Research Article

Multiplicative Isometries on Some $F$-Algebras of Holomorphic Functions

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Multiplicative (but not necessarily linear) isometries of $M^p(X)$ onto $M^p(X)$ will be described, where $M^p(X)$ ($p \geq 1$) are $F$-algebras included in the Smirnov class $N^*_p(X)$.

1. Introduction

Let $n$ be a positive integer. The space of $n$-complex variables $z = (z_1, \ldots, z_n)$ is denoted by $\mathbb{C}^n$. The unit polydisk $\{ z \in \mathbb{C}^n : |z_j| < 1, 1 \leq j \leq n \}$ is denoted by $U^n$, and the distinguished boundary $T^n$ is $\{ z \in \mathbb{C}^n : |z_j| = 1, 1 \leq j \leq n \}$. The unit ball $\{ z \in \mathbb{C}^n : \sum_{j=1}^{n} |z_j|^2 < 1 \}$ is denoted by $B^n$ and $S_n$ its boundary.

In this paper, $X$ denotes the unit polydisk or the unit ball for $n \geq 1$, and $\partial X$ denotes $T^n$ for $X = U^n$ or $S_n$ for $X = B^n$. The normalized (in the sense that $\sigma(\partial X) = 1$) Lebesgue measure on $\partial X$ is denoted by $d\sigma$.

For each $0 < q \leq \infty$, the Hardy space on $X$ is denoted by $H^q(X)$ with the norm $\| \cdot \|_q$.

The Nevanlinna class $N(X)$ on $X$ is defined as the set of all holomorphic functions $f$ on $X$ such that

$$\sup_{0 < r < 1} \int_{\partial X} \log (1 + |f(rz)|) \ d\sigma(z) < \infty$$

holds. It is known that $f \in N(X)$ has a finite nontangential limit, also denoted by $f$, almost everywhere on $\partial X$.

The Smirnov class $N_p(X)$ is defined as the set of all $f \in N(X)$ which satisfy the equality

$$\sup_{0 < r < 1} \int_{\partial X} \log (1 + |f(rz)|) \ d\sigma(z)$$

$$= \int_{\partial X} \log (1 + |f(z)|) \ d\sigma(z).$$

Define a metric

$$d_{N_p(X)}(f, g) = \int_{\partial X} \log (1 + |f(z) - g(z)|) \ d\sigma(z)$$

for $f, g \in N_p(X)$. With the metric $d_{N_p(X)}(\cdot, \cdot)$, the Smirnov class $N_p(X)$ is an $F$-algebra. Recall that an $F$-algebra is a topological algebra in which the topology arises from a complete metric. Complex-linear isometries on the Smirnov class are characterized by Stephenson in [1].

The Privalov class $N^p(X)$, $1 < p < \infty$, is defined as the set of all holomorphic functions $f$ on $X$ such that

$$\sup_{0 < r < 1} \int_{\partial X} (\log (1 + |f(rz)|))^p \ d\sigma(z) < \infty$$

holds. It is well-known that $N^p(X)$ is a subalgebra of $N_p(X)$; hence, every $f \in N^p(X)$ has a finite nontangential limit almost everywhere on $\partial X$. Under the metric defined by

$$d_{N^p(X)}(f, g) = \left( \int_{\partial X} (\log (1 + |f(z) - g(z)|))^p \ d\sigma(z) \right)^{1/p}$$

for $f, g \in N^p(X)$, $N^p(X)$ becomes an $F$-algebra (cf. [2]). Complex-linear isometries on $N^p(X)$ are investigated by Iida and Mochizuki [3] for one-dimensional case and by Subbotin [2, 4] for a general case.
Now, we define the class $M^p(X)$. For $1 \leq p < \infty$, the class $M^p(X)$ is defined as the set of all holomorphic functions $f$ on $X$ such that

$$
\left( \log \left( 1 + \sup_{0 \leq r < 1} |f(rz)| \right) \right)^p d\sigma(z) < \infty.
$$

(6)

Define a metric

$$
d_{M^p(X)}(f, g) = \left\{ \int_X \left( \log \left( 1 + \sup_{0 \leq r < 1} |f(rz) - g(rz)| \right) \right)^p d\sigma(z) \right\}^{1/p}.
$$

(7)

for $f, g \in M^p(X)$. With this metric, $M^p(X)$ is also an $F$-algebra (see [2]). Complex-linear surjective isometries on $M^p(X)$ are investigated by Subbotin [2, 4].

It is well-known that the following inclusion relations hold:

$$
H^q(X) \subset N^p(X) \subset M^1(X) \subset N_*(X)
$$

(8)

for $0 < q \leq +\infty$, $p > 1$.

As shown in [4], for any $p > 1$, the class $M^p(X)$ coincides with the class $N^p(X)$, and the metrics $d_{M^p(X)}$ and $d_{N^p(X)}$ are equivalent. Therefore, the topologies induced by these metrics are identical on the set $M^p(X) = N^p(X)$. But we note that [4, Theorems 1 and 4] implies that the sets of linear isometries on $M^p(X)$ and $N^p(X)$ are different. It is known that $H^\infty(X)$ is a dense subalgebra of $M^p(X)$. The convergence in the metric is stronger than uniform convergence on compact subsets of $X$.

In this paper, we consider surjective multiplicative (but not necessarily linear) isometries from the class $M^p(X)$ ($p \in \mathbb{N}$) on the open unit disk, the ball, or the polydisk onto itself.

### 2. The Results

**Proposition 1.** Let $n$ be a positive integer, and let $X$ be either $B_n$ or $U^n$. Let $p \in \mathbb{N}$, and suppose that $T : M^p(X) \to M^p(X)$ is a surjective isometry. If $T$ is 2-homogeneous in the sense that $T(2f) = 2T(f)$ holds for every $f \in M^p(X)$, then either

$$
T(f) = \alpha f \circ \Phi \quad \text{for every } f \in M^p(X),
$$

(9)

or

$$
T(f) = \alpha f \circ \overline{\Phi} \quad \text{for every } f \in M^p(X),
$$

(10)

where $\alpha$ is a complex number with the unit modulus and, for $X = B_n$, $\Phi(z_1, \ldots, z_n) = (\lambda_1 z_{i_1}, \ldots, \lambda_n z_{i_n})$, where $|\lambda_j| = 1$, $1 \leq j \leq n$ and $(i_1, \ldots, i_n)$ is some permutation of the integers from 1 through $n$.

**Proof.** We follow [5, Proposition 2.1] and [4, Theorem 3]. Let $f, g \in H^p(X)$. By 2-homogeneity of isometry $T$, the equation

$$
\int_X \left( \log \left( 1 + \sup_{0 \leq r < 1} \frac{|f(rz) - g(rz)|}{2^n} \right) \right)^p d\sigma(z)
$$

(11)

holds. In a way similar to the proof of Theorem 2.1 in [1], we see that

$$
\int_X \left( \log \left( 1 + \sup_{0 \leq r < 1} \frac{|(Tf)(rz) - (Tg)(rz)|}{2^n} \right) \right)^p d\sigma(z)
$$

(12)

This equality implies that $T$ is isometric in the norm

$$
\|f\|_{H^p_m} := \left\{ \int_X \left( \sup_{0 \leq r < 1} |f(rz)|^p \right)^{1/p} d\sigma(z) \right\}.
$$

(13)

of the space $H^p_m(X)$, which is equivalent to the standard norm in $H^p_m(X)$. From (12) with $g = 0$, we obtain $T(H^p_m(X)) \subseteq H^p_m(X)$ since $T(0) = 0$, which is observed by just letting $f = 0$ in the equation $T(2f) = 2T(f)$. Furthermore, the restricted map $T|_{H^p_m(X)}$ is an isometry with respect to the metric induced by the $H^p$-norm $\| \cdot \|_p$. The same argument for $T^{-1}$ shows that $T^{-1}(H^p_m(X)) \subseteq H^p_m(X)$. Thus, we see that $T(H^p_m(X)) = H^p_m(X)$ by the Mazur-Ulam theorem [6], $T|_{H^p_m(X)}$ is a real-linear isometry since $T(0) = 0$.

Using the limit

$$
\lim_{\epsilon \to 0^+} \frac{1}{\epsilon^{p+1}} \left\{ \log(1 + \epsilon t)^p - (\log(1 + \epsilon t))^p \right\} = \frac{p^2}{2} \epsilon t^{p+1}, \quad t \geq 0,
$$

(14)

we show that $T$ is also isometric in the norm $\|f\|_{H^p_m}$. If $p > 0$, then for any $k \in \mathbb{N}$, there exists the following limit:

$$
\lim_{\epsilon \to 0^+} \frac{1}{\epsilon^{p+1}} \left\{ \log(1 + \epsilon t)^p - \sum_{n=0}^{k-1} c_n \epsilon^{p+n} \right\} = c_k t^{p+k}, \quad t \geq 0,
$$

(15)

where $c_k$ are the Taylor coefficients of the function $(\log(1+t))^p$. Using (15), we can prove that $T$ is isometric in $H^p_m^{p+k}$ for $p \in \mathbb{N}$ and all $k \in \mathbb{N}$ by induction.

Since $d\sigma$ is a finite measure, we verify that

$$
\lim_{p \to +\infty} \|f\|_{H^p_m} = \|f\|_{H^\infty_m},
$$

(16)

holds for every $f \in H^\infty(X)$, and it is clear that $\|f\|_{H^\infty_m} = \|f\|_{H^\infty}$. Moreover, $\|f\|_p = \|T(f)\|_p$ for every $f \in H^\infty_m(X)$ and

$$
\lim_{p \to +\infty} \|T(f)\|_p = \|f\|_{H^\infty_m} = \|f\|_{H^\infty},
$$

so we have $T(f) \in H^\infty_m(X)$, and

$$
\|f\|_{H^\infty_m} = \|T(f)\|_{H^\infty},
$$

for every $f \in H^\infty_m(X)$. Similarly, we see that $f \in H^\infty(X)$ if $T(f)$ belongs to $H^\infty(X)$. Therefore $T|_{H^\infty_m(X)}$ is a surjective isometry with respect to $\| \cdot \|_{H^\infty_m}$ from
\( H^{\alpha}(X) \) onto itself. We may suppose that \( H^{\alpha}(X) \) is a uniform algebra on the maximal ideal space and the maximal ideal space is connected by the Silov idempotent theorem; hence, we see that \( T|_{H^{\alpha}(X)} \) is complex-linear or conjugate linear by [7, Theorem].

If \( T|_{H^{\alpha}(X)} \) is complex-linear, then \( T \) is complex-linear on \( M^{p}(X) \), since \( H^{\alpha}(X) \) is dense in \( M^{p}(X) \) and the convergence in the original metric is stronger than uniform convergence on compact subsets of \( X \). Therefore, the first formula of the conclusion holds by Corollary 2.3 in [1].

If \( T|_{H^{\alpha}(X)} \) is conjugate linear, then \( T \) is conjugate linear on \( M^{p}(X) \) as before. Let \( \tilde{T} : M^{p}(X) \to M^{p}(X) \) be defined as \( \tilde{T}(f) = T(f) \) for every \( f \in M^{p}(X) \), where

\[
\tilde{f}(z_1, \ldots, z_n) = f(\overline{z}_1, \ldots, \overline{z}_n)
\]

for \( f \in M^{p}(X) \). Then, \( \tilde{T} \) is complex-linear isometry from \( M^{p}(X) \) onto itself. Applying Corollary 2.3 in [1] to \( \tilde{T} \), the second formula of the conclusion holds.

Let \( 1 \leq p < \infty \). We say a map \( T : M^{p}(X) \to M^{p}(X) \) is multiplicative if \( T(fg) = T(f)T(g) \) for every \( f, g \in M^{p}(X) \). Next, we characterize multiplicative isometries from \( M^{p}(X) \) \((p \in \mathbb{N})\) onto itself. Let \( \Phi \) be a transformation described in Proposition 1. Then, \( T(f) = f \circ \Phi \) defines a complex-linear multiplicative isometry from \( M^{p}(X) \) onto itself, and \( T(f) = f \circ \overline{\Phi} \) defines a conjugate linear multiplicative isometry from \( M^{p}(X) \) onto itself. We show that they are the only multiplicative isometries from \( M^{p}(X) \) onto itself.

**Theorem 2.** Let \( p \in \mathbb{N} \), and let \( T \) be a multiplicative (not necessarily linear) isometry from \( M^{p}(X) \) onto itself. Then, there exists a holomorphic automorphism \( \Phi \) on \( X \) such that either of the following holds:

\[
T(f) = f \circ \Phi \quad \text{forevery} \ f \in M^{p}(X)
\]

\( \quad \text{(18)} \)

or

\[
T(f) = f \circ \overline{\Phi} \quad \text{forevery} \ f \in M^{p}(X)
\]

\( \quad \text{(19)} \)

where \( \Phi \) is a unitary transformation for \( X = B_n \); \( \Phi(z_1, \ldots, z_n) = (\lambda_1z_1, \ldots, \lambda_nz_n) \) for \( X = U^n \), where \( |\lambda_j| = 1 \) for every \( 1 \leq j \leq n \) and \((i_1, \ldots, i_n)\) is some permutation of the integers from 1 through \( n \).

**Proof.** Since \( T \) is multiplicative, we see by the same way as in the proof of Theorem 2.2 in [5] that \( T(1) = 1, T(2) = 2, \) and \( T(1/2) = 1/2 \). Therefore, \( T \) is a surjective isometry which satisfies \( T(2f) = 2T(f) \) as \( T \) is multiplicative. It follows by Proposition 1 that

\[
T(f) = \alpha f \circ \Phi, \quad f \in M^{p}(X)
\]

\( \quad \text{(20)} \)

or

\[
T(f) = \alpha f \circ \overline{\Phi}, \quad f \in M^{p}(X)
\]

\( \quad \text{(21)} \)

holds for a complex number \( \alpha \) and the holomorphic automorphism \( \Phi \) as described in Proposition 1. The constant \( \alpha = 1 \) is observed as \( T(1) = 1 \); hence, the conclusion holds.

**Remark 3.** We note that surjective multiplicative isometries of the class \( M^{p}(X) \) \((p \in \mathbb{N})\) have the same form as surjective multiplicative isometries of the Smirnov class [5, Theorem 2.2] and the Privalov class [8, Corollary 3.4]. The authors do not know whether this result holds for noninteger \( p \).

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