Hindawi Publishing Corporation
Journal of Function Spaces and Applications
Volume 2013, Article ID 681637, 3 pages
http://dx.doi.org/10.1155/2013/681637

Research Article

Multiplicative Isometries on Some F-Algebras of Holomorphic Functions

Yasuo Iida[1] and Kazuhiro Kasuga[2]

[1] *Department of Mathematics, Iwate Medical University, Yahaba, Iwate 028-3694, Japan*
[2] *Academic Support Center, Kogakuin University, Tokyo 192-0015, Japan*

Correspondence should be addressed to Yasuo Iida; yiida@iwate-med.ac.jp

Received 30 November 2012; Accepted 24 May 2013

Academic Editor: Gestur Ólafsson

Multiplicative (but not necessarily linear) isometries of $M^p(X)$ onto $M^p(X)$ will be described, where $M^p(X)$ $(p \geq 1)$ are F-algebras included in the Smirnov class $N_*(X)$.

1. Introduction

Let n be a positive integer. The space of n-complex variables $z = (z_1, \ldots, z_n)$ is denoted by $\mathbb{C}^n$. The unit polydisk $\{z \in \mathbb{C}^n : |z_j| < 1, 1 \leq j \leq n\}$ is denoted by U^n, and the distinguished boundary $\mathbb{T}^n$ is $\{z \in \mathbb{C}^n : |z_j| = 1, 1 \leq j \leq n\}$. The unit ball $\{z \in \mathbb{C}^n : \sum_{j=1}^n |z_j|^2 < 1\}$ is denoted by B_n and S_n its boundary. In this paper, X denotes the unit polydisk or the unit ball for $n \geq 1$, and ∂X denotes $\mathbb{T}^n$ for $X = U^n$ or S_n for $X = B_n$. The normalized (in the sense that $\sigma(\partial X) = 1$) Lebesgue measure on ∂X is denoted by $d\sigma$.

For each $0 < q \leq \infty$, the Hardy space on X is denoted by $H^q(X)$ with the norm $\|\cdot\|_q$.

The Nevanlinna class $N(X)$ on X is defined as the set of all holomorphic functions f on X such that

$$\sup_{0 \leq r < 1} \int_{\partial X} \log\left(1 + \left|f(rz)\right|\right) d\sigma(z) < \infty \tag{1}$$

holds. It is known that $f \in N(X)$ has a finite nontangential limit, also denoted by f, almost everywhere on ∂X.

The Smirnov class $N_*(X)$ is defined as the set of all $f \in N(X)$ which satisfy the equality

$$\begin{aligned} &\sup_{0 \leq r < 1} \int_{\partial X} \log\left(1 + \left|f(rz)\right|\right) d\sigma(z) \\ &\quad = \int_{\partial X} \log\left(1 + \left|f(z)\right|\right) d\sigma(z). \end{aligned} \tag{2}$$

Define a metric

$$d_{N_*(X)}(f, g) = \int_{\partial X} \log\left(1 + \left|f(z) - g(z)\right|\right) d\sigma(z) \tag{3}$$

for $f, g \in N_*(X)$. With the metric $d_{N_*(X)}(\cdot, \cdot)$, the Smirnov class $N_*(X)$ is an F-algebra. Recall that an F-algebra is a topological algebra in which the topology arises from a complete metric. Complex-linear isometries on the Smirnov class are characterized by Stephenson in [1].

The Privalov class $N^p(X)$, $1 < p < \infty$, is defined as the set of all holomorphic functions f on X such that

$$\sup_{0 \leq r < 1} \int_{\partial X} \left(\log\left(1 + \left|f(rz)\right|\right)\right)^p d\sigma(z) < \infty \tag{4}$$

holds. It is well-known that $N^p(X)$ is a subalgebra of $N_*(X)$; hence, every $f \in N^p(X)$ has a finite nontangential limit almost everywhere on ∂X. Under the metric defined by

$$d_{N^p(X)}(f, g) = \left(\int_{\partial X} \left(\log(1 + |f(z) - g(z)|)\right)^p d\sigma(z)\right)^{1/p} \tag{5}$$

for $f, g \in N^p(X)$, $N^p(X)$ becomes an F-algebra (cf. [2]). Complex-linear isometries on $N^p(X)$ are investigated by Iida and Mochizuki [3] for one-dimensional case and by Subbotin [2, 4] for a general case.

Now, we define the class $M^p(X)$. For $1 \le p < \infty$, the class $M^p(X)$ is defined as the set of all holomorphic functions f on X such that

$$\int_{\partial X} \left(\log \left(1 + \sup_{0 \le r < 1} |f(rz)| \right) \right)^p d\sigma(z) < \infty. \tag{6}$$

Define a metric

$$d_{M^p(X)}(f,g) = \left\{ \int_{\partial X} \left(\log \left(1 + \sup_{0 \le r < 1} |f(rz) - g(rz)| \right) \right)^p d\sigma(z) \right\}^{1/p} \tag{7}$$

for $f, g \in M^p(X)$. With this metric, $M^p(X)$ is also an F-algebra (see [2]). Complex-linear surjective isometries on $M^p(X)$ are investigated by Subbotin [2, 4].

It is well-known that the following inclusion relations hold:

$$H^q(X) \subsetneq N^p(X) \subsetneq M^1(X) \subsetneq N_*(X) \quad (0 < q \le +\infty,\ p > 1). \tag{8}$$

As shown in [4], for any $p > 1$, the class $M^p(X)$ coincides with the class $N^p(X)$, and the metrics $d_{M^p(X)}$ and $d_{N^p(X)}$ are equivalent. Therefore, the topologies induced by these metrics are identical on the set $M^p(X) = N^p(X)$. But we note that [4, Theorems 1 and 4] implies that the sets of linear isometries on $M^p(X)$ and $N^p(X)$ are different. It is known that $H^\infty(X)$ is a dense subalgebra of $M^p(X)$. The convergence in the metric is stronger than uniform convergence on compact subsets of X.

In this paper, we consider surjective multiplicative (but not necessarily linear) isometries from the class $M^p(X)$ ($p \in \mathbb{N}$) on the open unit disk, the ball, or the polydisk onto itself.

2. The Results

Proposition 1. *Let n be a positive integer, and let X be either B_n or U^n. Let $p \in \mathbb{N}$, and suppose that $T : M^p(X) \to M^p(X)$ is a surjective isometry. If T is 2-homogeneous in the sense that $T(2f) = 2T(f)$ holds for every $f \in M^p(X)$, then either*

$$T(f) = \alpha f \circ \Phi \quad \text{for every } f \in M^p(X), \tag{9}$$

or

$$T(f) = \alpha \overline{f \circ \Phi} \quad \text{for every } f \in M^p(X), \tag{10}$$

where α is a complex number with the unit modulus and, for $X = B_n$, Φ is a unitary transformation; for $X = U^n$, $\Phi(z_1, \ldots, z_n) = (\lambda_1 z_{i_1}, \ldots, \lambda_n z_{i_n})$, where $|\lambda_j| = 1$, $1 \le j \le n$ and $(i_1, \ldots, i_n)$ is some permutation of the integers from 1 through n.

Proof. We follow [5, Proposition 2.1] and [4, Theorem 3]. Let $f, g \in H^p(X)$. By 2-homogenuity of isometry T, the equation

$$\int_{\partial X} \left(\log \left(1 + \sup_{0 \le r < 1} \left| \frac{f(rz)}{2^n} - \frac{g(rz)}{2^n} \right| \right) \right)^p d\sigma(z) = \int_{\partial X} \left(\log \left(1 + \sup_{0 \le r < 1} \left| \frac{(Tf)(rz)}{2^n} - \frac{(Tg)(rz)}{2^n} \right| \right) \right)^p d\sigma(z) \tag{11}$$

holds. In a way similar to the proof of Theorem 2.1 in [1], we see that

$$\int_{\partial X} \sup_{0 \le r < 1} |f(rz) - g(rz)|^p d\sigma(z) = \int_{\partial X} \sup_{0 \le r < 1} |(Tf)(rz) - (Tg)(rz)|^p d\sigma(z). \tag{12}$$

This equality implies that T is isometric in the norm

$$\|f\|_{H_m^p} := \left\{ \int_{\partial X} \sup_{0 \le r < 1} |f(rz)|^p \, d\sigma(z) \right\}^{1/p} \tag{13}$$

of the space $H^p(X)$, which is equivalent to the standard norm in $H^p(X)$. From (12) with $g = 0$, we obtain $T(H^p(X)) \subseteq H^p(X)$ since $T(0) = 0$, which is observed by just letting $f = 0$ in the equation $T(2f) = 2T(f)$. Furthermore, the restricted map $T|_{H^p(X)}$ is an isometry with respect to the metric induced by the H^p-norm $\|\cdot\|_p$. The same argument for T^{-1} shows that $T^{-1}(H^p(X)) \subseteq H^p(X)$. Thus, we see that $T(H^p(X)) = H^p(X)$. By the Mazur-Ulam theorem [6], $T|_{H^p(X)}$ is a real-linear isometry since $T(0) = 0$.

Using the limit

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon^{p+1}} \left\{ (\varepsilon t)^p - (\log(1 + \varepsilon t))^p \right\} = \frac{p}{2} t^{p+1}, \quad t \ge 0, \tag{14}$$

we show that T is also isometric in the norm $\|f\|_{H_m^{p+1}}$. If $p > 0$, then for any $k \in \mathbb{N}$, there exists the following limit:

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon^{p+k}} \left\{ (\log(1 + \varepsilon t))^p - \sum_{n=0}^{k-1} c_n (\varepsilon t)^{n+p} \right\} = c_k t^{p+k}, \quad t \ge 0, \tag{15}$$

where c_n are the Taylor coefficients of the function $(\log(1+t)/t)^p$. Using (15), we can prove that T is isometric in H_m^{p+k} for $p \in \mathbb{N}$ and all $k \in \mathbb{N}$ by induction.

Since $d\sigma$ is a finite measure, we verify that

$$\lim_{p \to \infty} \|f\|_{H_m^p} = \|f\|_{H_m^\infty} \tag{16}$$

holds for every $f \in H^\infty(X)$, and it is clear that $\|f\|_{H_m^\infty} = \|f\|_\infty$. Moreover, $\|f\|_p = \|T(f)\|_p$ for every $f \in H^\infty(X)$ and $\lim_{p \to \infty} \|T(f)\|_p = \|T(f)\|_\infty$, so we have $T(f) \in H^\infty(X)$, and $\|f\|_\infty = \|T(f)\|_\infty$ for every $f \in H^\infty(X)$. Similarly, we see that $f \in H^\infty(X)$ if $T(f)$ belongs to $H^\infty(X)$. Therefore $T|_{H^\infty(X)}$ is a surjective isometry with respect to $\|\cdot\|_\infty$ from

$H^\infty(X)$ onto itself. We may suppose that $H^\infty(X)$ is a uniform algebra on the maximal ideal space and the maximal ideal space is connected by the Šilov idempotent theorem; hence, we see that $T|_{H^\infty(X)}$ is complex-linear or conjugate linear by [7, Theorem].

If $T|_{H^\infty(X)}$ is complex-linear, then T is complex-linear on $M^p(X)$, since $H^\infty(X)$ is dense in $M^p(X)$ and the convergence in the original metric is stronger than uniform convergence on compact subsets of X. Therefore, the first formula of the conclusion holds by Corollary 2.3 in [1].

If $T|_{H^\infty(X)}$ is conjugate linear, then T is conjugate linear on $M^p(X)$ as before. Let $\widetilde{T} : M^p(X) \to M^p(X)$ be defined as $\widetilde{T}(f) = T(\widetilde{f})$ for every $f \in M^p(X)$, where

$$\widetilde{f}(z_1, \ldots, z_n) = \overline{f(\overline{z_1}, \ldots, \overline{z_n})} \tag{17}$$

for $f \in M^p(X)$. Then, $\widetilde{T}$ is complex-linear isometry from $M^p(X)$ onto itself. Applying Corollary 2.3 in [1] to $\widetilde{T}$, the second formula of the conclusion holds. □

Let $1 \le p < \infty$. We say a map $T : M^p(X) \to M^p(X)$ is *multiplicative* if $T(fg) = T(f)T(g)$ for every $f, g \in M^p(X)$. Next, we characterize multiplicative isometries from $M^p(X)$ ($p \in \mathbb{N}$) onto itself. Let Φ be a transformation described in Proposition 1. Then, $T(f) = f \circ \Phi$ defines a complex-linear multiplicative isometry from $M^p(X)$ onto itself, and $T(f) = \overline{f \circ \overline{\Phi}}$ defines a conjugate linear multiplicative isometry from $M^p(X)$ onto itself. We show that they are the only multiplicative isometries from $M^p(X)$ onto itself.

Theorem 2. *Let $p \in \mathbb{N}$, and let T be a multiplicative (not necessarily linear) isometry from $M^p(X)$ onto itself. Then, there exists a holomorphic automorphism Φ on X such that either of the following holds:*

$$T(f) = f \circ \Phi \quad \text{for every } f \in M^p(X) \tag{18}$$

or

$$T(f) = \overline{f \circ \overline{\Phi}} \quad \text{for every } f \in M^p(X), \tag{19}$$

where Φ is a unitary transformation for $X = B_n$; $\Phi(z_1, \ldots, z_n) = (\lambda_1 z_{i_1}, \ldots, \lambda_n z_{i_n})$ for $X = U^n$, where $|\lambda_j| = 1$ for every $1 \le j \le n$ and $(i_1, \ldots, i_n)$ is some permutation of the integers from 1 through n.

Proof. Since T is multiplicative, we see by the same way as in the proof of Theorem 2.2 in [5] that $T(1) = 1$, $T(2) = 2$, and $T(1/2) = 1/2$. Therefore, T is a surjective isometry which satisfies $T(2f) = 2T(f)$ as T is multiplicative. It follows by Proposition 1 that

$$T(f) = \alpha f \circ \Phi, \quad f \in M^p(X), \tag{20}$$

or

$$T(f) = \alpha \overline{f \circ \overline{\Phi}}, \quad f \in M^p(X), \tag{21}$$

holds for a complex number α and the holomorphic automorphism Φ as described in Proposition 1. The constant $\alpha = 1$ is observed as $T(1) = 1$; hence, the conclusion holds. □

Remark 3. We note that surjective multiplicative isometries of the class $M^p(X)$ ($p \in \mathbb{N}$) have the same form as surjective mutiplicative isometries of the Smirnov class [5, Theorem 2.2] and the Privalov class [8, Corollary 3.4]. The authors do not know whether this result holds for noninteger p.

Acknowledgments

The authors wish to express their sincere gratitude to Professor O. Hatori, who introduced this subject and kindly directed them. The authors also would like to thank the referee for the detailed comments and valuable suggestions. The first author was partly supported by the Grant from Keiryokai Research Foundation no. 97.

References

[1] K. Stephenson, "Isometries of the Nevanlinna class," *Indiana University Mathematics Journal*, vol. 26, no. 2, pp. 307–324, 1977.

[2] A. V. Subbotin, "Isometries of Privalov spaces of holomorphic functions of several variables," *Journal of Mathematical Sciences*, vol. 135, no. 1, pp. 2794–2802, 2006.

[3] Y. Iida and N. Mochizuki, "Isometries of some F-algebras of holomorphic functions," *Archiv der Mathematik*, vol. 71, no. 4, pp. 297–300, 1998.

[4] A. V. Subbotin, "Linear isometry groups of Privalov's spaces of holomorphic functions of several variables," *Doklady Mathematics*, vol. 60, no. 1, pp. 77–79, 1999.

[5] O. Hatori and Y. Iida, "Multiplicative isometries on the Smirnov class," *Central European Journal of Mathematics*, vol. 9, no. 5, pp. 1051–1056, 2011.

[6] S. Mazur and S. Ulam, "Sur les transformationes isométriques d'espaces vectoriels normés," *Comptes Rendus de l'Académie des Sciences Paris*, vol. 194, pp. 946–948, 1932.

[7] A. J. Ellis, "Real characterizations of function algebras amongst function spaces," *Bulletin of the London Mathematical Society*, vol. 22, pp. 381–385, 1990.

[8] O. Hatori, Y. Iida, S. Stević, and S.-I. Ueki, "Multiplicative isometries on F-Algebras of holomorphic functions," *Abstract and Applied Analysis*, vol. 2012, Article ID 125987, 16 pages, 2012.