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Research Article

A Minimax Theorem for $\overline{L}^0$-Valued Functions on Random Normed Modules

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We generalize the well-known minimax theorems to $\overline{L}^0$-valued functions on random normed modules. We first give some basic properties of an L^0-valued lower semicontinuous function on a random normed module under the two kinds of topologies, namely, the (ε, λ)-topology and the locally L^0-convex topology. Then, we introduce the definition of random saddle points. Conditions for an L^0-valued function to have a random saddle point are given. The most greatest difference between our results and the classical minimax theorems is that we have to overcome the difficulty resulted from the lack of the condition of compactness. Finally, we, using relations between the two kinds of topologies, establish the minimax theorem of $\overline{L}^0$-valued functions in the framework of random normed modules and random conjugate spaces.

1. Introduction

The classical minimax theorem, which originated from game theory, is an important content of nonlinear analysis. It has been applied in many fields, such as optimization theory, different equations, and fixed point theory. The first mathematical formulation was established by Neumann in 1928 [1]. Since then, various generalizations of Neumann's minimax theorem have been given by several scholars; see [2–7]. The classical minimax theorems for extended real-valued functions $L : A \times B \to \overline{R}$ show that, under some suitable conditions of compactness, convexity, and continuity, the equality

$$\inf_{y\in B}\sup_{x\in A} f(x, y) = \sup_{x\in A}\inf_{y\in B} f(x, y) \tag{1}$$

holds. In 1980s, to meet the needs of vectorial optimization, minimax problems in this more general setting have been investigated; see [4–7]. In this paper, we generalize the well-known minimax theorems to $\overline{L}^0$-valued functions on random normed modules (briefly RN modules).

Random metric theory is based on the idea of randomizing the classical space theory of functional analysis. All the basic notions such as RN modules and random inner product modules (briefly RIP modules) and random locally convex modules (briefly RLC modules) together with their random conjugate spaces were naturally presented by Guo in the course of the development of random functional analysis, (cf. [8–12]). In the last ten years, random metric theory and its applications in the theory of conditional risk measures have undergone a systematic and deep development. Especially after 2009, in [13] Guo gives the relations between the basic results currently available derived from the two kinds of topologies, namely, the (ε, λ)-topology and the locally L^0-convex topology. In [14], Guo gives some basic results on L^0-convex analysis together with some applications to conditional risk measures and studies the relations among the three kinds of conditional convex risk measures. Furthermore, in [15] Guo et al. establish a complete random convex analysis over RN modules and RLC modules by simultaneously considering the two kinds of topologies in order to provide a solid analytic foundation for the module approach to conditional risk measures. These results pave the way for further research of the theory of random convex analysis and conditional risk measures.

Motivated by the recent applications of random metric theory to conditional risk measures [13, 16, 17], in this paper,

we establish a minimax theorem for $\overline{L}^0$-valued functions on random normed modules. Theorem 1, which is the main result of this paper, can be seen as a natural extension of the classical minimax theorems and has potential applications in the further study of conditional risk measures.

To introduce the main result of this paper, let us first recall some notation and terminology as follows:

K: the scalar field R of real numbers or C of complex numbers;

$(\Omega, \mathcal{F}, P)$: a probability space;

$L^0(\mathcal{F}, K)$ = the algebra of equivalence classes of K-valued $\mathcal{F}$-measurable random variables on $(\Omega, \mathcal{F}, P)$;

$L^0(\mathcal{F}) = L^0(\mathcal{F}, R)$;

$\overline{L}^0(\mathcal{F})$ = the set of equivalence classes of extended real-valued $\mathcal{F}$-measurable random variables on $(\Omega, \mathcal{F}, P)$.

Theorem 1. *Let $(E, \|\cdot\|)$ be a random strictly convex and random reflexive random normed module over R with base $(\Omega, \mathcal{F}, P)$, A and B $\mathcal{T}_c$-closed, $L^0(\mathcal{F})$-convex subset with the countable concatenation property of E and $L : E \times E \to \overline{L}^0(\mathcal{F})$. If L satisfies the following:*

(1) *for any fixed $p \in B$, $L(\cdot, p) : E \to \overline{L}^0(\mathcal{F})$ is proper $L^0(\mathcal{F})$-convex, $\mathcal{T}_c$-lower semicontinuous function on A and has the local property;*

(2) *for any fixed $u \in A$, $-L(u, \cdot) : E \to \overline{L}^0(\mathcal{F})$ is proper $L^0(\mathcal{F})$-convex, $\mathcal{T}_c$-lower semicontinuous function on B and has the local property;*

(3) *A and B are a.s. bounded,*

then there exists a random saddle point $(u_0, p_0) \in A \times B$ of L with respect to $A \times B$, namely,

$$\bigwedge_{u\in A} \bigvee_{p\in B} L(u, p) = L(u_0, p_0) = \bigvee_{p\in B} \bigwedge_{u\in A} L(u, p). \tag{2}$$

Theorem 1 has the same shape as the classical minimax theorems, and its proof follows a known pattern in [18], but it is not trivial since the complicated stratification structure in the random setting needs to be considered. Besides, the most greatest difference between our results and the classical minimax theorems is that we have to overcome the difficulty resulted from the lack of the condition of compactness. In order to overcome this obstacle, we make full use of the respective advantages of the (ε, λ)-topology and the locally L^0-convex topology. In [13], Guo pointed out that these two kinds of topologies can complement each other (see also Propositions 14 and 15 in this paper), and we can consider them simultaneously in some cases. Specifically, on one hand, in Theorem 1 we require the functions to be $\mathcal{T}_c$-lower semicontinuous; namely, the functions are lower semicontinuous under the locally L^0-convex topology, because we need a very important inequality to establish this theorem; see Definition 21 and Proposition 22 of this paper for details. On the other hand, in the process of the proof of Theorem 1 we must employ the (ε, λ)-topology also, because the (ε, λ)-topology is very natural from the viewpoint of probability theory, and under this type of topology we can use the relations between random normed modules and classical normed spaces to prove the main result; see the proof of Theorem 1 in Section 4.

The remainder of this paper is organized as follows: in Section 2 we will briefly collect some necessary known facts; in Section 3 we will give some basic properties of an $\overline{L}^0$-valued lower semicontinuous function on a random normed module under the two kinds of topologies, namely, Theorems 26 and 28; in Section 4 we will present the definition of random saddle points and prove our main result.

2. Preliminaries

It is well known from [19] that $\overline{L}^0(\mathcal{F})$ is a complete lattice under the ordering $\leq$: $\xi \leq \eta$ if and only if $\xi^0(\omega) \leq \eta^0(\omega)$, for almost all ω in Ω (briefly, a.s.), where ξ^0 and η^0 are arbitrarily chosen representatives of ξ and η, respectively. Furthermore, every subset G of $\overline{L}^0(\mathcal{F})$ has a supremum, denoted by $\bigvee G$, and an infimum, denoted by $\bigwedge G$. Finally $L^0(\mathcal{F})$, as a sublattice of $\overline{L}^0(\mathcal{F})$, is also a complete lattice in the sense that every subset with upper bound has a supremum. The pleasant properties of $\overline{L}^0(\mathcal{F})$ are summarized as follows.

Proposition 2 (see [19]). *For every subset G of $\overline{L}^0(\mathcal{F})$, there exist countable subsets $\{a_n \mid n \in N\}$ and $\{b_n \mid n \in N\}$ of G such that $\bigvee G = \bigvee_{n\geq 1} a_n$ and $\bigwedge G = \bigwedge_{n\geq 1} b_n$. Further, if G is directed (dually directed) with respect to $\leq$, then the above $\{a_n \mid n \in N\}$ (accordingly, $\{b_n \mid n \in N\}$) can be chosen as nondecreasing (correspondingly, nonincreasing) with respect to $\leq$.*

Specially, $L^0_+ = \{\xi \in L^0(\mathcal{F}) \mid \xi \geq 0\}$, $L^0_{++} = \{\xi \in L^0(\mathcal{F}) \mid \xi > 0 \text{ on } \Omega\}$, where for $A \in \mathcal{F}$, "$\xi > \eta$" on A means $\xi^0(\omega) > \eta^0(\omega)$ a.s. on A for any chosen representatives ξ^0 and η^0 of ξ and η, respectively. As usual, $\xi > \eta$ means $\xi \geq \eta$ and $\xi \neq \eta$. For any $A \in \mathcal{F}$, A^c denotes the complement of A, and $\widetilde{A} = \{B \in \mathcal{F} \mid P(A\Delta B) = 0\}$ denotes the equivalence class of A, where Δ is the symmetric difference operation, I_A is the characteristic function of A, and $\widetilde{I}_A$ is used to denote the equivalence class of I_A; given two ξ and η in $L^0(\mathcal{F})$, and $A = \{\omega \in \Omega \mid \xi^0 \neq \eta^0\}$, where ξ^0 and η^0 are arbitrarily chosen representatives of ξ and η respectively, then we always write $[\xi \neq \eta]$ for the equivalence class of A and $I_{[\xi\neq\eta]}$ for $\widetilde{I}_A$; one can also understand the implication of such notation as $I_{[\xi\leq\eta]}$, $I_{[\xi<\eta]}$ and $I_{[\xi=\eta]}$.

For an arbitrarily chosen representative ξ^0 of $\xi \in L^0(\mathcal{F}, K)$, define the two random variables $(\xi^0)^{-1}$ and $|\xi^0|$ by $(\xi^0)^{-1}(\omega) = 1/\xi^0(\omega)$ if $\xi^0(\omega) \neq 0$, and $(\xi^0)^{-1}(\omega) = 0$ otherwise, and by $|\xi^0|(\omega) = |\xi^0(\omega)|$, for all $\omega \in \Omega$. Then the equivalent class ξ^{-1} of $(\xi^0)^{-1}$ is called the generalized inverse of ξ, and the equivalent class $|\xi|$ of $|\xi^0|$ is called the absolute value of ξ.

Now, we introduce the definition of a random normed module, which is a random generalization of an ordinary normed space, and give some important examples.

Definition 3 (see [11, 20]). An ordered pair $(E, \|\cdot\|)$ is called a random normed space (briefly, an RN space) over K with base $(\Omega, \mathcal{F}, P)$ if E is a linear space over K, and $\|\cdot\|$ is a mapping from E to $L^0_+(\mathcal{F})$ such that the following are satisfied:

(RN-1) $\|\alpha x\| = |\alpha|\|x\|$, for all $\alpha \in K$ and $x \in E$;

(RN-2) $\|x\| = 0$ implies $x = \theta$ (the null element of E);

(RN-3) $\|x + y\| \leq \|x\| + \|y\|$, for all $x, y \in E$.

Here $\|\cdot\|$ is called the random norm on E and $\|x\|$ the random norm of $x \in E$ (if $\|\cdot\|$ only satisfies (RN-1) and (RN-3) above, it is called a random seminorm on E).

Furthermore, if, in addition, E is a left module over the algebra $L^0(\mathcal{F}, K)$ (briefly, an $L^0(\mathcal{F}, K)$-module) such that

(RNM-1) $\|\xi x\| = |\xi|\|x\|$, for all $\xi \in L^0(\mathcal{F}, K)$ and $x \in E$,

then $(E, \|\cdot\|)$ is called a random normed module (briefly, an RN module) over K with base $(\Omega, \mathcal{F}, P)$, and the random norm $\|\cdot\|$ with the property (RNM-1) is also called an L^0-norm on E (a mapping only satisfying (RN-3) and (RNM-1) above is called an L^0-seminorm on E).

Example 4. Let $L^0(\mathcal{F}, B)$ be the $L^0(\mathcal{F}, K)$-module of equivalence classes of $\mathcal{F}$-random variables (or strongly $\mathcal{F}$-measurable functions) from $(\Omega, \mathcal{F}, P)$ to a normed space $(B, \|\cdot\|)$ over K. $\|\cdot\|$ induces an L^0-norm (still denoted by $\|\cdot\|$) on $L^0(\mathcal{F}, B)$ by $\|x\|$:= the equivalence class of $\|x^0(\cdot)\|$ for all $x \in L^0(\mathcal{F}, B)$, where $x^0(\cdot)$ is a representative of x. Then $(L^0(\mathcal{F}, B), \|\cdot\|)$ is an RN module over K with base $(\Omega, \mathcal{F}, P)$. Specially, $L^0(\mathcal{F}, K)$ is an RN module, and the L^0-norm $\|\cdot\|$ on $L^0(\mathcal{F}, K)$ is still denoted by $|\cdot|$.

The next example of RN modules $L^p_{\mathcal{F}}(\mathcal{E})(1 \leq p \leq +\infty)$ is constructed by Filipović et al. in [16].

Example 5. Let $(\Omega, \mathcal{E}, P)$ be a probability space and $\mathcal{F}$ a σ-subalgebra of $\mathcal{E}$. Define $|||\cdot|||_p : L^0(\mathcal{E}) \to \overline{L}^0_+(\mathcal{F})$ by

$$|||x|||_p = \begin{cases} E[|x|^p \mid \mathcal{F}]^{1/p}, & \text{when } 1 \leq p < \infty, \\ \bigwedge \left\{\xi \in \overline{L}^0_+(\mathcal{F}) \mid |x| \leq \xi\right\}, & \text{when } p = +\infty, \end{cases} \tag{3}$$

for all $x \in L^0(\mathcal{E})$.

Denote $L^p_{\mathcal{F}}(\mathcal{E}) = \{x \in L^0(\mathcal{E}) | |||x|||_p \in L^0_+(\mathcal{F})\}$, then $(L^p_{\mathcal{F}}(\mathcal{E}), |||\cdot|||_p)$ is an RN module over R with base $(\Omega, \mathcal{F}, P)$ and $L^p_{\mathcal{F}}(\mathcal{E}) = L^0(\mathcal{F}) \cdot L^p(\mathcal{E}) = \{\xi x \mid \xi \in L^0(\mathcal{F})$ and $x \in L^p(\mathcal{E})\}$.

To put some important classes of stochastic processes into the framework of RN modules, Guo constructed a more general RN module $L^p_{\mathcal{F}}(S)$ in [13] for each $p \in [1, +\infty]$ as follows.

Example 6. Let $(E, \|\cdot\|)$ be an RN module over K with base $(\Omega, \mathcal{E}, P)$ and $\mathcal{F}$ a σ-subalgebra. Define $|||\cdot|||_p : E \to \overline{L}^0_+(\mathcal{F})$ by

$$|||\cdot|||_p = \begin{cases} E[\|x\|^p \mid \mathcal{F}]^{1/p}, & \text{when } 1 \leq p < \infty, \\ \bigwedge \left\{\xi \in \overline{L}^0_+(\mathcal{F}) \mid \|x\| \leq \xi\right\}, & \text{when } p = +\infty, \end{cases} \tag{4}$$

for all $x \in E$.

Denote $L^p_{\mathcal{F}}(E) = \{x \in S \mid |||x|||_p \in L^0_+(\mathcal{F})\}$; then $(L^p_{\mathcal{F}}(E), |||\cdot|||_p)$ is an RN module over K with base $(\Omega, \mathcal{F}, P)$. When $E = L^0(\mathcal{E})$, $L^p_{\mathcal{F}}(E)$ is exactly $L^p_{\mathcal{F}}(\mathcal{E})$.

Remark 7. For a given RN module $(E, \|\cdot\|)$ over K with base (Ω, F, P) and a given real or extended real number p such that $1 \leq p \leq +\infty$, define $\|\cdot\|_p : E \to [0, +\infty]$ by

$$\|g\|_p = \begin{cases} \left(\int_\Omega (\|g\|)^p\right)^{1/p}, & \text{if } 1 \leq p < +\infty, \\ \text{the } P\text{-essential supremum}, & \text{if } p = +\infty. \end{cases} \tag{5}$$

Let $L^p(E) = \{g \in E : \|g\|_p < +\infty\}$. As mentioned in [13], $(L^p(E), \|\cdot\|_p)$ is a normed space over K and is further a Banach space if $(E, \|\cdot\|)$ is complete.

For each RN module $(E, \|\cdot\|)$ over K with base $(\Omega, \mathcal{F}, P)$, $\|\cdot\|$ can induce two kinds of topologies, namely, the (ε, λ)-topology and the locally L^0-convex topology.

Definition 8 (see [12–14]). Let $(E, \|\cdot\|)$ be an RN module over K with base $(\Omega, \mathcal{F}, P)$. For any positive real numbers ε and λ such that $0 < \lambda < 1$, let $N_\theta(\varepsilon, \lambda) = \{x \in E \mid P\{\omega \in \Omega \mid \|x\|(\omega) < \varepsilon\} > 1 - \lambda\}$; then $\{N_\theta(\varepsilon, \lambda) \mid \varepsilon > 0, 0 < \lambda < 1\}$ is easily verified to be a local base at the null vector θ of some Hausdorff linear topology. The linear topology is called the (ε, λ)-topology for E induced by $\|\cdot\|$.

From now on, the (ε, λ)-topology for each RN module is always denoted by $\mathcal{T}_{\varepsilon,\lambda}$ when no confusion occurs.

Proposition 9 (see [12–14]). *Let $(E, \|\cdot\|)$ be an RN module over K with base $(\Omega, \mathcal{F}, P)$. Then one has the following statements.*

(1) *The (ε, λ)-topology for $L^0(\mathcal{F}, K)$ is exactly the topology of convergence in probability P, and $(L^0(\mathcal{F}, K), \mathcal{T}_{\varepsilon,\lambda})$ is a topological algebra over K.*

(2) *If $(E, \|\cdot\|)$ is an RN modules, then $(E, \mathcal{T}_{\varepsilon,\lambda})$ is a topological module over the topological algebra $L^0(F, K)$.*

(3) *A net $\{x_\delta, \delta \in \Gamma\}$ converges in the (ε, λ)-topology to some x in E if and only if $\{\|x_\delta - x\|, \delta \in \Gamma\}$ converges in probability P to 0.*

The following locally L^0-convex topology is easily seen to be much stronger than the (ε, λ)-topology and was first introduced by Filipović et al. in [16].

Definition 10 (see [14, 16]). Let $(E, \|\cdot\|)$ be an RN module over K with base $(\Omega, \mathcal{F}, P)$. For any $\varepsilon \in L^0_{++}$, let $N_\theta(\varepsilon) = \{x \in E \mid \|x\| \leq \varepsilon\}$. A subset G of E is called $\mathcal{T}_c$-open if for each $x \in G$ there exists some $N_\theta(\varepsilon)$ such that $x + N_\theta(\varepsilon) \subset G$, and $\mathcal{T}_c$ denotes the family of $\mathcal{T}_c$-open subsets of E. Then it is easy to see that $(E, \mathcal{T}_c)$ is a Hausdorff topological group with respect to the addition on E. $\mathcal{T}_c$ is called the locally L^0-convex topology for E induced by $\|\cdot\|$.

From now on, the locally L^0-convex topology for each random locally convex space is always denoted by $\mathcal{T}_c$ when no confusion occurs.

Proposition 11 (see [13, 14, 16]). *Let $(E, \|\cdot\|)$ be an RN module over K with base $(\Omega, \mathcal{F}, P)$. Then*

(1) *$L^0(\mathcal{F}, K)$ is a topological ring endowed with its locally L^0-convex topology;*

(2) *E is a topological module over the topological ring $L^0(\mathcal{F}, K)$ when E and $L^0(\mathcal{F}, K)$ are endowed with their respective locally L^0-convex topologies;*

(3) *a net $\{x_\alpha \mid \alpha \in \Gamma\}$ in E converges in the locally L^0-convex topology to $x \in E$ if and only if $\{\|x_\alpha - x\| \mid \alpha \in \Gamma\}$ converges in the locally L^0-convex topology of $L^0(\mathcal{F}, K)$ to 0.*

$\mathcal{T}_c$ is called locally L^0-convex because it has a striking local base $\mathcal{U}_\theta = \{B(\varepsilon) \mid \varepsilon \in L^0_{++}\}$, each member U of which is as follows:

(i) L^0-convex: $\xi \cdot x + (1 - \xi) \cdot y \in U$ for any $x, y \in U$ and $\xi \in L^0_+$ such that $0 \leqslant \xi \leqslant 1$;

(ii) L^0-absorbent: there is $\xi \in L^0_{++}$ for each $x \in E$ such that $x \in \xi \cdot U$;

(iii) L^0-balanced: $\xi \cdot x \in U$ for any $x \in U$ and any $\xi \in L^0(F, K)$ such that $|\xi| \leqslant 1$.

Remark 12. Let $(E, \|\cdot\|)$ be an RN module over K with base $(\Omega, \mathcal{F}, P)$ endowed with the locally L^0-convex topology $\mathcal{T}_c$. Although E can be viewed as a linear space over K with scalar multiplication $\alpha \cdot x := (\alpha \cdot 1) \cdot x$ for $\alpha \in K$, $1 \in L^0$ and $x \in E$, $(E, \mathcal{T}_c)$ is not a topological linear space since the map $K \to (E, \mathcal{T}_c)$, $\alpha \to \alpha \cdot x$, is not necessarily continuous for $x \neq \theta$; see [16] for details.

In the sequel of this paper, for a subset G of an RN module $(E, \|\cdot\|)$, $\overline{G}_{\varepsilon,\lambda}$ denotes the $\mathcal{T}_{\varepsilon,\lambda}$-closure of G, and $\overline{G}_c$ denotes the $\mathcal{T}_c$-closure of G.

For giving the relations of the two kinds of topologies, which Guo has studied the [13], we need to introduce the definition of the countable concatenation property.

Definition 13 (see [13]). Let E be a left module over the algebra $L^0(\mathcal{F}, K)$. A formal sum $\Sigma_{n\geq 1} \widetilde{I}_{A_n} x_n$ for some countable partition $\{A_n, n \in N\}$ of Ω to $\mathcal{F}$ and some sequence $\{x_n \mid n \in N\}$ in E is called a countable concatenation of $\{x_n \mid n \in N\}$ with respect to $\{A_n, n \in N\}$. Furthermore a countable concatenation $\Sigma_{n\geq 1} \widetilde{I}_{A_n} x_n$ is well defined or $\Sigma_{n\geq 1} \widetilde{I}_{A_n} x_n \in E$ if there is $x \in E$ such that $\widetilde{I}_{A_n} x = \widetilde{I}_{A_n} x_n$, for all $n \in N$. A subset G of E is said to have the countable concatenation property if every countable concatenation $\Sigma_{n\geq 1} \widetilde{I}_{A_n} x_n$ with $x_n \in G$ for each $n \in N$ still belongs to G; namely, $\Sigma_{n\geq 1} \widetilde{I}_{A_n} x_n$ is well defined and there exists $x \in G$ such that $x = \Sigma_{n\geq 1} \widetilde{I}_{A_n} x_n$.

Proposition 14 (see [13]). *Let $(E, \|\cdot\|)$ be an RN module over K with base $(\Omega, \mathcal{F}, P)$. Then E is $\mathcal{T}_{\varepsilon,\lambda}$-complete if and only if E is $\mathcal{T}_c$-complete and has the countable concatenation property.*

Proposition 15 (see [13]). *Let $(E, \|\cdot\|)$ be an RN module over K with base $(\Omega, \mathcal{F}, P)$ and $G \subset E$ a subset with the countable concatenation property. Then $\overline{G}_{\varepsilon,\lambda} = \overline{G}_c$.*

Now, we introduce the definition of random conjugate spaces of RN modules.

Definition 16 (see [7, 10, 11, 13]). Let $(E, \|\cdot\|)$ be an RN module over K with base $(\Omega, \mathcal{F}, P)$. A linear operator f from E to $L^0(\mathcal{F}, K)$ is said to be an a.s. bounded random linear functional on E if there exists some ξ in $L^0_+(\mathcal{F}, R)$ such that $|f(x)| \leq \xi \cdot \|x\|$, for all $x \in E$. Denote by E^* the linear space of a.s. bounded random linear functionals on E with the pointwise addition and scalar multiplication on linear operators; define $\|\cdot\|^* : E^* \to L^0_+(\mathcal{F}, R)$ by $\|f\|^* = \bigwedge\{\xi \in L^0_+(\mathcal{F}) \mid |f(x)| \leq \xi\|x\|, \text{for all } x \in E\}$ for all $f \in E^*$ and define $\cdot : L^0(\mathcal{F}, K) \times E^* \to E^*$ by $(\eta \cdot f)(x) = \eta(f(x))$ for all $\eta \in L^0(\mathcal{F}, K)$, $f \in E^*$, and $x \in E$; then it is easy to check that $(E^*, \|\cdot\|^*)$ is also an RN module over K with base $(\Omega, \mathcal{F}, P)$, called the random conjugate space of $(E, \|\cdot\|)$.

Guo et al. gave the topological characterizations of an a.s. bounded random linear functional in [10, 11, 16] as follows: let $(E, \|\cdot\|)$ be an RN module over K with base $(\Omega, \mathcal{F}, P)$, $E^*_{\varepsilon,\lambda}$ the $L^0(\mathcal{F}, K)$-module of continuous module homomorphisms from $(E, \mathcal{T}_{\varepsilon,\lambda})$ to $(L^0(\mathcal{F}, K), \mathcal{T}_{\varepsilon,\lambda})$, and E^*_c the $L^0(F, K)$-module of continuous module homomorphisms from $(E, \mathcal{T}_c)$ to $(L^0(F, K), \mathcal{T}_c)$, then it was proved that $E^*_{\varepsilon,\lambda} = E^*_c$. In fact, Guo et al. also proved in [10, 11, 13] $\|f\|^* = \bigvee\{|f(x)| \mid x \in E \text{ and } \|x\| \leq 1\}$ for any $f \in E$.

Let $(E, \|\cdot\|)$ be an RN module, E^{**} denotes $(E^*)^*$, and the canonical embedding mapping $J : E \to E^{**}$ defined by $(Jx)(f) = f(x)$, for all $x \in E$ and for all $f \in E^*$, is random-norm preserving. If J is subjective, then E is called random reflexive. In [13] Guo proved that the random reflexivity is independent of a special choice of $\mathcal{T}_{\varepsilon,\lambda}$ and $\mathcal{T}_c$. The following propositions are very essential relations, which are established by Guo in [12, 21], between classical reflexive spaces and random reflexive RN modules.

Proposition 17 (see [21]). *$L^0(\mathcal{F}, B)$ is random reflexive if and only if B is a reflexive Banach space.*

Proposition 18 (see [12]). *Let $(E, \|\cdot\|)$ be an RN module over K with base $(\Omega, \mathcal{F}, P)$. Then E is random reflexive if and only if $(L^p(E), \|\cdot\|_p)$ is reflexive, where $1 < p < +\infty$.*

Proposition 19 (see [12]). *Let $(E, \|\cdot\|)$ be an RN module over K with base $(\Omega, \mathcal{F}, P)$, $1 \le p < +\infty$ and $1 < q \le +\infty$ a pair of Hölder conjugate numbers. Then $(L^q(E^*), \|\cdot\|_q)$ is isometrically isomorphic with the classical conjugate space of $(L^p(E), \|\cdot\|)$, denoted by $(L^p(E))'$, under the canonical mapping $T : L^q(E^*) \to (L^p(E))'$ defined as follows. For each $f \in L^q(E^*)$, T_f (denoting $T(f)$): $L^p(E) \to K$ is defined by $T_f(g) = \int_\Omega f(g)dP$ for all $g \in L^p(E)$.*

3. Some Basic Properties of $\overline{L}^0$-Valued Lower Semicontinuous Functions

In this section, we give some basic properties of $\overline{L}^0$-valued lower semicontinuous functions. First, we recall the definition of $\overline{L}^0$-valued lower semicontinuous functions under two kinds of topologies, which was presented by Guo in [14] for the first time.

Let E be a left module over the algebra $L^0(\mathcal{F})$. The effective domain of function $f : E \to \overline{L}^0(\mathcal{F})$ is denoted by $\mathrm{dom}(f) := \{x \in E \mid f(x) \in L^0(\mathcal{F})\}$. The epigraph of f is denoted by $\mathrm{epi}(f) := \{(x, y) \in E \times L^0(\mathcal{F}) \mid f(x) \le y\}$. The function f is called proper if $f(x) > -\infty$ on Ω for every $x \in E$ and $\mathrm{dom}(f) \ne \emptyset$.

Definition 20 (see [14]). Let E be a left module over the algebra $L^0(\mathcal{F})$ and $f : E \to \overline{L}^0(\mathcal{F})$.

(1) f is $L^0(\mathcal{F})$-convex if $f(\xi x + (1-\xi)y) \le \xi f(x) + (1-\xi)f(y)$ for all x and y in E and $\xi \in L^0_+$ such that $0 \le \xi \le 1$ (here we make the convention that $0 \cdot (\pm\infty) = 0$ and $\infty - \infty = \infty$).

(2) f has the local property if $\tilde{I}_A f(x) = \tilde{I}_A f(\tilde{I}_A x)$ for all $x \in E$ and $A \in \mathcal{F}$.

(3) f is regular if $\tilde{I}_A f(x) = f(\tilde{I}_A x)$ for all $x \in E$ and $A \in \mathcal{F}$.

Definition 21 (see [14]). Let $(E, \|\cdot\|)$ be an RN module over R with base $(\Omega, \mathcal{F}, P)$. A function $f : E \to \overline{L}^0(\mathcal{F})$ is called $\mathcal{T}_c$-lower semicontinuous if $\mathrm{epi}(f)$ is closed in $(E, \mathcal{T}_c) \times (L^0(\mathcal{F}), \mathcal{T}_c)$. A function $f : E \to \overline{L}^0(\mathcal{F})$ is called $\mathcal{T}_{\varepsilon,\lambda}$-lower semicontinuous if $\mathrm{epi}(f)$ is closed in $(E, \mathcal{T}_{\varepsilon,\lambda}) \times (L^0(\mathcal{F}), \mathcal{T}_{\varepsilon,\lambda})$.

Proposition 22 (see [14]). *Let $(E, \|\cdot\|)$ be an RN module over R with base $(\Omega, \mathcal{F}, P)$ such that E has the countable concatenation property and $f : E \to \overline{L}^0(\mathcal{F})$ a function with the local property. Then the following are equivalent to each other:*

(1) *f is $\mathcal{T}_c$-lower semicontinuous;*

(2) *$\{x \in E \mid f(x) \le r\}$ is $\mathcal{T}_c$-closed for each $r \in L^0(\mathcal{F})$;*

(3) *$\underline{\lim}_\alpha f(x_\alpha) \ge f(x_0)$ for each $x_0 \in E$ and each net $\{x_\alpha, \alpha \in \Gamma\}$ in E such that $\{x_\alpha, \alpha \in \Gamma\}$ is $\mathcal{T}_c$-convergent to x_0, where $\underline{\lim}_\alpha f(x_\alpha) = \bigvee_{\alpha\in\Gamma}(\bigwedge_{\beta\ge\alpha} f(x_\beta))$.*

Remark 23. Proposition 22 first occurred in [16] where the countable concatenation property of E was not assumed, but this condition should be added (see [14] for details).

For $\mathcal{T}_{\varepsilon,\lambda}$-lower semicontinuous functions, we only have the following proposition.

Proposition 24 (see [14]). *Let $(E, \|\cdot\|)$ be an RN module over R with base $(\Omega, \mathcal{F}, P)$ and $f : E \to \overline{L}^0(\mathcal{F})$ a function. Then one has the following statements:*

(1) *f is $\mathcal{T}_{\varepsilon,\lambda}$-lower semicontinuous if $\underline{\lim}_\alpha f(x_\alpha) \ge f(x_0)$ for each $x_0 \in E$ and each net $\{x_\alpha, x \in \Gamma\}$ in E such that $\{x_\alpha, \alpha \in \Gamma\}$ is $\mathcal{T}_{\varepsilon,\lambda}$-convergent to x_0;*

(2) *$\{x \in E \mid f(x) \le r\}$ is $\mathcal{T}_{\varepsilon,\lambda}$-closed for each $r \in L^0(\mathcal{F})$ if f is $\mathcal{T}_{\varepsilon,\lambda}$-lower semicontinuous.*

If we define f to be lower semicontinuous via "$\underline{\lim}_\alpha f(x_\alpha) \ge f(x)$ for all net $\{x_\alpha, \alpha \in \Lambda\}$ in E such that it converges in the (ε, λ)-topology to some $x \in E$", the notion is, however, meaningless in the random setting, since we can construct an RN module E and a $\mathcal{T}_{\varepsilon,\lambda}$-continuous L^0-convex function f from E to $L^0(\mathcal{F})$, whereas f is not a $\mathcal{T}_{\varepsilon,\lambda}$-lower semicontinuous function. Hence, we cannot use this inequality for $\mathcal{T}_{\varepsilon,\lambda}$-lower semicontinuous functions. Since this inequality is very important for the proof of Theorem 1 (see Section 4 for details), we can only establish $\overline{L}^0$-valued minimax theorems for $\mathcal{T}_c$-lower semicontinuous functions.

Proposition 25 (see [14]). *Let $(E, \|\cdot\|)$ be an RN module over K with base $(\Omega, \mathcal{F}, P)$ such that E has the countable concatenation property and $f : E \to \overline{L}^0(\mathcal{F})$ a function with the local property. Then f is $\mathcal{T}_{\varepsilon,\lambda}$-lower semicontinuous if and only if f is $\mathcal{T}_c$-lower semicontinuous, specially this is true for an $L^0(\mathcal{F})$-convex function f.*

Now, we give some important properties of $\overline{L}^0$-valued lower semicontinuous functions on RN modules. To pave the way for Theorem 26, we first introduce some notation: if $(E, \|\cdot\|)$ is an RN module over K with base $(\Omega, \mathcal{F}, P)$, Π denotes the set of all probability measures equivalent to P on $(\Omega, \mathcal{F})$, $L^p_Q(E) = \{x \in E \mid \int_\Omega \|x\|^p dQ < +\infty\}$, where $Q \in \Pi$, and $\|\cdot\|^Q_p$ denotes the norm on $L^p_Q(E)$, namely, $\|x\|^Q_p = (\int_\Omega \|x\|^p dQ)^{1/p}$ for any $x \in L^p_Q(E)$.

Theorem 26. *Let $(E, \|\cdot\|)$ be a random reflexive RN module over R with base $(\Omega, \mathcal{F}, P)$, $G \subset E$ a $\mathcal{T}_c$-closed, $L^0(\mathcal{F})$-convex, and a.s. bounded set with the countable concatenation property and $f : E \to \overline{L}^0(\mathcal{F})$ a $\mathcal{T}_c$-lower semicontinuous function with the local property. If $f|_G$ is proper, then f is bounded from below on G.*

Proof. Let $\eta = \vee\{\|x\| : x \in G\}$; then it is easy to see that $\eta \in L^0_+$. If f is not bounded from below on G, then there exists $B \in \mathscr{F}$ such that $P(B) > 0$ and

$$\bigwedge \{f(x) \mid x \in G\} = -\infty \tag{6}$$

on B. Since G is $L^0(\mathscr{F})$-convex and f has the local property, it is easy to see that $\{f(x) \mid x \in G\}$ is directed. Hence there exists a sequence $\{x_n, n \in N\}$ such that $\{f(x_n), n \in N\} \searrow \bigwedge\{f(x) \mid x \in G\}$. Let $G_n = \{x \in E \mid f(x) \le f(x_n)\} \bigcap G$; then it is clear that G_n is $\mathscr{T}_c$-closed by Definition 21. Since f has the local property and G has the countable concatenation property, for any $n \in N$, we have that G_n has the countable concatenation property and G_n is $\mathscr{T}_{\varepsilon,\lambda}$-closed by Proposition 15. By the fact that G is a.s. bounded, we can define a probability measure Q on $(\Omega, \mathscr{F})$ by $dQ/dP = 1/c(1+\eta)^2$, where $c = E[1/(1+\eta)^2]$. Then Q is equivalent to P and $\int_\Omega \|x\|^2 dQ \le \int_\Omega (\eta^2/c(1+\eta)^2) dP < +\infty$, for any $x \in G$, which means that G is bounded in $(L^2_Q(E), \|\cdot\|^Q_2)$. Noting that replacing the probability measure P of the base space $(\Omega, \mathscr{F}, P)$ with a probability measure Q does not change the (ε, λ)-topology of E, for any given $n \in N$, we can obtain that G_n is norm-closed and convex in $(L^2_Q(E), \|\cdot\|^Q_2)$. Since $(E, \|\cdot\|)$ is a random reflexive RN module, we have that $(L^2_Q(E), \|\cdot\|^Q_2)$ is reflexive normed space from Proposition 18. Hence G_n is compact under the weak topology of the normed space $L^2_Q(E)$. Let $\mathscr{O} = \{G_n \mid n \in N\}$, then one can obtain that $\mathscr{O}$ has the finite intersection property and $\bigcap \mathscr{O} \ne \emptyset$. Let $x^* \in \bigcap \mathscr{O}$; then

$$f(x^*) = -\infty \tag{7}$$

on B, which it contradicts to the fact that $f|_G$ is proper. □

For giving Theorem 28, we need to introduce the following Proposition 27, which was established by Guo and Yang in [22] for studying Ekeland's variational principle for $\overline{L}^0$-valued functions on RN modules.

Proposition 27 (see [22]). *Let $(E, \|\cdot\|)$ be an RN module over R with base $(\Omega, \mathscr{F}, P)$, $G \subset E$ a subset with the countable concatenation property and $f : E \to \overline{L}^0(\mathscr{F})$ have the local property. If $f|_G$ is proper and bounded from below on G (resp., bounded from above on G), then for each $\varepsilon \in L^0_{++}(\mathscr{F})$, there exists $x_\varepsilon \in G$ such that $f(x_\varepsilon) \le \bigwedge f(G) + \varepsilon$ (accordingly, $f(x_\varepsilon) \ge \bigvee f(G) - \varepsilon$).*

Theorem 28. *Let $(E, \|\cdot\|)$ be a random reflexive RN module over R with base $(\Omega, \mathscr{F}, P)$, $G \subset E$ a $\mathscr{T}_c$-closed, $L^0(\mathscr{F})$-convex and a.s. bounded set with the countable concatenation property and $f : E \to \overline{L}^0(\mathscr{F})$ a $\mathscr{T}_c$-lower semicontinuous and $L^0(\mathscr{F})$-convex function with the local property. If $f|_G$ is proper, then there exists $x^* \in G$ such that $f(x^*) = \bigwedge f(G)$.*

Proof. Let $\xi = \bigvee\{\|x\| \mid x \in G\}$ and $\eta = \bigwedge f(G)$. It is clear that $\xi \in L^0_+$ and $\eta \in L^0(\mathscr{F})$ by Theorem 26. Take $B_j = [j-1 \le \xi < j]$, for all $j \in N$, then $\{B_j \mid j = 1, 2, \ldots\}$ is a countable partition of Ω to $\mathscr{F}$. For any $j \in N$; define a function $f_j : \tilde{I}_{B_j} \cdot E \to \overline{L}^0(\mathscr{F})$ as follows:

$$f_j\left(\tilde{I}_{B_j} x\right) = \tilde{I}_{B_j} f(x), \quad \forall x \in E. \tag{8}$$

Since f has the local property, for any $j \in N$, we have that $f_j(\tilde{I}_{B_j} x) = \tilde{I}_{B_j} f(\tilde{I}_{B_j} x)$ and $\tilde{I}_{B_j} \cdot \bigwedge f(G) = \bigwedge f_j(\tilde{I}_{B_j} \cdot G)$. Because $f|_G$ is proper and f is $\mathscr{T}_c$-lower semicontinuous and $L^0(\mathscr{F})$-convex, it is clear that $f_j|_{\tilde{I}_{B_j} \cdot G}$ is proper, $\mathscr{T}_c$-lower semicontinuous, and $L^0(\mathscr{F})$-convex. Next, we prove that f_j has the local property. We need only to prove that

$$\tilde{I}_A f_j\left(\tilde{I}_A \tilde{I}_{B_j} x\right) = \tilde{I}_A f_j\left(\tilde{I}_{B_j} x\right), \quad \forall A \in \mathscr{F} \bigcap B_j. \tag{9}$$

In fact, since f has the local property, for any $A \in \mathscr{F} \bigcap B_j$, we have that

$$\begin{aligned} \tilde{I}_A f_j\left(\tilde{I}_{B_j} x\right) &= \tilde{I}_A \tilde{I}_{B_j} f(x) = \tilde{I}_A \tilde{I}_{B_j} f\left(\tilde{I}_{B_j} x\right) \\ &= \tilde{I}_A \tilde{I}_{B_j} f\left(\tilde{I}_A \tilde{I}_{B_j} x\right) \\ &= \tilde{I}_A f_j\left(\tilde{I}_A \tilde{I}_{B_j} x\right). \end{aligned} \tag{10}$$

Let $\eta_j = \tilde{I}_{B_j} \cdot \eta$ for any $j \in N$. It is easy to see that $\tilde{I}_j G \subseteq (L^2(E), \|\cdot\|_2)$ is a bounded and convex set. Since $\tilde{I}_{B_j} G$ is $L^0(\mathscr{F})$-convex, $\mathscr{T}_c$-closed, and a.s. bounded in E and has the countable concatenation, we can obtain that $\tilde{I}_{B_j} G$ is $\mathscr{T}_{\varepsilon,\lambda}$-closed in E by Proposition 15. It is easy to see that $\tilde{I}_{B_j} G$ is convex and $\|\cdot\|_2$-closed in $(L^2(E), \|\cdot\|_2)$ from the fact that the topology induced by $\|\cdot\|_2$ is stronger than the (ε, λ)-topology. Since $(E, \|\cdot\|)$ is random reflexive, $(L^2(E), \|\cdot\|_2)$ is reflexive normed space, and $\tilde{I}_{B_j} G$ is compact in $L^2(E)$ under the weak topology of $L^2(E)$. For any $\varepsilon \in L^0_{++}$, define $G_j(\varepsilon) = \{\tilde{I}_{B_j} x \mid f_j(\tilde{I}_{B_j} x) \le \tilde{I}_{B_j} \eta + \varepsilon, x \in G\}$. It is clear that $G_j(\varepsilon) \ne \emptyset$ by Proposition 27. Since $f_j|_{\tilde{I}_{B_j} \cdot G}$ is $\mathscr{T}_c$-lower semicontinuous, we have that $G_j(\varepsilon)$ is $\mathscr{T}_c$-closed. Thus, we have that $G_j(\varepsilon)$ is $L^0(\mathscr{F})$-convex and $\|\cdot\|_2$-closed in $L^2(E)$ by the fact that f_j has the local property and Proposition 15. By Hahn-Banach theorem, we have that $G_j(\varepsilon)$ is closed under the weak topology of $L^2(E)$. Take

$$\mathscr{O} = \left\{G_j(\varepsilon) \mid \varepsilon \in L^0_{++}\right\}, \tag{11}$$

it is easy to prove that $\mathscr{O}$ has the finite intersection property. Since $\tilde{I}_{B_j} G$ is compact under the weak topology of $L^2(E)$, we have that $\bigcap \mathscr{O} \ne \emptyset$. Let $x_j \in \bigcap \mathscr{O}$ for any $j \in N$ and

$$x^* = \sum_{j=1}^{\infty} \tilde{I}_{B_j} \cdot x_j. \tag{12}$$

We have that $f_j(x_j) = \eta_j$ and

$$f(x^*) = \bigwedge f(G). \tag{13}$$

This completes the proof. □

Definition 29. Let $(E, \|\cdot\|)$ be an RN module over R with base $(\Omega, \mathcal{F}, P)$, and G a $L^0(\mathcal{F})$-convex subset in E. $f : G \to \overline{L}^0(\mathcal{F})$ is called strictly $L^0(\mathcal{F})$-convex if

$$f(\alpha x + (1-\alpha) y) < \alpha f(x) + (1-\alpha) f(y), \quad (14)$$

for any $x, y \in G$, $x \neq y$ and $0 < \alpha < 1$ on Ω.

Corollary 30. *Let $(E, \|\cdot\|)$ be a random reflexive RN module over R with base $(\Omega, \mathcal{F}, P)$, $G \subset E$ a $\mathcal{T}_c$-closed, $L^0(\mathcal{F})$-convex and a.s. bounded set with the countable concatenation property, and $f : E \to \overline{L}^0(\mathcal{F})$ a $\mathcal{T}_c$-lower semicontinuous and strictly $L^0(\mathcal{F})$-convex function with the local property. If $f|_G$ is proper, then there exists an unique $x^* \in G$ such that $f(x^*) = \bigwedge f(G)$.*

4. Main Results

Now, we give the definition of random saddle points.

Definition 31. Let A and B be any two nonempty sets, $(u_0, p_0) \in A \times B$ and $L : A \times B \to \overline{L}^0(\mathcal{F})$. Then (u_0, p_0) is called a random saddle point of f with respect to $A \times B$ if

$$L(u_0, p) \leq L(u_0, p_0) \leq L(u, p_0), \quad \forall u \in A, \ p \in B. \quad (15)$$

Remark 32. Let A and B be any two nonempty sets, $L : A \times B \to \overline{L}^0(\mathcal{F})$. It is easy to see that the following statements are equivalent:

(1) $(u_0, p_0) \in A \times B$ is a random saddle point of f with respect to $A \times B$;

(2) $\bigwedge_{u\in A} \bigvee_{p\in B} L(u, p) \leq \bigvee_{p\in B} \bigwedge_{u\in A} L(u, p)$;

(3) $\bigwedge_{u\in A} \bigvee_{p\in B} L(u, p) = L(u_0, p_0) = \bigvee_{p\in B} \bigwedge_{u\in A} L(u, p)$.

Before giving the proof of main result in this paper, we first recall the definition of random strictly convex RN module, which is presented by Guo and Zeng in [23] for the first time.

Definition 33 (see [23]). An RN module $(E, \|\cdot\|)$ is said to be random strictly convex if for any x and $y \in E \setminus \{\theta\}$ such that $\|x + y\| = \|x\| + \|y\|$, then there exist $A \in \mathcal{F}$ and $\xi \in L^0_+$ such that $P(A) > 0$, $\xi > 0$ on A and $\widetilde{I}_A x = \xi(\widetilde{I}_A y)$.

Definition 34 (see [24]). Let $(E, \|\cdot\|)$ be an RN module over R with base $(\Omega, \mathcal{F}, P)$, $x, y \in E$ and $F \in \mathcal{F}$. Then x and y are called L^0-independent on F if $\xi\widetilde{I}_F = \eta\widetilde{I}_F = 0$ whenever $\xi, \eta \in L^0(\mathcal{F})$ such that $\xi\widetilde{I}_F x + \eta\widetilde{I}_F y = \theta$.

By Definitions 33 and 34, we can obtain the following lemma easily.

Lemma 35. *Let $(E, \|\cdot\|)$ be a random strictly convex RN module over R with base $(\Omega, \mathcal{F}, P)$. Then the mapping $f : E \to L^0(\mathcal{F})$*

$$f(x) = \|x\|^2 \quad (16)$$

is strictly $L^0(\mathcal{F})$-convex.

For giving the proof of Theorem 1, we need the following lemma and remark.

Lemma 36 (Mazur lemma). *Let $(X, \|\cdot\|)$ be a normed space, $\{u_n \in X, n \in N\}$ converge to $\overline{u}$ under the weak topology on X. Then there exists a sequence $\{v_n \in X, n \in N\}$ such that it converges to $\overline{u}$ in norm, where*

$$v_n = \sum_{k=n}^{N_n} \lambda_k u_k, \qquad \sum_{k=n}^{N_n} \lambda_k = 1, \quad \lambda_k \geq 0. \quad (17)$$

Remark 37. Let $\{\xi_n \in L^0(\mathcal{F}), n \in N\}$ converge to ξ uniformly. Then we can obtain a net $\{\xi_\varepsilon \in L^0(\mathcal{F}) \mid \varepsilon \in L^0_{++}, \varepsilon \leq 1\}$ such that it converges to ξ under the locally L^0-convex topology of $(L^0(\mathcal{F}, R), |\cdot|)$. In fact, for any $\varepsilon \in L^0_{++}$ let $A_1 = [\varepsilon > 1]$, $A_i = [1/(i+1) < \varepsilon \leq 1/i]$, for all $i \in N$. Then $\{A_n, n \in N\}$ is a countable partition of Ω to $\mathcal{F}$. Since the sequence $\{\xi_n, n \in N\}$ converges to ξ uniformly, thus for any number $k > 0$, there exists $N(k) \in N$ such that

$$|\xi_n - \xi| < \frac{1}{k} \quad (18)$$

for any $n > N(k)$. Let

$$\xi_\varepsilon = \sum_{i=1}^{\infty} \widetilde{I}_{A_i} \xi_{N(i)+1}; \quad (19)$$

then it is easy to see that $|\xi_\varepsilon - \xi| < \varepsilon$. Set $\wedge = \{\varepsilon \in L^0_{++} \mid \varepsilon \leq 1\}$; then $\wedge$ is directed with respect to $\leq$, and one can easy to see that the net $\{\xi_\varepsilon, \varepsilon \in \wedge\}$ converges to ξ under the locally L^0-convex topology of $(L^0(\mathcal{F}), |\cdot|)$.

With the above preparations, we now give the proof of Theorem 1.

Proof of Theorem 1. First, let us assume that for any $p \in B$, $L(\cdot, p)$ is strictly $L^0(\mathcal{F})$-convex on A. Set $F(u) = \bigvee_{p\in B} L(u, p)$ and $G(p) = \bigwedge_{u\in A} L(u, p)$. We show that the functional F has the local property and F is $L^0(\mathcal{F})$-convex and $\mathcal{T}_c$-lower semicontinuous on A. By conditions (1) and (2), it is clear that F has the local property. For any $x_1, x_2 \in A, \alpha \in L^0_+$, we have that

$$\begin{aligned} F(\alpha x_1 + (1-\alpha) x_2) &= \bigvee_{p\in B} L(\alpha x_1 + (1-\alpha) x_2, p) \\ &\leq \bigvee_{p\in B} [\alpha L(x_1, p) + (1-\alpha) L(x_2, p)] \\ &\leq \alpha \bigvee_{p\in B} L(x_1, p) + (1-\alpha) \bigvee_{p\in B} L(x_2, p) \\ &= \alpha F(x_1) + (1-\alpha) F(x_2). \end{aligned} \quad (20)$$

Thus, F is $L^0(\mathscr{F})$-convex on A. For any $r \in L^0(\mathscr{F}, R)$, let $A_r = \{u \in A \mid F(u) \le r\}$. Since F has the local property, we have that A_r has the countable concatenation property. Let $\{x_\alpha, \alpha \in \wedge\} \subset A_r$ converge to x_0 under the locally L^0-convex topology of E. By $F(x_\alpha) \le r$, we can obtain that

$$L(x_\alpha, p) \le r, \quad \forall p \in B. \tag{21}$$

Since $L(\cdot, p)$ is $\mathscr{T}_c$-lower semicontinuous, it is easy to see that

$$L(x_0, p) \le r, \quad \forall p \in B. \tag{22}$$

Hence, $F(x_0) \le r$ and $x_0 \in A_r$. So we have that F is $\mathscr{T}_c$-lower semicontinuous. Similarly, replacing L with $-L$, we see that $-G$ has the local property and $-G$ is $L^0(\mathscr{F})$-convex and $\mathscr{T}_c$-lower semicontinuous on B.

By Theorems 26 and 28, we have that $\bigvee_{p\in B} G(p) \in L^0(\mathscr{F})$, and there exists $p_0 \in B$ such that

$$G(p_0) = \bigvee_{p\in B} G(p). \tag{23}$$

Since for all $p \in B$, $L(\cdot, p)$ is strictly $L^0(\mathscr{F})$-convex on A, according to Corollary 30 there exists an unique $u_p \in A$ such that

$$G(p) = \bigwedge_{u\in A} L(u, p) = L(u_p, p), \tag{24}$$

for any $p \in B$. Let $u_0 = u_{p_0}$, we need only to prove that

$$G(p_0) \ge L(u_0, p), \quad \forall p \in B. \tag{25}$$

For every $p \in B$, let $p_n = (1 - n^{-1})p_0 + n^{-1}p$ and $u_n = u_{p_n}$, for all $n \in N$. It is clear that

$$G(p_0) \ge G(p_n) = L(u_n, p_n). \tag{26}$$

By condition (2), we have that

$$\begin{gathered} G(p_0) \ge L(u_n, p_n) \ge \left(1 - n^{-1}\right) L(u_n, p_0) + n^{-1} L(u_n, p), \\ G(p_0) \ge \left(1 - n^{-1}\right) G(p_0) + n^{-1} L(u_n, p), \end{gathered} \tag{27}$$

namely, $G(p_0) \ge L(u_n, p)$. Let $\eta = \bigvee\{\|x\| \mid x \in A\}$ and $C_n = \{\omega \in \Omega \mid n - 1 \le \eta < n\}$, for all $n \in N$, according to the condition that A is a.s. bounded in E, and it is easy to see that $\{C_n, n \in N\}$ is a countable partition of Ω to $\mathscr{F}$.

For any fixed $i \in N$, we have that $\widetilde{I}_{C_i} \cdot u_n \in L^2(E)$, for all $n \in N$. It is easy to see that $\widetilde{I}_{C_i} \cdot A$ is a bounded and closed subset of $(L^2(E), \|\cdot\|_2)$. Thus, there exists a subsequence $\{\widetilde{I}_{C_i} \cdot u_{n_k}, k \in N\}$ and $\omega_i \in \widetilde{I}_{C_i} \cdot A$ such that $\{\widetilde{I}_{C_i} \cdot u_{n_k}, k \in N\}$ converges to ω_i under the weak topology of $L^2(E)$. Without loss of generality, denote this subsequence by $\{\widetilde{I}_{C_i} \cdot u_n, n \in N\}$. By Lemma 36, there exists a sequence $\{v_n \in \widetilde{I}_{C_i} \cdot A, n \in N\}$ such that it converges to ω_i in norm, where

$$v_n = \sum_{k=n}^{N_n} \lambda_k u_k, \tag{28}$$

$\sum_{k=n}^{N_n} \lambda_k = 1, \lambda_k \ge 0$. Since $\|v_n - \omega_i\|_2 \to 0$, we have that $\{\|v_n - \omega_i\|, n \in N\}$ converges in probability P to θ. By Egoroff theorem, for any number $\delta > 0$, there exists $C_\delta \in \mathscr{F}$ such that $P(C_\delta) < \delta$ and $\{\|v_n - \omega_i\|, n \in N\}$ converges uniformly to 0 on C_δ^c. Then there is a net

$$\{\|v_\varepsilon - \omega_i\|, \varepsilon \in \wedge\} \tag{29}$$

as in Remark 37, which converges 0 on C_δ^c under the locally L^0-convex topology of $(L^0(\mathscr{F}), |\cdot|)$. By the construction of $\{\|v_\varepsilon - \omega_i\|, \varepsilon \in \wedge\}$ as in Remark 37, we have that

$$\varliminf_{n\to\infty} L(v_n, p) \ge \varliminf_{\varepsilon\in\wedge} L(v_\varepsilon, p). \tag{30}$$

Since $L(\cdot, p)$ is $L^0(\mathscr{F})$-convex for any $p \in B$, we have that

$$\begin{aligned} \widetilde{I}_{C_\delta^c} \widetilde{I}_{C_i} G(p_0) &\ge \widetilde{I}_{C_\delta^c} \widetilde{I}_{C_i} \varliminf_{n\to\infty} \sum_{k=n}^{N_n} \lambda_k L(u_k, p) \\ &\ge \widetilde{I}_{C_\delta^c} \widetilde{I}_{C_i} \varliminf_{n\to\infty} L(v_n, p) \\ &\ge \widetilde{I}_{C_\delta^c} \widetilde{I}_{C_i} \varliminf_{\varepsilon\in\wedge} L(v_\varepsilon, p) \ge \widetilde{I}_{C_\delta^c} \widetilde{I}_{C_i} L(\omega_i, p). \end{aligned} \tag{31}$$

Hence, one can obtain that $\widetilde{I}_{C_\delta^c} \widetilde{I}_{C_i} G(p_0) \ge \widetilde{I}_{C_\delta^c} \widetilde{I}_{C_i} L(\omega_i, p)$. Because δ is an arbitrary nonnegative number and $L(\cdot, p)$ has the local property, we have that

$$\widetilde{I}_{C_i} G(p_0) \ge \widetilde{I}_{C_i} L(\omega_i, p), \tag{32}$$

for any $p \in B$.

Now, we prove that $\omega_i = \widetilde{I}_{C_i} u_0$ for any $i \in N$. By the definition of u_n, it is clear that

$$L(u_n, p_n) \le L(u, p_n), \quad \forall u \in A. \tag{33}$$

By condition (2), we have that

$$\left(1 - n^{-1}\right) L(u_n, p_0) + n^{-1} L(u_n, p) \le L(u, p_n), \quad \forall p \in B. \tag{34}$$

Hence, we can obtainthat

$$\varlimsup_{n\to\infty} \left(1 - n^{-1}\right) L(u_n, p_0) \le \varlimsup_{n\to\infty} L(u, p_n), \quad \forall p \in B. \tag{35}$$

Since $L(u_n, p) \ge G(p)$, we can obtain that $(1 - n^{-1})L(u_n, p_0) + n^{-1}G(p) \le L(u, p_n)$ and $(1 - n^{-1})L(v_n, p_0) + n^{-1}G(p) \le \widetilde{I}_{C_i} \cdot \varlimsup_{n\to\infty}((1 - n^{-1})L(u_n, p_0) + n^{-1}L(u_n, p))$. According to

$$\begin{aligned} \varliminf_{n\to\infty} L(v_n, p_0) &= \varliminf_{n\to\infty} \left(1 - n^{-1}\right) L(v_n, p_0) \\ &\le \varliminf_{n\to\infty} \left(1 - n^{-1}\right) \sum_{k=n}^{N_n} \lambda_k L(u_k, p_0) \\ &\le \varlimsup_{n\to\infty} \left(1 - n^{-1}\right) L(u_n, p_0) \\ &\le \varlimsup_{n\to\infty} L(u, p_n), \end{aligned} \tag{36}$$

where $\sum_{k=n}^{N_n} \lambda_k = 1$, it is obvious that

$$\begin{aligned}\widetilde{I}_{C_i} L(\omega_i, p_0) &\le \varliminf_{n\to\infty} \widetilde{I}_{C_i} L(v_n, p_0) \le \varlimsup_{n\to\infty} \widetilde{I}_{C_i} L(u_n, p_0) \\ &\le \varlimsup_{n\to\infty} \widetilde{I}_{C_i} L(u, p_n).\end{aligned} \tag{37}$$

Since $\{\|p_n - p_0\|, n \in N\}$ converges in probability to 0, by Egoroff theorem, we can obtain that for any $\sigma > 0$, there exists $C_\sigma \in \mathscr{F}$ such that $P(C_\sigma) < \sigma$ and $\{\|p_n - p_0\|, n \in N\}$ converges to 0 uniformly on C_σ^c. By Remark 37, we can construct a net $\{p_\alpha \in B, \alpha \in \wedge\}$ as in Remark 37 such that $\|p_\alpha - p_0\| \to 0$ under the locally L^0-convex topology of E. Hence, by (2) we can obtain that

$$\widetilde{I}_{C_\sigma^c} \widetilde{I}_{C_i} L(\omega_i, p_0) \le \widetilde{I}_{C_\sigma^c} \widetilde{I}_{C_i} \varlimsup_{\alpha\in\wedge} L(u, p_\alpha) \le \widetilde{I}_{C_\sigma^c} \widetilde{I}_{C_i} L(u, p_0). \tag{38}$$

Since σ is an arbitrary nonnegative number, it is clear that

$$\widetilde{I}_{C_i} L(\omega_i, p_0) \le \widetilde{I}_{C_i} L(u, p_0). \tag{39}$$

Therefore, for any $i \in N$, $\omega_i = \widetilde{I}_{C_i} u_0$ and $G(p_0) \ge L(u_0, p)$, for all $p \in B$; namely, (u_0, p_0) is a random saddle point of L with respect to $A \times B$.

If there is $p \in B$ such that $L(\cdot, p)$ is not strictly $L^0(\mathscr{F})$-convex on A, define

$$L_n(u, p) = L(u, p) + n^{-1}\|u\|^2, \quad \forall n \in N. \tag{40}$$

Since E is random strictly convex RN module, we can obtain that L_n is strictly $L^0(\mathscr{F})$-convex from Lemma 35. By the similar method, we have that for any $n \in N$, there exists $(u_n, p_n) \in A\times B$ such that it is a saddle point of L_n with respect to $A \times B$.

Let $\zeta = \bigvee\{\|y\| \mid y \in B\}$ and D_m be any representation element of $[m - 1 \le \eta < m]$, for all $m \in N$, according to the condition that B is a.s. bounded in E, and it is easy to see that $\{D_m, m \in N\}$ is a countable partition of Ω to $\mathscr{F}$. It is easy to see that $\{C_i \bigcap D_j, i, j \in N\}$ is also a countable partition of Ω to $\mathscr{F}$. For any $i, j \in N$, let $H_{ij} = C_i \bigcap D_j$. We can suppose that, without loss of generality, $\widetilde{I}_{H_{ij}} u_n \in L^2(E)$ converge to μ_{ij} under the weak topology of $L^2(E)$. Then we have that there exists a net $\{\widetilde{I}_{H_{ij}} u_\alpha \in E \mid \alpha \in \Lambda\}$ such that it converges to μ_{ij} under the locally L^0-convex topology of E. Thus, we have that

$$\begin{aligned}\widetilde{I}_{H_{ij}} L(\mu_{ij}, p) &\le \varliminf_{\alpha} \widetilde{I}_{H_{ij}} L\left(\widetilde{I}_{H_{ij}} u_\alpha, p\right) \\ &\le \varliminf_{n\to\infty} \widetilde{I}_{H_{ij}} L\left(\widetilde{I}_{H_{ij}} u_n, p\right) + n^{-1}\|u_n\|^2 \\ &\le \varlimsup_{n\to\infty} \left(\widetilde{I}_{H_{ij}} L\left(u, \widetilde{I}_{H_{ij}} p_n\right) + n^{-1}\|u\|^2\right)\end{aligned} \tag{41}$$

for all $u \in A$, $p \in B$. Similarly, for any $i, j \in N$, one can have a net $\{\widetilde{I}_{H_{ij}} p_\beta \in E \mid \beta \in \Gamma\}$ such that it converges to ν_{ij} under the locally L^0-convex topology of E and

$$\begin{aligned}&\varlimsup_{n\to\infty} \left(\widetilde{I}_{H_{ij}} L\left(u, \widetilde{I}_{H_{ij}} p_n\right) + n^{-1}\|u\|^2\right) \\ &\quad\le \varlimsup_{n\to\infty} \widetilde{I}_{H_{ij}} L\left(u, \widetilde{I}_{H_{ij}} p_n\right) + \varlimsup_{n\to\infty} n^{-1}\|u\|^2 \\ &\quad\le \varlimsup_{\beta\in\Gamma} \widetilde{I}_{H_{ij}} L\left(u, \widetilde{I}_{H_{ij}} p_\beta\right) + \varlimsup_{n\to\infty} n^{-1}\|u\|^2 \\ &\quad\le \widetilde{I}_{H_{ij}} L\left(u, \nu_{ij}\right).\end{aligned} \tag{42}$$

Let $u_0 = \sum_{i,j\in N} \widetilde{I}_{H_{ij}} \mu_{ij}$ and $p_0 = \sum_{i,j\in N} \widetilde{I}_{H_{ij}} \nu_{ij}$; it is easy to check that

$$L(u_0, p) \le L(u_0, p_0) \le L(u, p_0) \tag{43}$$

for all $u \in A$, $p \in B$; namely, (u_0, p_0) is a random saddle point of L with respect to $A \times B$.

This completes the proof. □

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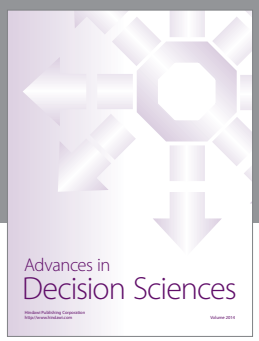

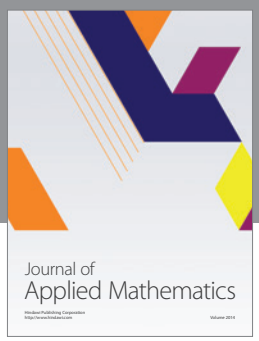

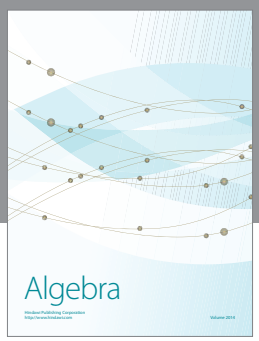

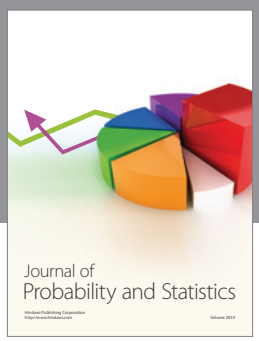

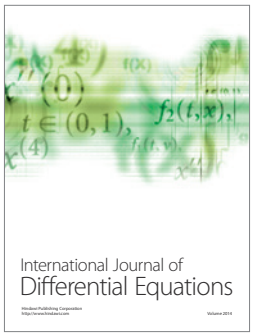

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