Research Article

Solution and Stability of a General Mixed Type Cubic and Quartic Functional Equation

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We consider the following mixed type cubic and quartic functional equation $\lambda[f(x + \lambda y) + f(x - \lambda y)] = \lambda^3[f(x + y) + f(x - y)] - 2\lambda^3(\lambda + 1)f(y) - 2\lambda(\lambda^2 - 1)f(x) + 2(\lambda + 1)f(\lambda y)$, where λ is a fixed integer. We establish the general solution of the functional equation when the integer $\lambda \neq 0, \pm 1$, and then, by using the fixed point alternative, we investigate the generalized Hyers-Ulam-Rassias stability for this functional equation when the integer $\lambda \geq 2$.

1. Introduction

In 1940, Ulam [1] asked the fundamental question for the stability for the group homomorphisms.

Let $(G_1, *)$ be a group, and let $(G_2, \diamond, d)$ be a metric group with the metric $d(\cdot, \cdot)$. Given $\varepsilon > 0$, does there exist a $\delta(\varepsilon) > 0$ such that if a mapping $h: G_1 \rightarrow G_2$ satisfies the inequality

$$d(h(x * y), h(x) \diamond h(y)) < \delta \quad (1)$$

for all $x, y \in G_1$, then there is a homomorphism $H: G_1 \rightarrow G_2$ with

$$d(h(x), H(x)) < \varepsilon \quad (2)$$

for all $x \in G_1$?

In other words, under what conditions, does there exist a homomorphism near an approximately homomorphism? In the next year, Hyers [2] gave the first affirmative answer to the question of Ulam for Cauchy equation in the Banach spaces. Then, Rassias [3] generalized Hyers' result by considering an unbounded Cauchy difference, and this stability phenomenon is known as generalized Hyers-Ulam-Rassias stability or Hyers-Ulam-Rassias stability. During the last three decades, the stability problem for several functional

equations has been extensively investigated by many mathematicians; see, for example, [4–9] and the references therein. We also refer the readers to the books [10–13].

In [14], Jun and Kim introduced the following functional equation

$$f(2x + y) + f(2x - y) = 2f(x + y) + 2f(x - y) + 12f(x). \quad (3)$$

It is easy to see that the function $f(x) = x^3$ satisfies the functional equation (3). Thus, it is natural that (3) is called a cubic functional equation and every solution of the cubic functional equation is said to be a cubic function. In [14], Jun and Kim established the general solution and the generalized Hyers-Ulam-Rassias stability for (3). They proved that a function $f: X \rightarrow Y$ between real vector spaces is a solution of the functional equation (3) if and only if there exists a function $G: X \times X \times X \rightarrow Y$ such that $f(x) = G(x, x, x)$ for all $x \in X$ and G is symmetric for each fixed one variable and is additive for fixed two variables.

In [15], Lee et al. considered the following quartic functional equation

$$f(2x + y) + f(2x - y) = 4[f(x + y) + f(x - y)] + 24f(x) - 6f(y). \quad (4)$$

Since the function $f(x) = x^4$ satisfies the functional equation (4), the functional equation (4) is called a quartic functional equation and every solution of the quartic functional equation is said to be a quartic function. In [15], the authors solved the functional equation (4) and proved the stability for it. Actually, they obtained that a function $f : X \rightarrow Y$ between real vector spaces satisfies the functional equation (4) if and only if there exists a symmetric biquadratic function $B : X \times X \rightarrow Y$ such that $f(x) = B(x, x)$ for all $x \in X$. A function $f : X \rightarrow Y$ between real vector spaces is said to be quadratic if it satisfies the following functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \quad (5)$$

for all $x, y \in X$, and a function $B : X \times X \rightarrow Y$ is said to be bi-quadratic if B is quadratic for each fixed one variable (see [8]). The following mixed type cubic and quartic functional equation was introduced by Eshaghi Gordji et al. [16]

$$\begin{aligned} f(x+2y) + f(x-2y) \\ = 4[f(x+y) + f(x-y)] \\ - 24f(y) - 6f(x) + 3f(2y). \end{aligned} \quad (6)$$

It can be verified that the function $f(x) = x^3 + x^4$ satisfies the functional equation (6). In [16], the authors obtained the general solution and the generalized Hyers-Ulam-Rassias stability of (6) in quasi-Banach space. The literature on the stability of the mixed type functional equations is very rich; see [17–22].

In the present paper, we extend (6) and consider the following functional equation

$$\begin{aligned} \lambda[f(x+\lambda y) + f(x-\lambda y)] \\ = \lambda^3[f(x+y) + f(x-y)] - 2\lambda^3(\lambda+1)f(y) \\ - 2\lambda(\lambda^2-1)f(x) + 2(\lambda+1)f(\lambda y), \end{aligned} \quad (7)$$

where λ is a fixed integer. One can see that the functional equation (6) is a special case of (7) when we take the integer $\lambda = 2$.

In 2003, Radu [23] noticed that the fixed point theorem, which was established by Diaz and Margolis [24], plays an important part in solving the stability problem of functional equations. Subsequently, this method has been successfully used by many mathematicians to investigate the stability of several functional equations; see, for example, [21, 25, 26] and the references therein.

In this paper, we first establish the general solution of functional equation (7) when the integer $\lambda \neq 0, \pm 1$ and f is a mapping between vector spaces. Then, by using the fixed point method, we prove the generalized Hyers-Ulam-Rassias stability of the functional equation (7) when the integer $\lambda \geq 2$ and f is a mapping from the normed space to the Banach space.

2. Solution of the Functional Equation (7)

Recall from [14, 15] that every solution of the cubic functional equation (3) and the quartic functional equation (4) is said to

be a cubic function and a quartic function, respectively. In this section, we investigate the general solution of the mixed type cubic and quartic functional equation (7). Throughout this section, let X and Y be two real vector spaces, and we always assume that the integer λ in the functional equation (7) is different from 0, -1 , and 1. Before proving our main theorem, we first give the following two lemmas.

Lemma 1. *If an odd mapping $f : X \rightarrow Y$ satisfies (7), then f is cubic.*

Proof. Note that, in view of the oddness of f , we have $f(0) = 0$ and $f(-x) = -f(x)$ for all $x \in X$. Letting $x = 0$ in (7), we get

$$f(\lambda y) = \lambda^3 f(y) \quad (8)$$

for all $y \in X$. Applying (8) to (7), we obtain

$$\begin{aligned} f(x+\lambda y) + f(x-\lambda y) \\ = \lambda^2[f(x+y) + f(x-y)] - 2(\lambda^2-1)f(x) \end{aligned} \quad (9)$$

for all $x, y \in X$. Replacing y by $x+y$ in (9), we get

$$\begin{aligned} f((\lambda+1)x+\lambda y) - f((\lambda-1)x+\lambda y) \\ = \lambda^2[f(2x+y) - f(y)] - 2(\lambda^2-1)f(x) \end{aligned} \quad (10)$$

for all $x, y \in X$. Now, if we replace x by $\lambda x + y$ in (9) and use (8), we see that

$$\begin{aligned} f(\lambda x + (\lambda+1)y) + f(\lambda x - (\lambda-1)y) \\ = \lambda^2[f(\lambda x + 2y) + \lambda^3 f(x)] - 2(\lambda^2-1)f(\lambda x + y) \end{aligned} \quad (11)$$

for all $x, y \in X$. Interchanging x with y in (11) and using the oddness of f , we get the relation

$$\begin{aligned} f((\lambda+1)x+\lambda y) - f((\lambda-1)x-\lambda y) \\ = \lambda^2[f(2x+\lambda y) + \lambda^3 f(y)] - 2(\lambda^2-1)f(x+\lambda y) \end{aligned} \quad (12)$$

for all $x, y \in X$. Then, subtracting (12) from (10), one has

$$\begin{aligned} f((\lambda-1)x-\lambda y) - f((\lambda-1)x+\lambda y) \\ = \lambda^2[f(2x+y) - f(2x+\lambda y)] \\ - (\lambda^2+\lambda^5)f(y) + 2(\lambda^2-1)(f(x+\lambda y) - f(x)) \end{aligned} \quad (13)$$

for all $x, y \in X$. Now, replacing y by $-y$ in (13) and noting that f is odd, we have

$$\begin{aligned} f((\lambda-1)x+\lambda y) - f((\lambda-1)x-\lambda y) \\ = \lambda^2[f(2x-y) - f(2x-\lambda y)] + (\lambda^2+\lambda^5)f(y) \\ + 2(\lambda^2-1)(f(x-\lambda y) - f(x)) \end{aligned} \quad (14)$$

for all $x, y \in X$. Adding (13) to (14) gives

$$\begin{aligned} & \lambda^2 [f(2x + y) + f(2x - y)] \\ & + 2(\lambda^2 - 1) [f(x + \lambda y) + f(x - \lambda y)] \\ & = \lambda^2 [f(2x + \lambda y) + f(2x - \lambda y)] + 4(\lambda^2 - 1) f(x) \end{aligned} \quad (15)$$

for all $x, y \in X$. Replacing x by $2x$ in (9) and using (8), we get

$$\begin{aligned} & f(2x + \lambda y) + f(2x - \lambda y) \\ & = \lambda^2 [f(2x + y) + f(2x - y)] - 16(\lambda^2 - 1) f(x). \end{aligned} \quad (16)$$

Applying (9) and (16) to (15) yields that

$$\begin{aligned} & f(2x + y) + f(2x - y) \\ & = 2[f(x + y) + f(x - y)] + 12f(x) \end{aligned} \quad (17)$$

for all $x, y \in X$. Thus, the mapping $f: X \rightarrow Y$ is cubic. This completes the proof. $\square$

Lemma 2. *If an even mapping $f: X \rightarrow Y$ satisfies (7) for all $x, y \in X$, then f is quartic.*

Proof. In view of the evenness of f , we have $f(-x) = f(x)$ for all $x \in X$. Putting $x = y = 0$ in (7), we get $f(0) = 0$. Then, let $x = 0$ in (7), we obtain

$$f(\lambda y) = \lambda^4 f(y) \quad (18)$$

for all $y \in X$. Combining (7) and (18) implies the following equation

$$\begin{aligned} & f(x + \lambda y) + f(x - \lambda y) \\ & = \lambda^2 [f(x + y) + f(x - y)] \\ & - 2(\lambda^2 - 1) f(x) + 2\lambda^2 (\lambda^2 - 1) f(y) \end{aligned} \quad (19)$$

for all $x, y \in X$. Replacing y by $x + y$ in (19) and note that f is even, we get

$$\begin{aligned} & f((\lambda + 1)x + \lambda y) + f((\lambda - 1)x + \lambda y) \\ & = \lambda^2 [f(2x + y) + f(y)] - 2(\lambda^2 - 1) f(x) \\ & + 2\lambda^2 (\lambda^2 - 1) f(x + y) \end{aligned} \quad (20)$$

for all $x, y \in X$. Replacing x by $\lambda x + y$ in (19) and using (18), we get

$$\begin{aligned} & f(\lambda x + (\lambda + 1)y) + f(\lambda x - (\lambda - 1)y) \\ & = \lambda^2 [f(\lambda x + 2y) + \lambda^4 f(x)] \\ & - 2(\lambda^2 - 1) f(\lambda x + y) + 2\lambda^2 (\lambda^2 - 1) f(y) \end{aligned} \quad (21)$$

for all $x, y \in X$. Interchanging the roles of x and y in (21), we obtain

$$\begin{aligned} & f((\lambda + 1)x + \lambda y) + f((\lambda - 1)x - \lambda y) \\ & = \lambda^2 [f(2x + \lambda y) + \lambda^4 f(y)] \\ & - 2(\lambda^2 - 1) f(x + \lambda y) + 2\lambda^2 (\lambda^2 - 1) f(x) \end{aligned} \quad (22)$$

for all $x, y \in X$. If we subtract (22) from (20), we obtain

$$\begin{aligned} & f((\lambda - 1)x + \lambda y) - f((\lambda - 1)x - \lambda y) \\ & = \lambda^2 [f(2x + y) - f(2x + \lambda y)] \\ & + 2(\lambda^2 - 1) f(x + \lambda y) + (\lambda^2 - \lambda^6) f(y) \\ & + 2\lambda^2 (\lambda^2 - 1) f(x + y) - 2(\lambda^2 + 1)(\lambda^2 - 1) f(x) \end{aligned} \quad (23)$$

for all $x, y \in X$. Replacing y by $-y$ in (23), we get

$$\begin{aligned} & f((\lambda - 1)x - \lambda y) - f((\lambda - 1)x + \lambda y) \\ & = \lambda^2 [f(2x - y) - f(2x - \lambda y)] \\ & + 2(\lambda^2 - 1) f(x - \lambda y) + (\lambda^2 - \lambda^6) f(y) \\ & + 2\lambda^2 (\lambda^2 - 1) f(x - y) - 2(\lambda^2 + 1)(\lambda^2 - 1) f(x) \end{aligned} \quad (24)$$

for all $x, y \in X$. If we add (23) to (24), we have

$$\begin{aligned} & \lambda^2 [f(2x + y) + f(2x - y)] \\ & - \lambda^2 [f(2x + \lambda y) + f(2x - \lambda y)] \\ & + 2(\lambda^2 - 1) [f(x + \lambda y) + f(x - \lambda y)] \\ & + 2\lambda^2 (\lambda^2 - 1) [f(x + y) + f(x - y)] \\ & = 4(\lambda^2 + 1)(\lambda^2 - 1) f(x) + 2(\lambda^6 - \lambda^2) f(y) \end{aligned} \quad (25)$$

for all $x, y \in X$. Replacing x by $2x$ in (19) and using (18), we get

$$\begin{aligned} & f(2x + \lambda y) + f(2x - \lambda y) \\ & = \lambda^2 [f(2x + y) + f(2x - y)] \\ & - 32(\lambda^2 - 1) f(x) + 2\lambda^2 (\lambda^2 - 1) f(y) \end{aligned} \quad (26)$$

for all $x, y \in X$. Applying (19) and (26) to (25), we obtain that

$$\begin{aligned} & f(2x + y) + f(2x - y) \\ & = 4[f(x + y) + f(x - y)] + 24f(x) - 6f(y) \end{aligned} \quad (27)$$

for all $x, y \in X$. Therefore, the mapping $f: X \rightarrow Y$ is quartic and the proof is complete. $\square$

Now, we are ready to find out the general solution of (7).

Theorem 3. A mapping $f: X \rightarrow Y$ satisfies (7) for all $x, y \in X$ if and only if there exist a symmetric multiadditive mapping $G: X \times X \times X \rightarrow Y$ and a symmetric bi-quadratic mapping $B: X \times X \rightarrow Y$ such that $f(x) = G(x, x, x) + B(x, x)$ for all $x \in X$.

Proof. First, we assume that there exist a symmetric multi-additive mapping $G: X \times X \times X \rightarrow Y$ and a symmetric bi-quadratic mapping $B: X \times X \rightarrow Y$ such that $f(x) = G(x, x, x) + B(x, x)$ for all $x \in X$. We need to show that the function f satisfies (7). By a simple computation, one can obtain that the function $x \mapsto G(x, x, x)$ satisfies (7). Thus, to show that the function f satisfies (7), we only need to show that the function $x \mapsto B(x, x)$ also satisfies (7); namely,

$$\begin{aligned} & \lambda [B(x + \lambda y, x + \lambda y) + B(x - \lambda y, x - \lambda y)] \\ &= \lambda^3 [B(x + y, x + y) + B(x - y, x - y)] \\ & \quad - 2\lambda^3 (\lambda + 1) B(y, y) \\ & \quad - 2\lambda (\lambda^2 - 1) B(x, x) + 2(\lambda + 1) B(\lambda y, \lambda y) \end{aligned} \quad (28)$$

for all $x, y \in X$. Since $B: X \times X \rightarrow Y$ is a bi-quadratic mapping, it can be verified that

$$B(\lambda x, \lambda y) = \lambda^4 B(x, y) \quad (29)$$

for all integer λ and all $x, y \in X$. Then, (28) becomes

$$\begin{aligned} & B(x + \lambda y, x + \lambda y) + B(x - \lambda y, x - \lambda y) \\ &= \lambda^2 [B(x + y, x + y) + B(x - y, x - y)] \\ & \quad - 2(\lambda^2 - 1) B(x, x) + 2\lambda^2 (\lambda^2 - 1) B(y, y). \end{aligned} \quad (30)$$

To establish (28), it suffices to show (30). Note that if (30) holds for some integer λ , then so does $-\lambda$. Thus, in the following, we will show that (30) holds for all positive integers λ with $\lambda \geq 2$ and all $x, y \in X$. To do this, we use induction on λ . Fix any $x, y \in X$. In the case when $\lambda = 2$, we have

$$\begin{aligned} & B(x + 2y, x + 2y) + B(x - 2y, x - 2y) \\ &= [B(x + y + y, x + 2y) + B(x + y - y, x + 2y)] \\ & \quad + [B(x - y + y, x - 2y) + B(x - y - y, x - 2y)] \\ & \quad - [B(x, x + 2y) + B(x, x - 2y)] \\ &= 2 [B(x + y, x + 2y) + B(y, x + 2y)] \\ & \quad + 2 [B(x - y, x - 2y) + B(y, x - 2y)] \\ & \quad - [2B(x, x) + 2B(x, 2y)] \\ &= 2 [B(x + y, x + y + y) + B(x + y, x + y - y)] \\ & \quad + 2 [B(x - y, x - y - y) + B(x - y, x - y + y)] \\ & \quad - 2 [B(x + y, x) + B(x - y, x)] \\ & \quad + 2 [B(y, x + 2y) + B(y, x - 2y)] \\ & \quad - [2B(x, x) + 8B(x, y)] \end{aligned}$$

$$\begin{aligned} &= 4 [B(x + y, x + y) + B(x + y, y)] \\ & \quad + 4 [B(x - y, x - y) + B(x - y, y)] \\ & \quad - 4 [B(x, x) + B(y, x)] + 4 [B(y, x) + B(y, 2y)] \\ & \quad - [2B(x, x) + 8B(x, y)] \\ &= 4 [B(x + y, x + y) + B(x - y, x - y)] \\ & \quad + 4 [B(x + y, y) + B(x - y, y)] \\ & \quad - 6B(x, x) - 8B(x, y) + 16B(y, y) \\ &= 4 [B(x + y, x + y) + B(x - y, x - y)] \\ & \quad + 8 [B(x, y) + B(y, y)] \\ & \quad - 6B(x, x) - 8B(x, y) + 16B(y, y) \\ &= 4 [B(x + y, x + y) + B(x - y, x - y)] \\ & \quad - 6B(x, x) + 24B(y, y). \end{aligned} \quad (31)$$

So (30) is true for $\lambda = 2$. Here, we have used the bi-quadratic property of the function B and (29). Now, assume that (30) is true for all positive integers that are less than or equal to some integer $\lambda (> 2)$. Then,

$$\begin{aligned} & B(x + (\lambda + 1)y, x + (\lambda + 1)y) \\ & \quad + B(x - (\lambda + 1)y, x - (\lambda + 1)y) \\ &= [B(x + y + \lambda y, x + y + \lambda y) \\ & \quad + B(x + y - \lambda y, x + y - \lambda y)] \\ & \quad + [B(x - y + \lambda y, x - y + \lambda y) \\ & \quad + B(x - y - \lambda y, x - y - \lambda y)] \\ & \quad - [B(x + (\lambda - 1)y, x + (\lambda - 1)y) \\ & \quad + B(x - (\lambda - 1)y, x - (\lambda - 1)y)] \\ &= \lambda^2 [B(x + 2y, x + 2y) + B(x, x)] \\ & \quad - 2(\lambda^2 - 1) B(x + y, x + y) + 2\lambda^2 (\lambda^2 - 1) B(y, y) \\ & \quad + \lambda^2 [B(x, x) + B(x - 2y, x - 2y)] \\ & \quad - 2(n^2 - 1) B(x - y, x - y) + 2\lambda^2 (\lambda^2 - 1) B(y, y) \\ & \quad - \{(\lambda - 1)^2 [B(x + y, x + y) + B(x - y, x - y)] \\ & \quad - 2((\lambda - 1)^2 - 1) B(x, x) \\ & \quad + 2(\lambda - 1)^2 ((\lambda - 1)^2 - 1) B(y, y)\}. \end{aligned} \quad (32)$$

Applying (31) to (32), one can obtain that

$$\begin{aligned}
 & B(x + (\lambda + 1)y, x + (\lambda + 1)y) \\
 & + B(x - (\lambda + 1)y, x - (\lambda + 1)y) \\
 & = (\lambda + 1)^2 [B(x + y, x + y) + B(x - y, x - y)] \\
 & - 2((\lambda + 1)^2 - 1)B(x, x) \\
 & + 2(\lambda + 1)^2((\lambda + 1)^2 - 1)B(y, y).
 \end{aligned} \tag{33}$$

This means that (30) is true for $\lambda + 1$, and we have showed that the function $x \mapsto B(x, x)$ satisfies (7). Therefore, the mapping $f: X \rightarrow Y$ satisfies (7).

Conversely, we decompose f into the odd part and the even part by putting

$$f_o(x) = \frac{f(x) - f(-x)}{2}, \quad f_e(x) = \frac{f(x) + f(-x)}{2} \tag{34}$$

for all $x \in X$. Then, $f(x) = f_o(x) + f_e(x)$ for all $x \in X$. It is easy to show that the mappings f_o and f_e satisfy (7). Hence, it follows from Lemmas 1 and 2 that the function f_o is cubic and f_e is quartic, respectively. Therefore, there exist a symmetric multi-additive mapping $G: X \times X \times X \rightarrow Y$ such that $f_o(x) = G(x, x, x)$ for all $x \in X$ (see [14, Theorem 2.1]) and a symmetric bi-quadratic mapping $B: X \times X \rightarrow Y$ such that $f_e(x) = B(x, x)$ for all $x \in X$ (see [15, Theorem 2.1]). Hence, we get $f(x) = G(x, x, x) + B(x, x)$ for all $x \in X$. The proof is complete. $\square$

3. Generalized Hyers-Ulam-Rassias Stability of the Functional Equation (7)

In this section, we will investigate the stability of the functional equation (7) by using the fixed point alternative. Throughout this section, let X be a real normed space and Y be a real Banach space, and we always assume that the integer λ used in the section is greater than or equal to 2. For convenience, we use the following abbreviation for a given function $f: X \rightarrow Y$:

$$\begin{aligned}
 Df(x, y) &:= \lambda[f(x + \lambda y) + f(x - \lambda y)] \\
 &\quad - \lambda^3[f(x + y) + f(x - y)] \\
 &\quad + 2\lambda^3(\lambda + 1)f(y) + 2\lambda(\lambda^2 - 1)f(x) \\
 &\quad - 2(\lambda + 1)f(\lambda y)
 \end{aligned} \tag{35}$$

for all $x, y \in X$.

Let us recall the following result by Diaz and Margolis.

Proposition 4 (see [24]). *Let (E, d) be a complete generalized metric space (i.e., one for which d may assume infinite value), and let $J: E \rightarrow E$ be a strictly contractive mapping with Lipschitz constant $L < 1$; that is,*

$$d(Jx, Jy) \leq Ld(x, y) \quad \forall x, y \in X. \tag{36}$$

Then, for each fixed element $x_0 \in E$, either

$$d(J^n x_0, J^{n+1} x_0) = +\infty \tag{37}$$

for all nonnegative integers n or there exists a non-negative integer n_0 such that

- (a) $d(J^n x_0, J^{n+1} x_0) < +\infty$ for all $n \geq n_0$;
- (b) the sequence $\{J^n x_0\}$ converges to a fixed point x^* of J ;
- (c) x^* is the unique fixed point of J in the set $\mathcal{F} := \{x \in E \mid d(J^{n_0} x_0, x) < +\infty\}$;
- (d) $d(x, x^*) \leq (1/(1 - L))d(x, Jx)$ for all $x \in \mathcal{F}$.

Lemma 5. *Let $f: X \rightarrow Y$ be an odd function for which there exists a function $\varphi: X \times X \rightarrow [0, +\infty)$ such that*

$$\|Df(x, y)\| \leq \varphi(x, y) \tag{38}$$

for all $x, y \in X$. If there exists a constant $0 < L < 1$ such that

$$\varphi(\lambda x, \lambda y) \leq \lambda^3 L \varphi(x, y) \tag{39}$$

for all $x, y \in X$, then there exists a unique cubic mapping $C: X \rightarrow Y$ such that

$$\|f(x) - C(x)\| \leq \frac{L}{1 - L} \phi(x) \tag{40}$$

for all $x \in X$, where $\phi(x) := (1/2(\lambda + 1))\varphi(0, x/\lambda)$.

Proof. It follows from (39) that

$$\lim_{r \rightarrow \infty} \frac{\varphi(\lambda^r x, \lambda^r y)}{\lambda^{3r}} = 0 \tag{41}$$

for all $x, y \in X$. Letting $x = 0$ in (38) and replacing y by x , we have

$$\|\lambda^3 f(x) - f(\lambda x)\| \leq \phi(\lambda x) \tag{42}$$

for all $x \in X$. By (39), we have $\phi(\lambda x) \leq \lambda^3 L \phi(x)$ for all $x \in X$. This, together with (42), implies that

$$\left\| f(x) - \frac{1}{\lambda^3} f(\lambda x) \right\| \leq L \phi(x) \tag{43}$$

for all $x \in X$. Let Ω be the set of all odd mappings $g: X \rightarrow Y$. We introduce the generalized metric on Ω :

$$\begin{aligned}
 & d(g, h) \\
 & := \inf \{K \in [0, \infty) \mid \|g(x) - h(x)\| \leq K\phi(x), x \in X\}.
 \end{aligned} \tag{44}$$

It is easy to show that (Ω, d) is complete. Now, we define a function $T: \Omega \rightarrow \Omega$ by

$$Tg(x) := \frac{1}{\lambda^3} g(\lambda x) \tag{45}$$

for all $g \in \Omega$ and all $x \in X$. Note that, for all $g, h \in \Omega$,

$$\begin{aligned} d(g, h) &\leq K \\ \implies \|g(x) - h(x)\| &\leq K\phi(x), \quad x \in X \\ \implies \left\| \frac{1}{\lambda^3}g(\lambda x) - \frac{1}{\lambda^3}h(\lambda x) \right\| &\leq \frac{1}{\lambda^3}K\phi(\lambda x), \quad x \in X \\ \implies \left\| \frac{1}{\lambda^3}g(\lambda x) - \frac{1}{\lambda^3}h(\lambda x) \right\| &\leq LK\phi(x), \quad x \in X \\ \implies d(Tg, Th) &\leq LK. \end{aligned} \quad (46)$$

Hence, we obtain that

$$d(Tg, Th) \leq Ld(g, h) \quad (47)$$

for all $g, h \in \Omega$; that is, T is a strictly contractive mapping of Ω with Lipschitz constant L . It follows from (43) that $d(Tf, f) \leq L$. Therefore, according to Proposition 4, the sequence $\{T^r f\}$ converges to a fixed point C of T ; that is,

$$C: X \longrightarrow Y, \quad C(x) = \lim_{r \rightarrow \infty} (T^r f)(x) = \lim_{r \rightarrow \infty} \frac{f(\lambda^r x)}{\lambda^{3r}} \quad (48)$$

and $C(\lambda x) = \lambda^3 C(x)$ for all $x \in X$. Also C is the unique fixed point of T in the set $\Delta = \{g \in \Omega \mid d(f, g) < \infty\}$ and

$$d(f, C) \leq \frac{1}{1-L} d(Tf, f) \leq \frac{L}{1-L}, \quad (49)$$

which yields the inequality (40). It follows from the definition of C , (38), and (41) that

$$\|DC(x, y)\| = \lim_{r \rightarrow \infty} \frac{\|Df(\lambda^r x, \lambda^r y)\|}{\lambda^{3r}} \leq \lim_{r \rightarrow \infty} \frac{\varphi(\lambda^r x, \lambda^r y)}{\lambda^{3r}} = 0 \quad (50)$$

for all $x, y \in X$; that is, the mapping $C: X \rightarrow Y$ satisfies (7). Since f is odd, C is odd. Therefore, Lemma 1 guarantees that C is cubic. Finally, it remains to prove the uniqueness of C . Let $S: X \rightarrow Y$ be another cubic function satisfying (40). Since $d(f, S) \leq L/(1-L)$ and S is cubic, we get $S \in \Delta$ and $TS(x) = (1/\lambda^3)S(\lambda x) = S(x)$ for all $x \in X$; that is, S is a fixed point of T . Since C is the unique fixed point of T in Δ , it follows that $S = C$. $\square$

Lemma 6. Let $f: X \rightarrow Y$ be an odd function for which there exists a function $\varphi: X \times X \rightarrow [0, +\infty)$ such that

$$\|Df(x, y)\| \leq \varphi(x, y) \quad (51)$$

for all $x, y \in X$. If there exists a constant $0 < L < 1$ such that

$$\lambda^3 \varphi(x, y) \leq L\varphi(\lambda x, \lambda y) \quad (52)$$

for all $x, y \in X$, then there exists a unique cubic mapping $C: X \rightarrow Y$ such that

$$\|f(x) - C(x)\| \leq \frac{1}{1-L} \psi(x) \quad (53)$$

for all $x \in X$, where $\psi(x) := (1/2(\lambda + 1))\varphi(0, x/\lambda)$.

Proof. It follows from (52) that

$$\lim_{r \rightarrow \infty} \lambda^{3r} \varphi\left(\frac{x}{\lambda^r}, \frac{y}{\lambda^r}\right) = 0 \quad (54)$$

for all $x, y \in X$. Letting $x = 0$ in (51) and replacing y by x , we have

$$\left\| \lambda^3 f\left(\frac{x}{\lambda}\right) - f(x) \right\| \leq \psi(x) \quad (55)$$

for all $x \in X$. We introduce the same definitions for Ω and d as in the proof of Lemma 5 (by replacing ϕ by ψ) such that (Ω, d) becomes a generalized complete metric space. Let $T: \Omega \rightarrow \Omega$ be the mapping defined by

$$Tg(x) := \lambda^3 g\left(\frac{x}{\lambda}\right) \quad (56)$$

for all $g \in \Omega$ and all $x \in X$. One can show that $d(Tg, Th) \leq Ld(g, h)$ for all $g, h \in \Omega$. It follows from (55) that $d(Tf, f) \leq 1$. Due to Proposition 4, the sequence $\{T^r f\}$ converges to a fixed point C of T ; that is,

$$C: X \longrightarrow Y, \quad C(x) = \lim_{r \rightarrow \infty} (T^r f)(x) = \lim_{r \rightarrow \infty} \lambda^{3r} f\left(\frac{x}{\lambda^r}\right) \quad (57)$$

and $C(\lambda x) = \lambda^3 C(x)$ for all $x \in X$. Also,

$$d(f, C) \leq \frac{1}{1-L} d(Tf, f) \leq \frac{1}{1-L}, \quad (58)$$

which yields the inequality (53). The rest of the proof is similar to the proof of Lemma 5, and we omit the details. $\square$

Similarly, we can prove the following two lemmas on even functions.

Lemma 7. Let $f: X \rightarrow Y$ be an even function with $f(0) = 0$ for which there exists a function $\varphi: X \times X \rightarrow [0, +\infty)$ such that

$$\|Df(x, y)\| \leq \varphi(x, y) \quad (59)$$

for all $x, y \in X$. If there exists a constant $0 < L < 1$ such that

$$\varphi(\lambda x, \lambda y) \leq \lambda^4 L \varphi(x, y) \quad (60)$$

for all $x, y \in X$, then there exists a unique quartic mapping $Q: X \rightarrow Y$ such that

$$\|f(x) - Q(x)\| \leq \frac{L}{1-L} \phi(x) \quad (61)$$

for all $x \in X$, where $\phi(x) := (1/2)\varphi(0, x/\lambda)$.

Lemma 8. Let $f: X \rightarrow Y$ be an even function with $f(0) = 0$ for which there exists a function $\varphi: X \times X \rightarrow [0, +\infty)$ such that

$$\|Df(x, y)\| \leq \varphi(x, y) \quad (62)$$

for all $x, y \in X$. If there exists a constant $0 < L < 1$ such that

$$\lambda^4 \varphi(x, y) \leq L\varphi(\lambda x, \lambda y) \quad (63)$$

for all $x, y \in X$, then there exists a unique quartic mapping $Q: X \rightarrow Y$ such that

$$\|f(x) - Q(x)\| \leq \frac{1}{1-L} \psi(x) \quad (64)$$

for all $x \in X$, where $\psi(x) := (1/2)\varphi(0, x/\lambda)$.

Now, we are ready to give our main theorems in this section.

Theorem 9. Let $f: X \rightarrow Y$ be a function with $f(0) = 0$ for which there exists a function $\varphi: X \times X \rightarrow [0, +\infty)$ such that

$$\|Df(x, y)\| \leq \varphi(x, y) \quad (65)$$

for all $x, y \in X$. If there exists a constant $0 < L < 1$ such that

$$\varphi(\lambda x, \lambda y) \leq \lambda^3 L \varphi(x, y) \quad (66)$$

for all $x, y \in X$, then there exist a unique cubic mapping $C: X \rightarrow Y$ and a unique quartic mapping $Q: X \rightarrow Y$ such that

$$\|f(x) - C(x) - Q(x)\| \leq \frac{L}{1-L} [\phi_o(x) + \phi_e(x)] \quad (67)$$

for all $x \in X$, where

$$\begin{aligned} \phi_o(x) &:= \frac{1}{2(\lambda+1)} \Phi\left(0, \frac{x}{\lambda}\right), & \phi_e(x) &:= \frac{1}{2} \Phi\left(0, \frac{x}{\lambda}\right), \\ \Phi(x, y) &:= \frac{1}{2} [\varphi(x, y) + \varphi(-x, -y)]. \end{aligned} \quad (68)$$

Proof. Let f_o and f_e denote the odd and the even part of f , respectively. Then, it can be verified from (65) that

$$\|Df_o(x, y)\| \leq \Phi(x, y), \quad \|Df_e(x, y)\| \leq \Phi(x, y) \quad (69)$$

for all $x, y \in X$. Moreover, by (66), it is easy to compute that

$$\Phi(\lambda x, \lambda y) \leq \lambda^3 L \Phi(x, y) \quad (70)$$

for all $x, y \in X$. Thus, by applying Lemmas 5 and 7, one can obtain that there exist a unique cubic mapping $C: X \rightarrow Y$ and a unique quartic mapping $Q: X \rightarrow Y$ such that

$$\|f_o(x) - C(x)\| \leq \frac{L}{1-L} \phi_o(x), \quad (71)$$

$$\|f_e(x) - Q(x)\| \leq \frac{L}{1-L} \phi_e(x) \quad (72)$$

for all $x \in X$, where $\phi_o(x) := (1/2)(\lambda+1)\Phi(0, x/\lambda)$ and $\phi_e(x) := (1/2)\Phi(0, x/\lambda)$. Moreover, combining (71) and (72) yields the inequality (67). The proof is complete. $\square$

Theorem 10. Let $f: X \rightarrow Y$ be a function with $f(0) = 0$ for which there exists a function $\varphi: X \times X \rightarrow [0, +\infty)$ such that

$$\|Df(x, y)\| \leq \varphi(x, y) \quad (73)$$

for all $x, y \in X$. If there exists a constant $0 < L < 1$ such that

$$\lambda^4 \varphi(x, y) \leq L\varphi(\lambda x, \lambda y) \quad (74)$$

for all $x, y \in X$, then there exist a unique cubic mapping $C: X \rightarrow Y$ and a unique quartic mapping $Q: X \rightarrow Y$ such that

$$\|f(x) - C(x) - Q(x)\| \leq \frac{1}{1-L} [\psi_o(x) + \psi_e(x)] \quad (75)$$

for all $x \in X$, where

$$\begin{aligned} \psi_o(x) &:= \frac{1}{2(\lambda+1)} \Psi\left(0, \frac{x}{\lambda}\right), & \psi_e(x) &:= \frac{1}{2} \Psi\left(0, \frac{x}{\lambda}\right), \\ \Psi(x, y) &:= \frac{1}{2} [\varphi(x, y) + \varphi(-x, -y)]. \end{aligned} \quad (76)$$

Proof. Similar to the proof of Theorem 9, the result follows from Lemmas 6 and 8. $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

References

- [1] S. M. Ulam, *Problems in Modern Mathematics*, chapter 6, John Wiley & Sons, New York, NY, USA, 1964.
- [2] D. H. Hyers, "On the stability of the linear functional equation," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 27, pp. 222–224, 1941.
- [3] Th. M. Rassias, "On the stability of the linear mapping in Banach spaces," *Proceedings of the American Mathematical Society*, vol. 72, no. 2, pp. 297–300, 1978.
- [4] S.-M. Jung, "Hyers-Ulam-Rassias stability of Jensen's equation and its application," *Proceedings of the American Mathematical Society*, vol. 126, no. 11, pp. 3137–3143, 1998.
- [5] K.-W. Jun, H.-M. Kim, and I.-S. Chang, "On the Hyers-Ulam stability of an Euler-Lagrange type cubic functional equation," *Journal of Computational Analysis and Applications*, vol. 7, no. 1, pp. 21–33, 2005.
- [6] K. W. Jun, S. B. Lee, and W. G. Park, "Solution and stability of a cubic functional equation," *Acta Mathematica Sinica*, vol. 26, no. 7, pp. 1255–1262, 2010.
- [7] G. H. Kim, "A stability of the generalized sine functional equations," *Journal of Mathematical Analysis and Applications*, vol. 331, no. 2, pp. 886–894, 2007.
- [8] W.-G. Park and J.-H. Bae, "On a bi-quadratic functional equation and its stability," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 62, no. 4, pp. 643–654, 2005.
- [9] J. Sikorska, "Generalized stability of the Cauchy and the Jensen functional equations on spheres," *Journal of Mathematical Analysis and Applications*, vol. 345, no. 2, pp. 650–660, 2008.

- [10] J. Aczél and J. Dhombres, *Functional Equations in Several Variables*, vol. 31 of *Encyclopedia of Mathematics and Its Applications*, Cambridge University Press, 1989.
- [11] S. Czerwik, *Functional Equations and Inequalities in Several Variables*, World Scientific Publishing, River Edge, NJ, USA, 2002.
- [12] D. H. Hyers, G. Isac, and Th. M. Rassias, *Stability of Functional Equations in Several Variables*, Birkhäuser, Boston, Mass, USA, 1998.
- [13] Pl. Kannappan, *Functional Equations and Inequalities with Applications*, Springer Monographs in Mathematics, Springer, New York, NY, USA, 2009.
- [14] K.-W. Jun and H.-M. Kim, "The generalized Hyers-Ulam-Rassias stability of a cubic functional equation," *Journal of Mathematical Analysis and Applications*, vol. 274, no. 2, pp. 267–278, 2002.
- [15] S. H. Lee, S. M. Im, and I. S. Hwang, "Quartic functional equations," *Journal of Mathematical Analysis and Applications*, vol. 307, no. 2, pp. 387–394, 2005.
- [16] M. Eshaghi Gordji, S. Zolfaghari, J. M. Rassias, and M. B. Savadkouhi, "Solution and stability of a mixed type cubic and quartic functional equation in quasi-Banach spaces," *Abstract and Applied Analysis*, vol. 2009, Article ID 417473, 14 pages, 2009.
- [17] M. E. Gordji, S. K. Gharetapeh, J. M. Rassias, and S. Zolfaghari, "Solution and stability of a mixed type additive, quadratic, and cubic functional equation," *Advances in Difference Equations*, vol. 2009, Article ID 826130, 17 pages, 2009.
- [18] M. Eshaghi Gordji and H. Khodaei, "Solution and stability of generalized mixed type cubic, quadratic and additive functional equation in quasi-Banach spaces," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 71, no. 11, pp. 5629–5643, 2009.
- [19] M. Eshaghi Gordji and M. B. Savadkouhi, "Stability of a mixed type cubic-quartic functional equation in non-Archimedean spaces," *Applied Mathematics Letters*, vol. 23, no. 10, pp. 1198–1202, 2010.
- [20] F. Moradlou, H. Vaezi, and G. Z. Eskandani, "Hyers-Ulam-Rassias stability of a quadratic and additive functional equation in quasi-Banach spaces," *Mediterranean Journal of Mathematics*, vol. 6, no. 2, pp. 233–248, 2009.
- [21] A. Najati and Th. M. Rassias, "Stability of a mixed functional equation in several variables on Banach modules," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 72, no. 3-4, pp. 1755–1767, 2010.
- [22] A. Najati and G. Z. Eskandani, "Stability of a mixed additive and cubic functional equation in quasi-Banach spaces," *Journal of Mathematical Analysis and Applications*, vol. 342, no. 2, pp. 1318–1331, 2008.
- [23] V. Radu, "The fixed point alternative and the stability of functional equations," *Fixed Point Theory*, vol. 4, no. 1, pp. 91–96, 2003.
- [24] J. B. Diaz and B. Margolis, "A fixed point theorem of the alternative, for contractions on a generalized complete metric space," *Bulletin of the American Mathematical Society*, vol. 74, pp. 305–309, 1968.
- [25] L. Cădariu and V. Radu, "Fixed point methods for the generalized stability of functional equations in a single variable," *Fixed Point Theory and Applications*, vol. 2008, Article ID 749392, 15 pages, 2008.
- [26] A. K. Mirmostafaei, "A fixed point approach to almost quartic mappings in quasi fuzzy normed spaces," *Fuzzy Sets and Systems*, vol. 160, no. 11, pp. 1653–1662, 2009.