &= \int_{B^m} f\left(X_{\mathscr{H}}\left(\sum_{k=1}^m A_{g_k}x_k\right)\right) d\nu^m(x_1,\dots,x_m) \\ &= \int_{B^m} f\left(\sum_{k=1}^m X_{\mathscr{A}_k}(x_k)\right) d\nu^m(x_1,\dots,x_m) \\ &= \int_B f\left(X_{\mathscr{W}_{(m)}}(x)\right) d\nu(x), \end{aligned} \tag{40}$$

where $\mathscr{W}_{(m)} = \{w_1,\dots,w_n\}$ is an orthogonal set in H, such that

$$\left|w_j\right|^2 = \sum_{k=1}^m \left|\alpha_j^k\right|^2 = \sum_{k=1}^m \left|h_j \odot g_k\right|^2 \tag{41}$$

for each $j \in \{1,\dots,n\}$.

Next, we note that, for each $j,l \in \{1,\dots,n\}$ with $j \neq l$,

$$\begin{aligned} \left\langle h_j \odot s(\mathscr{G}), h_l \odot s(\mathscr{G})\right\rangle &= \left\langle h_j \odot s(\mathscr{G}) \odot s(\mathscr{G}), h_l\right\rangle \\ &= \left\langle h_j \odot \sum_{k=1}^m g_k^{\odot 2}, h_l\right\rangle \\ &= \left\langle \sum_{k=1}^m h_j \odot g_k^{\odot 2}, h_l\right\rangle \\ &= \sum_{k=1}^m \left\langle h_j \odot g_k^{\odot 2}, h_l\right\rangle \\ &= \sum_{k=1}^m \left\langle h_j \odot g_k, h_l \odot g_k\right\rangle = 0. \end{aligned} \tag{42}$$

Hence, from (35) and (42), we see that $\{h_1 \odot s(\mathscr{G}),\dots,h_n \odot s(\mathscr{G})\}$ is an orthogonal set in H with $|h_j \odot s(\mathscr{G})|^2 = \sum_{k=1}^m |h_j \odot g_k|^2$ for all $j \in \{1,\dots,n\}$, and so we can choose $\mathscr{W}_{(m)}$ to be the orthogonal set $\{h_1 \odot s(\mathscr{G}),\dots,h_n \odot s(\mathscr{G})\}$. In this case, we see that

$$\begin{aligned} \int_B &f\left(X_{\mathscr{W}_{(m)}}(x)\right) d\nu(x) \\ &= \int_B f\left(\left(h_1 \odot s(\mathscr{G}), x\right)^{\sim},\dots,\left(h_n \odot s(\mathscr{G}), x\right)^{\sim}\right) d\nu(x) \\ &= \int_B f\left(\left(h_1, A_{s(\mathscr{G})}x\right)^{\sim},\dots,\left(h_n, A_{s(\mathscr{G})}x\right)^{\sim}\right) d\nu(x) \\ &= \int_B F\left(A_{s(\mathscr{G})}x\right) d\nu(x). \end{aligned} \tag{43}$$

Equation (37) follows from (40) and (43). □

Example 12. Let us return to the classical Wiener space $(C_0[0,T], C_0'[0,T], m_w)$ again. We introduce the family of functions $\Gamma \equiv \{\gamma_\tau : \tau \in [0,T]\}$ from $C_0'[0,T]$:

$$\gamma_\tau(s) = \begin{cases} s, & s \in [0,\tau] \\ \tau, & s \in [\tau,T]. \end{cases} \tag{44}$$

These functions have the reproducing property

$$\begin{aligned} \langle h, \gamma_\tau\rangle &= \int_0^T Dh(s)\, D\gamma_\tau(s)\, ds \\ &= \int_0^T Dh(s)\, \chi_{[0,\tau]}(s)\, ds \\ &= \int_0^\tau Dh(s)\, ds = h(\tau) \end{aligned} \tag{45}$$

for all $h \in C_0'[0,T]$. In fact, $\Gamma \subset C_0^*[0,T]$.

For $\gamma_\tau \in \Gamma$, we consider the admixable operator A_{γ_τ} given by (21). We note that, for each $g \in C_0^*[0,T]$,

$$\begin{aligned}\left(g, A_{\gamma_\tau} x\right)^{\sim} &= \left(g \odot \gamma_\tau, x\right)^{\sim} \\ &= \int_0^T Dg(s)\, D\gamma_\tau(s)\, \tilde{d}x(s) \\ &= \int_0^\tau Dg(s)\, \tilde{d}x(s) \\ &= \mathcal{Z}_g(x,\tau) = A_g x(\tau).\end{aligned} \tag{46}$$

From this, we see that, for $g_I(t) = t = \int_0^t ds \in C_0^*[0,T]$ and $a \in \mathbb{R}$,

$$\left(g_I, A_{a\gamma_\tau} x\right)^{\sim} = ax(\tau), \tag{47}$$

and so, using (37), we find that, for any $f \in L_1(\mathbb{R})$ and every $a, b \in \mathbb{R}$,

$$\begin{aligned}&\int_{(C_0[0,T])^2} f\left(ax_1(\tau) + bx_2(\tau)\right) d\left(m_w \times m_w\right)(x_1, x_2) \\ &= \int_{(C_0[0,T])^2} f\left(\left(g_I, A_{a\gamma_\tau} x_1\right)^{\sim}\right. \\ &\qquad \left. + \left(g_I, A_{b\gamma_\tau} x_2\right)^{\sim}\right) d\left(m_w \times m_w\right)(x_1, x_2) \\ &= \int_{C_0[0,T]} f\left(\left(g_I, A_{s(a\gamma_\tau, b\gamma_\tau)} x\right)^{\sim}\right) dm_w(x).\end{aligned} \tag{48}$$

Using (16), we now have

$$\begin{aligned}s(a\gamma_\tau, b\gamma_\tau)(t) &= \sqrt[\odot]{(a\gamma_\tau)^{\odot 2} + (b\gamma_\tau)^{\odot 2}}\,(t) \\ &= \sqrt{(a\gamma_\tau)^{\odot 2} + (b\gamma_\tau)^{\odot 2}}\,(t) \\ &= \sqrt{a^2 + b^2}\,\gamma_\tau(t)\end{aligned} \tag{49}$$

on $[0,T]$. Thus, by (48), (49), and (47), we have

$$\begin{aligned}&\int_{(C_0[0,T])^2} f\left(ax_1(\tau) + bx_2(\tau)\right) d\left(m_w \times m_w\right)(x_1, x_2) \\ &= \int_{C_0[0,T]} f\left(\sqrt{a^2 + b^2}\, x(\tau)\right) dm_w(x).\end{aligned} \tag{50}$$

Proposition 13 (Cameron and Storvick, [2]). *Let $F(\sqrt{a^2+b^2}x)$ be Wiener integrable on $C_0[0,T]$ for $a, b \in \mathbb{R}$. As a result, $F(ax_1 + bx_2)$ is integrable on $C_0^2[0,T]$, and*

$$\begin{aligned}&\int_{C_0^2[0,T]} F\left(ax_1 + bx_2\right) d\left(m_w \times m_w\right)(x_1, x_2) \\ &= \int_{C_0[0,T]} F\left(\sqrt{a^2 + b^2}\, x\right) dm_w(x).\end{aligned} \tag{51}$$

Proof. Let n be any positive integer, and let $0 = \tau_0 < \tau_1 < \cdots < \tau_n \le T$ be any partition of $[0,T]$. It suffices to show that (51) holds for any tame function

$$F(x) = f\left(x(\tau_1), x(\tau_2) - x(\tau_1), \ldots, x(\tau_n) - x(\tau_{n-1})\right) \tag{52}$$

with $f \in L_1(\mathbb{R}^n)$.

Let $\odot$ be defined by (16) between $C_0'[0,T]$ and $C_0^*[0,T]$. For $g_I(t) = t = \int_0^t ds \in C_0^*[0,T]$, let the $g_I^{\odot}$-admixable operator A_{g_I} be given by (21), and, for each $j \in \{1, \ldots, n\}$, let

$$h_j(t) = \int_0^t \chi_{[\tau_{j-1}, \tau_j]}(s)\, ds \tag{53}$$

on $[0,T]$. For any $a, b \in \mathbb{R} - \{0\}$,

(c1) $\{h_1, \ldots, h_n\}$ is an orthogonal set in $C_0'[0,T]$, and

(c2) $\{h_1 \odot ag_I, \ldots, h_n \odot ag_I\}$ and $\{h_1 \odot bg_I, \ldots, h_n \odot bg_I\}$ are orthogonal sets in $C_0'[0,T]$.

Also, for any $a, b \in \mathbb{R} - \{0\}$, $A_{s(ag_I, bg_I)} = A_{\sqrt{a^2+b^2} g_I}$, and, for each $j \in \{1, \ldots, n\}$,

$$\begin{aligned}\left(h_j, A_{ag_I} x\right)^{\sim} &= a \int_0^T \chi_{[\tau_{j-1}, \tau_j]}(s)\, dx(s) \\ &= a\left(x(\tau_j) - x(\tau_{j-1})\right).\end{aligned} \tag{54}$$

Thus, the left side of (51) with F given by (52) is rewritten by

$$\begin{aligned}&\int_{(C_0[0,T])^2} F\left(ax_1 + bx_2\right) d\left(m_w \times m_w\right)(x_1, x_2) \\ &= \int_{(C_0[0,T])^2} f\left(\left(h_1, A_{ag_I} x_1\right)^{\sim}\right. \\ &\qquad + \left(h_1, A_{bg_I} x_2\right)^{\sim}, \\ &\qquad \ldots, \left(h_n, A_{ag_I} x_1\right)^{\sim} \\ &\qquad \left. + \left(h_n, A_{bg_I} x_2\right)^{\sim}\right) d\left(m_w \times m_w\right)(x_1, x_2).\end{aligned} \tag{55}$$

Thus, from Theorem 11, we obtain

$$\begin{aligned}&\int_{(C_0[0,T])^2} F\left(ax_1 + bx_2\right) d\left(m_w \times m_w\right)(x_1, x_2) \\ &= \int_{C_0[0,T]} f\left(\left(h_1, A_{\sqrt{a^2+b^2} g_I} x\right)^{\sim},\right. \\ &\qquad \left. \ldots, \left(h_n, A_{\sqrt{a^2+b^2} g_I} x\right)^{\sim}\right) dm_w(x) \\ &= \int_{C_0[0,T]} F\left(\sqrt{a^2 + b^2}\, x\right) dm_w(x).\end{aligned} \tag{56}$$

Using standard methods, similar to those in [1], we can obtain the result for general functionals F on $C_0[0,T]$. □

4. Fourier-Feynman Transforms and Convolutions Associated with Admixable Operators

In this section, to apply our results from the previous section, we first define an L_p analytic Fourier-Feynman transform associated with admixable operators on B. Then, we establish the existence theorem and the inverse transform theorem of this transform for some classes of cylinder-type functionals on B having the form (9) for s-a.e. $x \in B$. Moreover, we present various relationships involving the convolution and the transforms.

Throughout the rest of this paper, let $\mathbb{C}$, $\mathbb{C}_+$, and $\widetilde{\mathbb{C}}_+$ denote, respectively, the complex numbers, the complex numbers with positive real part, and the nonzero complex numbers with nonnegative real part.

Let $g \in B^*$, and let A_g be the corresponding admixable operator on B. Let $F : B \to \mathbb{C}$ be a scale-invariant measurable functional such that

$$J_F(g;\lambda) = \int_B F\left(\lambda^{-1/2} A_g x\right) d\nu(x) \tag{57}$$

exists as a finite number for all $\lambda > 0$. If there exists a function $J_F^*(g;\lambda)$ analytic on $\mathbb{C}_+$ such that $J_F^*(g;\lambda) = J_F(g;\lambda)$ for all $\lambda > 0$, then $J_F^*(g;\lambda)$ is defined to be the analytic Wiener integral (associated with the $g^{\odot}$-admixable operator A_g) of F over B with parameter λ. For $\lambda \in \mathbb{C}_+$, we write

$$I_g^{\mathrm{anw}_\lambda}[F] = J_F^*(g;\lambda). \tag{58}$$

Let $q \neq 0$ be a real number, and let F be a functional such that $\int_B^{\mathrm{anw}_\lambda} F(A_g x) d\nu(x) = J_F^*(g;\lambda)$ exists for all $\lambda \in \mathbb{C}_+$. If the following limit exists, we call it the analytic Feynman integral of F with parameter q, and we write

$$I_g^{\mathrm{anf}_q}[F] = \lim_{\lambda \to -iq} I_g^{\mathrm{anw}_\lambda}[F], \tag{59}$$

where λ approaches $-iq$ through values in $\mathbb{C}_+$.

Note that if A_g is the identity operator on B, then these definitions agree with the previous definitions of the analytic Wiener integral and the analytic Feynman integral [18–20].

We are now ready to state the definition of the analytic Fourier-Feynman transform associated with admixable operator (admix-FFT).

Definition 14. Let (B, H, ν) be an abstract Wiener space. For $g \in B^*$, $\lambda \in \mathbb{C}_+$, and $y \in B$, let

$$T_{\lambda,g}(F)(y) = \int_B^{\mathrm{anw}_\lambda} F\left(y + A_g x\right) d\nu(x), \tag{60}$$

where A_g is the $g^{\odot}$-admixable operator on B. Let q be a nonzero real number. For $p \in (1,2]$, we define the L_p analytic $g^{\odot}$-admix-FFT, $T_{q,g}^{(p)}(F)$ of F, by the formula ($\lambda \in \mathbb{C}_+$),

$$T_{q,g}^{(p)}(F)(y) = \underset{\lambda \to -iq}{\mathrm{l.i.m.}}\, T_{\lambda,g}(F)(y), \tag{61}$$

if it exists; that is, for each $\rho > 0$,

$$\lim_{\lambda \to -iq} \int_B \left| T_{\lambda,g}(F)(\rho y) - T_{q,g}^{(p)}(F)(\rho y) \right|^{p'} d\nu(y) = 0, \tag{62}$$

where $1/p + 1/p' = 1$. We define the L_1 analytic $g^{\odot}$-admix-FFT, $T_{q,g}^{(1)}(F)$ of F, by the formula ($\lambda \in \mathbb{C}_+$),

$$T_{q,g}^{(1)}(F)(y) = \lim_{\lambda \to -iq} T_{\lambda,g}(F)(y), \tag{63}$$

if it exists.

We note that, for $p \in [1,2]$, $T_{q,g}^{(p)}(F)$ is defined only s-a.e.. We also note that if $T_{q,g}^{(p)}(F)$ exists and if $F \approx G$, then $T_{q,g}^{(p)}(G)$ exists and $T_{q,g}^{(p)}(G) \approx T_{q,g}^{(p)}(F)$.

Next, we give the definition of the convolution product (CP).

Definition 15. Let F and G be scale-invariant measurable functionals on B. For $\lambda \in \widetilde{\mathbb{C}}_+$ and $g \in B^*$, we define their CP with respect to A_g (if it exists) by

$$(F * G)_{\lambda,g}(y) = \begin{cases} \displaystyle\int_B^{\mathrm{anw}_\lambda} F\left(\frac{y + A_g x}{\sqrt{2}}\right) G\left(\frac{y - A_g x}{\sqrt{2}}\right) d\nu(x), & \lambda \in \mathbb{C}_+, \\ \displaystyle\int_B^{\mathrm{anf}_q} F\left(\frac{y + A_g x}{\sqrt{2}}\right) G\left(\frac{y - A_g x}{\sqrt{2}}\right) d\nu(x), & \lambda = -iq,\ q \in \mathbb{R},\ q \neq 0. \end{cases} \tag{64}$$

When $\lambda = -iq$, we denote $(F * G)_{\lambda,g}$ by $(F * G)_{q,g}$.

For any scale-invariant measurable functional F, we see that, for $\lambda > 0$,

$$\int_B^{\mathrm{anw}_\lambda} F\left(y + A_g x\right) d\nu(x) = \int_B F\left(y + \lambda^{-1/2} A_g x\right) d\nu(x) \tag{65}$$

if it exists.

Let $\mathscr{M}(\mathbb{R}^n)$ denote the space of complex-valued, countably additive (and hence finite) Borel measures on $\mathscr{B}(\mathbb{R}^n)$, the Borel σ-algebra of $\mathbb{R}^n$. It is well known that a complex-valued Borel measure τ necessarily has a finite total variation $\|\tau\|$, and $\mathscr{M}(\mathbb{R}^n)$ is a Banach algebra under the norm $\|\cdot\|$ and with convolution as multiplication.

For $\tau \in \mathscr{M}(\mathbb{R}^n)$, the Fourier transform $\widehat{\tau}$ of τ is a complex-valued function defined on $\mathbb{R}^n$ by the formula

$$\widehat{\tau}(\vec{u}) = \int_{\mathbb{R}^n} \exp\left\{ i \sum_{j=1}^n u_j v_j \right\} d\tau(\vec{v}), \tag{66}$$

where $\vec{u} = (u_1, \ldots, u_n)$ and $\vec{v} = (v_1, \ldots, v_n)$ are in $\mathbb{R}^n$.

Let $\mathscr{H} = \{h_1, \ldots, h_n\}$ be an orthonormal set in H. Define the functional $F : B \to \mathbb{C}$ by

$$F(x) = \widehat{\tau}\left(X_{\mathscr{H}}(x)\right) = \widehat{\tau}\left((h_1, x)^{\sim}, \ldots, (h_n, x)^{\sim}\right) \tag{67}$$

for s-a.e. $x \in B$, where $\hat{\tau}$ is the Fourier transform of τ in $\mathscr{M}(\mathbb{R}^n)$. Then F is a bounded cylinder-type functional because $|\hat{\tau}(\vec{u})| \le \|\tau\| < +\infty$.

Let $\mathfrak{T}_{\mathscr{H}}$ be the set of all functionals F on B having the form (67). Note that $F \in \mathfrak{T}_{\mathscr{H}}$ implies that F is scale-invariant measurable on B. Throughout this section, we fix the orthogonal set $\mathscr{H}$.

We now state the existence theorem for the analytic Feynman integral of the functionals in $\mathfrak{T}_{\mathscr{H}}$.

Theorem 16. *Let $F \in \mathfrak{T}_{\mathscr{H}}$ be given by* (67). *Then, for all $g \in B^*_{\mathscr{H}}$ and all nonzero real numbers q, the analytic Feynman integral $I_g^{anf_q}[F]$ of F exists and is given by the formula*

$$I_g^{anf_q}[F] = \int_{\mathbb{R}^n} \exp\left\{-\frac{i}{2q}\sum_{j=1}^{n}\left|h_j \odot g\right|^2 v_j^2\right\} d\tau(\vec{v}). \tag{68}$$

Proof. By (67), (66), the Fubini theorem, (23), and (24), we see that, for all $\lambda > 0$,

$$\begin{aligned}
&J_F(g;\lambda)\\
&= \int_B F\left(\lambda^{-1/2}A_g x\right) d\nu(x)\\
&= \int_{\mathbb{R}^n}\left[\int_B \exp\left\{i\lambda^{-1/2}\sum_{j=1}^{n}\left(h_j \odot g, x\right)^{\sim} v_j\right\} d\nu(x)\right] d\tau(\vec{v})\\
&= \left(\prod_{j=1}^{n}\frac{\lambda}{2\pi\left|h_j \odot g\right|^2}\right)^{1/2}\\
&\quad\times \int_{\mathbb{R}^n}\left[\int_{\mathbb{R}^n}\exp\left\{i\sum_{j=1}^{n}u_j v_j - \frac{\lambda}{2}\sum_{j=1}^{n}\frac{u_j^2}{\left|h_j \odot g\right|^2}\right\} d\vec{u}\right] d\tau(\vec{v})\\
&= \left(\prod_{j=1}^{n}\frac{\lambda}{2\pi\left|h_j \odot g\right|^2}\right)^{1/2}\\
&\quad\times \int_{\mathbb{R}^n}\left[\prod_{j=1}^{n}\int_{\mathbb{R}}\exp\left\{iu_j v_j - \frac{\lambda}{2}\frac{u_j^2}{\left|h_j \odot g\right|^2}\right\} du_j\right] d\tau(\vec{v})\\
&= \int_{\mathbb{R}^n}\exp\left\{-\frac{1}{2\lambda}\sum_{j=1}^{n}\left|h_j \odot g\right|^2 v_j^2\right\} d\tau(\vec{v}).
\end{aligned} \tag{69}$$

Now, let $J_F^*(g;\lambda) = \int_{\mathbb{R}^n}\exp\{-(1/2\lambda)\sum_{j=1}^{n}|h_j \odot g|^2 v_j^2\} d\tau(\vec{v})$ for $\lambda \in \mathbb{C}_+$. Then, $J_F^*(g;\lambda) = J_F(g;\lambda)$ for all $\lambda > 0$ and $|J_F^*(g;\lambda)| \le \int_{\mathbb{R}^n} d|\tau|(\vec{v}) \le \|\tau\| < \infty$ for all $\lambda \in \mathbb{C}_+$. Thus, applying the dominated convergence theorem, we see that $J_F^*(g;\lambda)$ is continuous on $\mathbb{C}_+$. Also, because $k(\lambda) \equiv \exp\{-(1/2\lambda)\sum_{j=1}^{n}|h_j \odot g|^2 v_j^2\}$ is analytic on $\mathbb{C}_+$, applying the Fubini theorem, we have

$$\int_\Delta J_F^*(g;\lambda)\, d\lambda = \int_{\mathbb{R}^n}\int_\Delta k(\lambda)\, d\lambda\, d\nu(\vec{v}) = 0 \tag{70}$$

for all rectifiable simple closed curve Δ lying in $\mathbb{C}_+$. Thus, by the Morera theorem, $J_F^*(g;\lambda)$ is analytic on $\mathbb{C}_+$. Therefore, the analytic Wiener integral $I_g^{\mathrm{anw}_\lambda}[F] = J_F^*(g;\lambda)$ exists. Finally, applying the dominated convergence theorem, we know that $I_g^{\mathrm{anf}_q}[F] = \lim_{\lambda\to -iq} I_g^{\mathrm{anw}_\lambda}[F]$ is given by the right side of (68). □

Next, we establish the existence of the admix-FFT for functionals in $\mathfrak{T}_{\mathscr{H}}$.

Theorem 17. *Let $F \in \mathfrak{T}_{\mathscr{H}}$ be given by* (67). *Then, for all $p \in [1,2]$ and all $g \in B^*$, the analytic L_p $g^{\odot}$-admix-FFT, $T_{q,g}^{(p)}(F)$, exists for all nonzero real numbers q, belongs to $\mathfrak{T}_{\mathscr{H}}$, and is given by the formula*

$$T_{q,g}^{(p)}(F)(y) = \int_{\mathbb{R}^n}\exp\left\{i\sum_{j=1}^{n}\left(h_j, y\right)^{\sim} v_j\right\} d\tau_g^q(\vec{v}) \tag{71}$$

for s-a.e. $y \in B$, where τ_g^q is the complex measure on $\mathbb{R}^n$ given by

$$\tau_g^q(E) = \int_E \exp\left\{-\frac{i}{2q}\sum_{j=1}^{n}\left|h_j \odot g\right|^2 v_j^2\right\} d\tau(\vec{v}) \tag{72}$$

for $E \in \mathscr{B}(\mathbb{R}^n)$.

Proof. Proceeding as in the proof of Theorem 16, we see that, for all $\lambda \in \mathbb{C}_+$, all $g \in B^*_{\mathscr{H}}$, and for s-a.e. $y \in B$,

$$\begin{aligned}
&T_{\lambda,g}(F)(y)\\
&\quad= \int_B F\left(y + \lambda^{-1/2}A_g x\right) d\nu(x)\\
&\quad= \int_{\mathbb{R}^n}\exp\left\{i\sum_{j=1}^{n}\left(h_j, y\right)^{\sim} v_j - \frac{1}{2\lambda}\sum_{j=1}^{n}\left|h_j \odot g\right|^2 v_j^2\right\} d\tau(\vec{v}),
\end{aligned} \tag{73}$$

is an analytic function of λ on $\mathbb{C}_+$, and that for any $q \in \mathbb{R} - \{0\}$ and s-a.e. $y \in B$,

$$\begin{aligned}
&T_{q,g}^{(1)}(F)(y)\\
&\quad= \lim_{\lambda\to -iq} T_{\lambda,g}(F)(y)\\
&\quad= \int_{\mathbb{R}^n}\exp\left\{i\sum_{j=1}^{n}\left(h_j, y\right)^{\sim} v_j - \frac{i}{2q}\sum_{j=1}^{n}\left|h_j \odot g\right|^2 v_j^2\right\} d\tau(\vec{v}).
\end{aligned} \tag{74}$$

Clearly, the set function τ_g^q given by (72) is a complex measure on $\mathscr{B}(\mathbb{R}^n)$, and so the right side of (74) can be rewritten as the right side of (71).

Next, we note that

$$\left|T_{\lambda,g}(F)(y)\right| \le \int_{\mathbb{R}^n} d|\tau|(\vec{v}) = \|\tau\| \tag{75}$$

for all $\lambda \in \mathbb{C}_+$ and

$$\left|T_{q,g}^{(1)}(F)(y)\right| \leq \int_{\mathbb{R}^n} d|\tau|(\vec{v}) = \|\tau\|. \tag{76}$$

Using these, we see that, for all $p \in (1,2]$, all $\rho > 0$, and all $\lambda \in \mathbb{C}_+$,

$$\left|T_{\lambda,g}(F)(\rho y) - T_{q,g}^{(1)}(F)(\rho y)\right|^{p'} \leq (2\|\tau\|)^{p'}, \tag{77}$$

and so, by the dominated convergence theorem, we see that, for any nonzero real q,

$$\lim_{\lambda \to -iq} \int_B \left|T_{\lambda,g}(F)(\rho y) - T_{q,g}^{(1)}(F)(\rho y)\right|^{p'} d\nu(y) = 0. \tag{78}$$

Hence, $T_{q,g}^{(p)}(F)(y)$ exists and is given by the right side of (74) for all desired values of p and q and all $g \in B_{\mathscr{H}}^*$. Thus, the theorem is proved. □

Theorem 18. *Let $F \in \mathfrak{T}_{\mathscr{H}}$ be given by (67). Then, for all $p \in [1,2]$, all $g \in B_{\mathscr{H}}^*$, and all nonzero real q,*

$$T_{-q,g}^{(p)}\left(T_{q,g}^{(p)}(F)\right) \approx F. \tag{79}$$

As such, the admix-FFT, $T_{q,g}^{(p)}$, has the inverse transform $\{T_{q,g}^{(p)}\}^{-1} = T_{-q,g}^{(p)}$.

Theorem 19. *Let F and G be elements of $\mathfrak{T}_{\mathscr{H}}$ with corresponding finite Borel measures τ and μ in $\mathscr{M}(H)$. Then, for all $k \in B_{\mathscr{H}}^*$, the CP, $(F * G)_{q,k}$, exists for all nonzero real numbers q, belongs to $\mathfrak{T}_{\mathscr{H}}$, and is given by the formula*

$$(F * G)_{q,k}(y) = \iint_{\mathbb{R}^n} \exp\left\{i\sum_{j=1}^n \left(h_j, y\right)^{\sim} r_j\right\} d\left(\varpi_k^q \circ \varphi^{-1}\right)(h) \tag{80}$$

for s-a.e. $y \in B$, where $\varphi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function defined by

$$\varphi(\vec{v}, \vec{w}) = \frac{(\vec{v} + \vec{w})}{\sqrt{2}}, \tag{81}$$

and ϖ_k^q is a complex measure on $\mathscr{B}(\mathbb{R}^n)$ given by

$$\varpi_k^q(E) = \int_E \exp\left\{-\frac{i}{4q}\sum_{j=1}^n \left|h_j \odot k\right|^2 \left(v_j - w_j\right)^2\right\} d\tau(\vec{v})\, d\mu(\vec{w}) \tag{82}$$

for $E \in \mathscr{B}(\mathbb{R}^n)$.

Proof. Using (64), the Fubini theorem, (23), and (24), we have that, for all $\lambda > 0$ and s-a.e. $y \in B$,

$$\begin{aligned}
&(F * G)_{\lambda,k}(y) \\
&= \int_B F\left(\frac{y + A_k x}{\sqrt{2}}\right) G\left(\frac{y - A_k x}{\sqrt{2}}\right) d\nu(x) \\
&= \iint_{\mathbb{R}^n} \exp\left\{\frac{i}{\sqrt{2}}\sum_{j=1}^n \left(h_j, y\right)^{\sim}\left(v_j + w_j\right)\right\} \\
&\quad \times \left[\int_B \exp\left\{\frac{i}{\sqrt{2\lambda}}\sum_{j=1}^n \left(h_j \odot k, x\right)^{\sim} \times \left(v_j - w_j\right)\right\} d\nu(x)\right] d\tau(\vec{v})\, d\mu(\vec{w}) \\
&= \iint_{\mathbb{R}^n} \exp\left\{\frac{i}{\sqrt{2}}\sum_{j=1}^n \left(h_j, y\right)^{\sim}\left(v_j + w_j\right) - \frac{1}{4\lambda}\sum_{j=1}^n \left|h_j \odot k\right|^2 \left(v_j - w_j\right)^2\right\} d\tau(\vec{v})\, d\mu(\vec{w}).
\end{aligned} \tag{83}$$

Using the same argument as in the proof of Theorem 17, we can show that the last expression in the previous equation is an analytic function of λ on $\mathbb{C}_+$ and is a bounded continuous function of λ on $\widetilde{\mathbb{C}}_+$ because τ and μ are finite Borel measures. Hence, $(F * G)_{q,k}$ exists and is given by

$$\begin{aligned}
&(F * G)_{q,k}(y) \\
&= \iint_{\mathbb{R}^n} \exp\left\{\frac{i}{\sqrt{2}}\sum_{j=1}^n \left(h_j, y\right)^{\sim}\left(v_j + w_j\right) - \frac{i}{4q}\sum_{j=1}^n \left|h_j \odot k\right|^2 \left(v_j - w_j\right)^2\right\} d\tau(\vec{v})\, d\mu(\vec{w})
\end{aligned} \tag{84}$$

for all $q \in \mathbb{R} - \{0\}$ and s-a.e. $y \in B$.

Consider the set function ϖ_k^q and the continuous function φ given by (82) and (81), respectively. Clearly, the set function ϖ_k^q is a complex measure on $\mathscr{B}(\mathbb{R}^n)$. Hence, $\varpi_k^q \circ \varphi$ is an element of $\mathscr{M}(\mathbb{R}^n)$, and so the right side of (84) can be rewritten as the right side of (80). Thus, the theorem is proved. □

Lemma 20. *Let $\mathscr{H} = \{h_1, \ldots, h_n\}$ be any orthogonal set in H. For every $g \in B_{\mathscr{H}}^*$, every $\vec{v} = (v_1, \ldots, v_n)$ and $\vec{w} = (w_1, \ldots, w_n)$ in $\mathbb{R}^n$, let $Y_{g,\vec{v}}^{\mathscr{H}}, Z_{g,\vec{w}}^{\mathscr{H}} : B^2 \to \mathbb{R}$ be given by*

$$\begin{aligned}
Y_{g,\vec{v}}^{\mathscr{H}}(x_1, x_2) &= \sum_{j=1}^n \frac{\left(h_j, A_g x_1 + A_g x_2\right)^{\sim}}{\sqrt{2}} v_j, \\
Z_{g,\vec{w}}^{\mathscr{H}}(x_1, x_2) &= \sum_{j=1}^n \frac{\left(h_j, A_g x_1 - A_g x_2\right)^{\sim}}{\sqrt{2}} w_j,
\end{aligned} \tag{85}$$

respectively. As a result, $Y^{\mathscr{H}}_{g,\vec{v}}$ and $Z^{\mathscr{H}}_{g,\vec{w}}$ are independent random variables.

Proof. Since the random variables $Y^{\mathscr{H}}_{g,\vec{v}}$ and $Z^{\mathscr{H}}_{g,\vec{w}}$ are Gaussian with mean zero, it suffices to show that

$$\int_{B^2} Y^{\mathscr{H}}_{g,\vec{v}}(x_1,x_2)\, Z^{\mathscr{H}}_{g,\vec{w}}(x_1,x_2)\, d(\nu\times\nu)(x_1,x_2) = 0. \quad (86)$$

We know that $\{h_j \odot g, \ldots, h_n \odot g\}$ is an orthogonal set in H; thus, $\{(h_j \odot g, x)^{\sim}, \ldots, (h_n \odot g, x)^{\sim}\}$ is a set of independent Gaussian random variables with mean zero on B. However, using the Fubini theorem, we obtain

$$\begin{aligned}
&\int_{B^2} Y^{\mathscr{H}}_{g,\vec{v}}(x_1,x_2)\, Z^{\mathscr{H}}_{g,\vec{w}}(x_1,x_2)\, d(\nu\times\nu)(x_1,x_2)\\
&= \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}\Big\{v_j w_k \iint_B \big(h_j\odot g, x_1\big)^{\sim}\\
&\qquad \times (h_k\odot g, x_1)^{\sim} d\nu(x_1)\, d\nu(x_2)\\
&\qquad + v_j w_k \int_B \big(h_j\odot g, x_1\big)^{\sim} d\nu(x_1)\\
&\qquad \times \int_B (h_k\odot g, x_2)^{\sim} d\nu(x_2)\\
&\qquad + v_j w_k \int_B \big(h_j\odot g, x_2\big)^{\sim} d\nu(x_2)\\
&\qquad \times \int_B (h_k\odot g, x_1)^{\sim} d\nu(x_1)\\
&\qquad - v_j w_k \iint_B \big(h_j\odot g, x_2\big)^{\sim}\\
&\qquad\qquad \times (h\odot g, x_2)^{\sim} d\nu(x_1)\, d\nu(x_2)\Big\}\\
&= \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}\Big\{v_j w_k \int_B \big(h_j\odot g, x_1\big)^{\sim}(h_k\odot g, x_1)^{\sim} d\nu(x_1)\\
&\qquad - v_j w_k \int_B \big(h_j\odot g, x_2\big)^{\sim}\\
&\qquad\qquad \times (h_k\odot g, x_2)^{\sim} d\nu(x_2)\Big\}\\
&= \frac{1}{2}\sum_{j=1}^{n}\Big\{v_j w_j \int_B \big[\big(h_j\odot g, x_1\big)^{\sim}\big]^2 d\nu(x_1)\\
&\qquad - v_j w_j \int_B \big[\big(h_j\odot g, x_2\big)^{\sim}\big]\, d\nu(x_2)\Big\}\\
&\quad + \frac{1}{2}\sum_{j\neq k}^{n}\Big\{v_j w_k \int_B \big(h_j\odot g, x_1\big)^{\sim}(h_k\odot g, x_1)^{\sim} d\nu(x_1)\\
&\qquad - v_j w_k \int_B \big(h_j\odot g, x_2\big)^{\sim}(h\odot g, x_2)^{\sim} d\nu(x_2)\Big\}\\
&= 0,
\end{aligned} \quad (87)$$

which concludes the proof of Lemma 20. □

Remark 21. For each $\vec{v}\in\mathbb{R}^n$, let

$$\psi(\vec{v}; x) = \exp\left\{i\sum_{j=1}^{n}\big(h_j, x\big)^{\sim} v_j\right\}, \quad x\in B. \quad (88)$$

As such, (67) is rewritten as

$$F(x) = \int_{\mathbb{R}^n} \psi(\vec{v}; x)\, d\tau(\vec{v}), \quad (89)$$

and the functional $\psi(\vec{v};\cdot)$ is an element of $\mathfrak{G}^{(1)}_{\mathscr{H}}$ for each $\vec{v}$.

Applying the same method as used in the proof of Theorem 3.5 in [4], we have the following theorem. For the proof of Theorem 22, we can apply Lemma 20 and Theorem 11 to the functional ψ given by (88).

Theorem 22. *Let F, G, τ, and μ be as in Theorem 19, and let g be an element of $B^*_{\mathscr{H}}$. Then, for all $p\in[1,2]$ and all nonzero real q,*

$$T^{(p)}_{q,g}\big((F*G)_{q,g}\big)(y) = T^{(p)}_{q,g}(F)\left(\frac{y}{\sqrt{2}}\right) T^{(p)}_{q,g}(G)\left(\frac{y}{\sqrt{2}}\right) \quad (90)$$

for s-a.e. $y\in B$.

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