Research Article

Subdifferential Properties of Minimal Time Functions Associated with Set-Valued Mappings with Closed Convex Graphs in Hausdorff Topological Vector Spaces

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For a set-valued mapping M defined between two Hausdorff topological vector spaces E and F and with closed convex graph and for a given point $(x, y) \in E \times F$, we study the minimal time function associated with the images of M and a bounded set $\Omega \subset F$ defined by $\mathcal{T}_{M,\Omega}(x, y) := \inf\{t \geq 0 : M(x) \cap (y + t\Omega) \neq \emptyset\}$. We prove and extend various properties on directional derivatives and subdifferentials of $\mathcal{T}_{M,\Omega}$ at those points of $(x, y) \in E \times F$ (both cases: points in the graph $\text{gph } M$ and points outside the graph). These results are used to prove, in terms of the minimal time function, various new characterizations of the convex tangent cone and the convex normal cone to the graph of M at points inside $\text{gph } M$ and to the graph of the enlargement set-valued mapping at points outside $\text{gph } M$. Our results extend many existing results, from Banach spaces and normed vector spaces to Hausdorff topological vector spaces (Bounkhel, 2012; Bounkhel and Thibault, 2002; Burke et al., 1992; He and Ng, 2006; and Jiang and He 2009). An application of the minimal time function $\mathcal{T}_{M,\Omega}$ to the calmness property of perturbed optimization problems in Hausdorff topological vector spaces is given in the last section of the paper.

1. Introduction

Let E and F be two Hausdorff topological vector spaces and Ω a bounded nonempty subset of F . Let $M : E \rightrightarrows F$ be a set-valued mapping with nonempty closed values. We associate to M and Ω the so-called minimal time function defined by

$$\mathcal{T}_{M,\Omega}(x, y) := \inf\{t \geq 0 : M(x) \cap (y + t\Omega) \neq \emptyset\}, \quad \forall (x, y) \in E \times F. \quad (1)$$

The case $M(x) = S$ is a fixed closed subset of F that coincides with the minimal time function associated with S and Ω and is defined by

$$T_{S,\Omega}(u) := \inf\{t \geq 0 : S \cap (u + t\Omega) \neq \emptyset\}, \quad \forall u \in F. \quad (2)$$

This function $T_{S,\Omega}$ plays an important role in variational analysis, optimization, control theory, Hamilton-Jacobi partial differential equations, approximation theory, and so

forth; the reader can find more discussions in [1–13] and the references therein. The value $\mathcal{T}_{M,\Omega}(x_0, y_0)$ can be seen as the minimal time $t \geq 0$ associated with the differential inclusion $\dot{y}(\tau) \in \Omega$, almost everywhere on $[0, t]$, with $y(0) = y_0$ and $y(t) \in M(x_0)$; in other words,

$$\mathcal{T}_{M,\Omega}(x_0, y_0) = \begin{cases} \inf\{t > 0 : \exists y(\cdot) \text{ s.t. } \dot{y}(\tau) \in \Omega, \\ \text{a.e. } [0, t], y(0) = y_0, \\ y(t) \in M(x_0)\}, & \forall y_0 \notin M(x_0), \\ 0, & \forall y_0 \in M(x_0). \end{cases} \quad (3)$$

We notice that the minimal time function $\mathcal{T}_{M,\Omega}$ covers many crucial functions in variational analysis: the distance function to images Δ_M , the indicator function, the usual distance function, and the Minkowski function, by taking some particular cases of M and Ω as follows.

- (1) If $M(x) = \{0\}$, then $\mathcal{T}_{M,\Omega}(x, y) = T_{\{0\},\Omega}(-y) = \rho_{\Omega}(y)$ which coincides with the Minkowski (gauge) function associated with Ω ; that is,

$$\rho_{\Omega}(y) = \inf \{t \geq 0 : y \in t\Omega\}, \quad \forall y \in F. \quad (4)$$

- (2) If $\Omega = \{0\}$, then $\mathcal{T}_{M,\Omega}(x, y) = T_{M(x),\{0\}}(y) = \psi_{M(x)}(y) = \psi_{\text{gph } M}(x, y)$, which coincides with the indicator function associated with $\text{gph } M$. Here, $\psi_S(u) = 0$ if $u \in S$ and $\psi_S(u) = +\infty$, otherwise.

- (3) If F is normed, $M(x) = \{0\}$, and $\Omega = \mathbb{B}_F$ is the closed unit ball in F , then $\mathcal{T}_{M,\Omega}(x, y) = T_{\{0\},\mathbb{B}_F}(y) = \|y\|$.

- (4) If F is normed and $\Omega = \mathbb{B}_F$, then

$$\mathcal{T}_{M,\Omega}(x, y) = \mathcal{T}_{M,\mathbb{B}}(x) = \inf_{s \in M(x)} \|y - s\| = \Delta_M(x, y) \quad (5)$$

which coincides with the distance function associated with the images of M (which has been successfully used in optimization theory, first by Clarke in [14] for Lipschitz set-valued mappings and later by various authors (see for instance the book [15] and the references therein) for set-valued mappings that are not necessarily locally Lipschitz).

- (5) If F is normed, $M(x) = S$, and $\Omega = \mathbb{B}_F$, then

$$\mathcal{T}_{M,\Omega}(x, y) = T_{S,\mathbb{B}}(y) = \inf_{s \in S} \|y - s\| = d_S(y), \quad (6)$$

which coincides with the usual distance function associated with S in F .

From the above cases, we can see the importance of the study of the minimal time function $\mathcal{T}_{M,\Omega}$ in normed vector spaces as well as in Hausdorff topological vector spaces. This type of study will unify the study of all the above functions.

The case of minimal time function associated with a closed set S has been the subject of many recent works [1–3, 5–10, 12, 13]. To the best of our knowledge, the unique work studying the function $\mathcal{T}_{M,\Omega}$ is [16], in which the author studied the Fréchet subdifferential of $\mathcal{T}_{M,\Omega}$ in Banach spaces. We mention that there are no results on the directional derivatives and subdifferentials of $\mathcal{T}_{M,\Omega}$ in the Hausdorff topological vector spaces. Starting from this point, as a goal, we will develop a thorough study of the minimal time function in Hausdorff topological vector spaces in the convex setting. The nonconvex case will be the subject of a series of forthcoming works by the author. In the present paper, we extend various existing results on directional derivatives and subdifferentials of $\mathcal{T}_{M,\Omega}$ and their relationships to tangent and normal cones in Hausdorff topological vector spaces. The paper is organized as follows. Section 2 is devoted to state the main notations and definitions used throughout the paper. In Section 3, we prove our main results for points on the graph of M . The case of points outside the graph of M is studied in Section 4. In the last section we state an application of the minimal time function $\mathcal{T}_{M,\Omega}$ to the study of the calmness property of optimization problems in Hausdorff topological vector spaces.

2. Notations and Preliminaries

Throughout the whole paper (unless otherwise specified), we assume that E and F are two Hausdorff topological vector spaces. We will denote by E^* and F^* the topological dual of E and F , respectively, and by $\langle (\cdot, \cdot), (\cdot, \cdot) \rangle$ the pairing between the spaces $E \times F$ and $E^* \times F^*$.

Let S be a nonempty closed convex subset of $E \times F$ and $(\bar{x}, \bar{y})$ a point in S . The convex tangent cone $T(S; (\bar{x}, \bar{y}))$ is defined by $T(S; (\bar{x}, \bar{y})) = \text{cl}[\mathbb{R}_+(S - (\bar{x}, \bar{y}))]$, where $\mathbb{R}_+$ denotes the set of all nonnegative real numbers. The convex normal cone to S at $(\bar{x}, \bar{y})$ is defined as the negative polar of $T(S; (\bar{x}, \bar{y}))$; that is, $N(S; (\bar{x}, \bar{y})) := (T(S; (\bar{x}, \bar{y})))^\circ$, where $A^\circ := \{(x^*, y^*) \in E^* \times F^* : \langle (x^*, y^*), (h, k) \rangle \leq 0, \text{ for all } (h, k) \in A\}$, for any subset $A \subset E \times F$. Recall also from [14] that $N(S; (\bar{x}, \bar{y}))$ is also characterized by $N(S; (\bar{x}, \bar{y})) = \{(x^*, y^*) \in E^* \times F^* : \langle (x^*, y^*), (x, y) - (\bar{x}, \bar{y}) \rangle \leq 0, \text{ for all } (x, y) \in S\}$.

Let f be a convex function from E into $\mathbb{R} \cup \{+\infty\}$ with $\bar{u} \in \text{dom } f := \{u \in E : f(u) < \infty\}$; the directional derivative (resp., the convex subdifferential) of f at $\bar{u}$ is defined by

$$f'(\bar{u}; v) := \lim_{t \downarrow 0} t^{-1} [f(\bar{u} + tv) - f(\bar{u})], \quad \forall v \in E.$$

(resp.,

$$\partial f(\bar{x})$$

$$:= \{u^* \in E^* : \langle u^*, v \rangle \leq f'(\bar{u}; v), \quad \forall v \in E\}$$

$$= \{u^* \in E^* : \langle u^*, u - \bar{u} \rangle \leq f(u) - f(\bar{u}), \quad \forall u \in E\}). \quad (7)$$

Note that for l.s.c convex functions we have $f'(\bar{u}; h) = \sigma_{\partial f(\bar{u})}(h)$, whenever $\bar{u} \in \text{core}(\text{dom } f)$ and E is a Hausdorff locally convex topological vector space. Here, σ_A denotes the support function associated with a closed subset $A \subset E^*$ defined by $\sigma_A(h) := \sup\{\langle x^*, h \rangle : x^* \in A\}$ and $\text{core}(S) := \{x \in E : \text{for all } h \in E, \exists t_{x,h} > 0, \text{ such that } x + [0, t_{x,h}]h \subset S\}$ is called the algebraic interior of S (for more details on the core, we refer the reader to [17]).

3. Points on the Graph of the Set-Valued Mapping

Before starting the study of minimal time functions for set-valued mappings with closed convex graphs, we need to prove some results for general set-valued mappings $M : E \rightrightarrows F$ with nonempty values (with graph not necessarily closed nor convex). These results have their own interests. We start with the following lemma which is needed in all the proofs of our work.

Lemma 1. *Let E and F be Hausdorff topological vector spaces. Assume that Ω is a bounded set in F and M is a set-valued mapping with nonempty values in F .*

- (1) *If $\mathcal{T}_{M,\Omega}(x, y) = 0$, then $y \in \text{cl}(M(x))$.*
- (2) *Conversely, for any $y \in M(x)$, one has $T_{M,\Omega}(x, y) = 0$.*
- (3) *If in addition $0 \in \text{int } \Omega$, then for any $y \in \text{cl}(M(x))$ one has $T_{M,\Omega}(x, y) = 0$.*

Proof. (1) Let $x \in E$ with $\mathcal{T}_{M,\Omega}(x, y) = 0$, then for all $n \geq 1$, $\exists t_n \geq 0$ such that $\mathcal{T}_{M,\Omega}(x, y) + 1/n > t_n \geq \mathcal{T}_{M,\Omega}(x, y)$ with $M(x) \cap (y + t_n \Omega) \neq \emptyset$; that is, there exists $u_n \in \Omega$ such that $y + t_n u_n \in M(x)$. Let V be any neighborhood of 0 in F . There exists W as a balanced neighborhood of 0 such that $W \subset V$. Since Ω is bounded, there exists $\alpha > 0$ such that $\Omega \subset \alpha W$. Hence, there exists $w_n \in W$ such that $u_n = \alpha w_n$. Clearly, for any $n > \alpha$, we have $t_n \alpha \in (0, 1)$, and so by the fact that W is balanced we obtain

$$t_n u_n = t_n \alpha w_n \in t_n \alpha W \subset W \subset V. \quad (8)$$

Since V is arbitrary, we get the convergence of the sequence $t_n u_n$ to zero, that is, the convergence of the sequence $y + t_n u_n$ to y , and since $y + t_n u_n \in M(x)$, we get $y \in \text{cl}(M(x))$.

(2) Let $y \in M(x)$ and let $w \in \Omega$. Clearly, we have $y = y + 0 \cdot w \in (y + 0 \cdot \Omega) \cap M(x)$. This ensures by definition that $\mathcal{T}_{M,\Omega}(x, y) = 0$.

(3) Assume now that $0 \in \text{int } \Omega$. Let $y \in \text{cl}(M(x))$ and fix any $\epsilon > 0$. Clearly, $\epsilon \Omega \in \mathcal{N}(0)$, and hence, $(y + \epsilon \Omega) \cap M(x) \neq \emptyset$. It follows that $0 \leq \mathcal{T}_{M,\Omega}(x, y) \leq \epsilon$ which ensures that $\mathcal{T}_{M,\Omega}(x, y) = 0$. $\square$

Note that the assumption $0 \in \text{int } \Omega$ in (3) cannot be removed when the values of M are open sets. Take $M(x)$ as any open set, and take $\Omega = \{0\}$. Clearly, for all $y \in \text{bd}(M(x)) := \text{cl}(M(x)) \setminus M(x)$, we have $\mathcal{T}_{M,\Omega}(x, y) = \psi_{M(x)}(y) = +\infty$.

As a corollary of parts (1) and (2) in Lemma 1 we have the following.

Corollary 2. *Let E and F be Hausdorff topological vector spaces, and let $M : E \rightrightarrows F$ be a set-valued mapping with nonempty closed values. Assume that Ω is a bounded set in F . Then $\mathcal{T}_{M,\Omega}(x, y) = 0$ if and only if $y \in M(x)$.*

The following lemma characterizes the convexity of the graph of set-valued mappings in terms of the convexity of its associated minimal time function in Hausdorff topological vector spaces. It extends the well known characterization of the convexity (see, for instance, the lemma on page 53 in [14]) in terms of the distance function in normed vector spaces as well as the one in terms of the indicator function.

Lemma 3. *Let E and F be two Hausdorff topological vector spaces, $M : E \rightrightarrows F$ a set-valued mapping with nonempty closed values, and Ω a bounded convex set in F . Then, $\mathcal{T}_{M,\Omega}(\cdot, \cdot)$ is convex on its domain if and only if the graph $\text{gph } M$ is convex.*

Proof. Assume that $\mathcal{T}_{M,\Omega}(\cdot, \cdot)$ is convex on its domain; that is,

$$\begin{aligned} \mathcal{T}_{M,\Omega}(\lambda(x_1, y_1) + (1-\lambda)(x_2, y_2)) \\ \leq \lambda \mathcal{T}_{M,\Omega}(x_1, y_1) + (1-\lambda) \mathcal{T}_{M,\Omega}(x_2, y_2), \end{aligned} \quad (9)$$

for all $(x_1, y_1), (x_2, y_2) \in \text{dom } \mathcal{T}_{M,\Omega}$, and all $\lambda \in [0, 1]$. Therefore, by Corollary 2, for any $(x_1, y_1), (x_2, y_2) \in \text{gph } M$, we have $\mathcal{T}_{M,\Omega}(x_1, y_1) = \mathcal{T}_{M,\Omega}(x_2, y_2) = 0$, and so by the previous inequality, we obtain $\mathcal{T}_{M,\Omega}(\lambda(x_1, y_1) + (1-\lambda)(x_2, y_2)) = 0$, and hence by Corollary 2, once again we get

$\lambda y_1 + (1-\lambda)y_2 \in M(\lambda x_1 + (1-\lambda)x_2)$; that is, $\lambda(x_1, y_1) + (1-\lambda)(x_2, y_2) \in \text{gph } M$, which means that $\text{gph } M$ is convex. Conversely, let $(x_i, y_i) \in \text{dom } \mathcal{T}_{M,\Omega}$ ($i = 1, 2$), and let any $\lambda \in [0, 1]$. Fix any $\epsilon > 0$. Since $\mathcal{T}_{M,\Omega}(x_1, y_1) < \infty$ and $\mathcal{T}_{M,\Omega}(x_2, y_2) < \infty$, we can find that $w_i \in \Omega$, ($i = 1, 2$) and $t_i \geq 0$ ($i = 1, 2$) such that $t_i < \mathcal{T}_{M,\Omega}(x_i, y_i) + \epsilon$ and $y_i + t_i w_i \in M(x_i)$ ($i = 1, 2$). So the convexity of $\text{gph } M$ ensures that $\lambda(x_1, y_1 + t_1 w_1) + (1-\lambda)(x_2, y_2 + t_2 w_2) \in \text{gph } M$. Hence,

$$\begin{aligned} [\lambda y_1 + (1-\lambda)y_2] + [\lambda t_1 + (1-\lambda)t_2] \bar{w} \\ \in M(\lambda x_1 + (1-\lambda)x_2), \end{aligned} \quad (10)$$

where $\bar{w} := [\lambda t_1 / (\lambda t_1 + (1-\lambda)t_2)] w_1 + [(1-\lambda)t_2 / (\lambda t_1 + (1-\lambda)t_2)] w_2$. On the other hand, by the convexity of Ω , we have

$$\bar{w} = \frac{\lambda t_1}{\lambda t_1 + (1-\lambda)t_2} w_1 + \frac{(1-\lambda)t_2}{\lambda t_1 + (1-\lambda)t_2} w_2 \in \Omega, \quad (11)$$

which ensures that

$$\begin{aligned} ([\lambda y_1 + (1-\lambda)y_2] + [\lambda t_1 + (1-\lambda)t_2] \bar{w}) \\ \cap M(\lambda x_1 + (1-\lambda)x_2) \neq \emptyset. \end{aligned} \quad (12)$$

This ensures that

$$\begin{aligned} \mathcal{T}_{M,\Omega}(\lambda(x_1, y_1) + (1-\lambda)(x_2, y_2)) \\ = \mathcal{T}_{M,\Omega}(\lambda x_1 + (1-\lambda)x_2, \lambda y_1 + (1-\lambda)y_2) \\ \leq \lambda t_1 + (1-\lambda)t_2 \\ < \lambda \mathcal{T}_{M,\Omega}(x_1, y_1) + (1-\lambda) \mathcal{T}_{M,\Omega}(x_2, y_2) + \epsilon, \end{aligned} \quad (13)$$

for any $\epsilon > 0$. Thus, taking $\epsilon \rightarrow 0$ completes the proof. $\square$

Note that the convexity of Ω , in the proof of Lemma 3, is needed only in one direction (reverse implication); that is, the convexity of $\mathcal{T}_{M,\Omega}$ ensures the convexity of the graph of M even when Ω is not convex.

Now, we are looking for the lower semicontinuity of the minimal time function in Hausdorff topological vector spaces.

Proposition 4. *Let E and F be two Hausdorff topological vector spaces. Assume that Ω is compact in F and $\text{gph } M$ is closed in $E \times F$. Then $\mathcal{T}_{M,\Omega}$ is lower semicontinuous at any $(\bar{x}, \bar{y}) \in \text{dom } \mathcal{T}_{M,\Omega}$.*

Proof. Let $(\bar{x}, \bar{y}) \in \text{dom } \mathcal{T}_{M,\Omega}$. We have to prove that

$$\mathcal{T}_{M,\Omega}(\bar{x}, \bar{y}) \leq \liminf_{(x,y) \rightarrow (\bar{x}, \bar{y})} \mathcal{T}_{M,\Omega}(x, y). \quad (14)$$

The case $\liminf_{(x,y) \rightarrow (\bar{x}, \bar{y})} \mathcal{T}_{M,\Omega}(x, y) = \infty$ is obvious, so we assume that

$$\alpha := \liminf_{(x,y) \rightarrow (\bar{x}, \bar{y})} \mathcal{T}_{M,\Omega}(x, y) < \infty. \quad (15)$$

Let $(x_j, y_j)_{j \in J}$ be a net satisfying the above $\liminf$; that is, $(x_j, y_j) \rightarrow (\bar{x}, \bar{y})$ and $\alpha = \lim_j \mathcal{T}_{M,\Omega}(x_j, y_j)$. Let $(t_j)_{j \in J}$ be a net of positive real numbers satisfying $t_j \rightarrow 0$.

By the definition of the minimal time function, we can find for any $j \in J$ a real number δ_j such that

$$\begin{aligned} 0 \leq \mathcal{T}_{M,\Omega}(x_j, y_j) &\leq \delta_j < \mathcal{T}_{M,\Omega}(x_j, y_j) + t_j, \\ (y_j + \delta_j \Omega) \cap M(x_j) &\neq \emptyset, \quad \forall j \in J. \end{aligned} \quad (16)$$

Hence, for any $j \in J$, there exists $w_j \in \Omega$ such that $(x_j, y_j + \delta_j w_j) \in \text{gph } M$. Using the compactness of Ω , we get the convergence of a subnet of $(w_j)_j$ to some point $w \in \Omega$. Without loss of generality the subnet is still denoted by $(w_j)_j$. Therefore, $(x_j, y_j + \delta_j w_j) \rightarrow (\bar{x}, \bar{y} + \alpha w) \in \text{gph } M$ (by the closedness of $\text{gph } M$). This shows that $(\bar{y} + \alpha \Omega) \cap M(\bar{x}) \neq \emptyset$, which ensures that $\mathcal{T}_{M,\Omega}(\bar{x}, \bar{y}) \leq \alpha$, and hence, the proof is complete. $\square$

An inspection of the proof of the previous proposition shows that the conclusion is still valid under the assumptions that Ω is weakly compact and $\text{gph } M$ is weakly closed. Consequently, we have the two following corollaries.

Corollary 5. *Let E and F be two Hausdorff topological vector spaces. Assume that Ω is weakly compact in F and $\text{gph } M$ is weakly closed in $E \times F$. Then, $\mathcal{T}_{M,\Omega}$ is lower semicontinuous at any $(\bar{x}, \bar{y}) \in \text{dom } \mathcal{T}_{M,\Omega}$.*

Corollary 6. *Assume that F is a reflexive Banach space, Ω is closed convex bounded set in F , and $\text{gph } M$ is closed convex set in $E \times F$. Then $\mathcal{T}_{M,\Omega}$ is lower semicontinuous at any $(\bar{x}, \bar{y}) \in \text{dom } \mathcal{T}_{M,\Omega}$.*

The following lemma is technical and is needed in some forthcoming proofs. Its proof is straightforward.

Lemma 7. *Let E and F be two Hausdorff topological vector spaces, $M : E \rightrightarrows F$ a set-valued mapping, and Ω a nonempty set in F . Then,*

$$\begin{aligned} t^{-1} \mathcal{T}_{M,\Omega}(x + th, y + tk) \\ = T_{t^{-1}[M(x+th)-y],\Omega}(k), \quad \forall (x, y), (h, k) \in E \times F, \forall t > 0. \end{aligned} \quad (17)$$

Now, we are going to establish our main results of this section. We prove some formulas and relationships between the directional derivative and the convex subdifferential of $\mathcal{T}_{M,\Omega}$ and the convex tangent cone and the convex normal cone of $\text{gph } M$ in Hausdorff topological vector spaces at points in $\text{gph } M$. We associate with the set-valued mapping M a new set-valued mapping $\tilde{T}_M : E \rightrightarrows F$ with graph $\text{gph } \tilde{T}_M := T(\text{gph } M; (\bar{x}, \bar{y}))$; that is, $k \in \tilde{T}_M(h)$ if and only if $(h, k) \in T(\text{gph } M; (\bar{x}, \bar{y}))$.

Theorem 8. *Let E and F be two Hausdorff topological vector spaces and $M : E \rightrightarrows F$ a set-valued mapping. Assume that $\text{gph } M$ is a nonempty closed convex set in $E \times F$, $(\bar{x}, \bar{y}) \in \text{gph } M$, and Ω is a bounded convex set in F . Then, one has*

(1)

$$\begin{aligned} \mathcal{T}'_{M,\Omega}((\bar{x}, \bar{y}); (h, k)) \\ = \lim_{t \rightarrow 0^+} t^{-1} \mathcal{T}_{M,\Omega}(\bar{x} + th, \bar{y} + tk) \\ = \lim_{t \rightarrow 0^+} T_{t^{-1}[M(\bar{x}+th)-\bar{y}],\Omega}(k), \\ \forall (h, k) \in E \times F; \end{aligned} \quad (18)$$

(2)

$$\begin{aligned} \partial \mathcal{T}_{M,\Omega}(\bar{x}, \bar{y}) \\ = N(\text{gph } M; (\bar{x}, \bar{y})) \\ \cap \{(x^*, y^*) \in E^* \times F^* : \sigma_\Omega(-y^*) \leq 1\}; \end{aligned} \quad (19)$$

(3) *if, in addition, $(\bar{x}, \bar{y}) \in \text{core}(\text{dom } \mathcal{T}_{M,\Omega})$, $E \times F$ is locally convex, then*

$$\begin{aligned} \mathcal{T}'_{M,\Omega}((\bar{x}, \bar{y}); (h, k)) &= \mathcal{T}_{\tilde{T}_M,\Omega}(h, k), \\ \forall (h, k) \in E \times F; \end{aligned} \quad (20)$$

(4)

$$\begin{aligned} T(\text{gph } M; (\bar{x}, \bar{k})) \\ = \{(h, k) \in E \times F : \mathcal{T}'_{M,\Omega}((\bar{x}, \bar{y}); (h, k)) = 0\}; \end{aligned} \quad (21)$$

(5) *and*

$$\begin{aligned} N(\text{gph } M; (\bar{x}, \bar{y})) \\ = \text{cl}_{w^*}(\mathbb{R}_+ \partial \mathcal{T}_{M,\Omega}((\bar{x}, \bar{y}))). \end{aligned} \quad (22)$$

Proof. (1) By Lemma 3, we have the convexity of $\mathcal{T}_{M,\Omega}$, and so $\mathcal{T}'_{M,\Omega}((\bar{x}, \bar{y}); (h, k))$ is well defined and hence, Corollary 2 and Lemma 7 complete the proof of part (1).

(2) Let $(x^*, y^*) \in N(\text{gph } M; (\bar{x}, \bar{y}))$ with $\sigma_\Omega(-y^*) \leq 1$. Then, by definition of the convex normal cone and by Corollary 2, we have

$$\begin{aligned} \langle (x^*, y^*), (x - \bar{x}, y - \bar{y}) \rangle \\ \leq 0 = \mathcal{T}_{M,\Omega}(x, y) - \mathcal{T}_{M,\Omega}(\bar{x}, \bar{y}), \\ \forall (x, y) \in \text{gph } M. \end{aligned} \quad (23)$$

Assume now that $(x, y) \notin \text{gph } M$, and let $\epsilon > 0$. If $\mathcal{T}_{M,\Omega}(x, y) = \infty$, then the previous inequality holds obviously. Assume that $\mathcal{T}_{M,\Omega}(x, y) < \infty$. By definition of minimal time function, there exists $\delta > 0$ and $w \in \Omega$ such that

$$\begin{aligned} 0 < \mathcal{T}_{M,\Omega}(x, y) \\ \leq \delta < \mathcal{T}_{M,\Omega}(x, y) + \epsilon \quad \text{with} \quad y + \delta w \in M(x). \end{aligned} \quad (24)$$

Hence, we get by the definition of convex normal cones that

$$\langle (x^*, y^*), (x, y + \delta w) - (\bar{x}, \bar{y}) \rangle \leq 0. \quad (25)$$

Therefore, combining this inequality with the assumption $\sigma_\Omega(-y^*) \leq 1$, we obtain that

$$\begin{aligned} & \langle (x^*, y^*), (x - \bar{x}, y - \bar{y}) \rangle \\ &= \langle (x^*, y^*), (x - \bar{x}, y + \delta w - \bar{y}) \rangle + \delta \langle -y^*, w \rangle \\ &\leq \delta \sigma_\Omega(-y^*) \leq \delta < \mathcal{F}_{M,\Omega}(x, y) + \epsilon, \end{aligned} \quad (26)$$

which ensures, by taking $\epsilon \rightarrow 0$, that

$$\begin{aligned} & \langle (x^*, y^*), (x - \bar{x}, y - \bar{y}) \rangle \\ &\leq \mathcal{F}_{M,\Omega}(x, y) - \mathcal{F}_{M,\Omega}(\bar{x}, \bar{y}), \quad \forall (x, y) \notin \text{gph } M. \end{aligned} \quad (27)$$

Thus, this inequality holds for any $(x, y) \in E \times F$ which means that $(x^*, y^*) \in \partial \mathcal{F}_{M,\Omega}(\bar{x}, \bar{y})$.

Conversely, let $(x^*, y^*) \in \partial \mathcal{F}_{M,\Omega}(\bar{x}, \bar{y})$. Then, by definition, we have

$$\begin{aligned} & \langle (x^*, y^*), (x, y) - (\bar{x}, \bar{y}) \rangle \\ &\leq \mathcal{F}_{M,\Omega}(x, y) - \mathcal{F}_{M,\Omega}(\bar{x}, \bar{y}), \quad \forall (x, y) \in E \times F. \end{aligned} \quad (28)$$

Using Corollary 2, we have

$$\langle (x^*, y^*), (x, y) - (\bar{x}, \bar{y}) \rangle \leq 0 \quad \forall (x, y) \in \text{gph } M. \quad (29)$$

This means that $(x^*, y^*) \in N(\text{gph } M; (\bar{x}, \bar{y}))$. We have to prove that $\sigma_\Omega(-y^*) \leq 1$. Let $w \in \Omega$ and $\delta > 0$. Then, $\bar{y} = \bar{y} - \delta w + \delta w \in M(\bar{x}) \cap (\bar{y} - \delta w + \delta \Omega)$; that is, $M(\bar{x}) \cap (\bar{y} - \delta w + \delta \Omega) \neq \emptyset$, which ensures that $\mathcal{F}_{M,\Omega}(\bar{x}, \bar{y} - \delta w) \leq \delta$. Therefore, by (28), we have

$$\begin{aligned} & \langle -y^*, \delta w \rangle = \langle (x^*, y^*), (\bar{x}, \bar{y} - \delta w) - (\bar{x}, \bar{y}) \rangle \\ &\leq \mathcal{F}_{M,\Omega}(\bar{x}, \bar{y} - \delta w) \leq \delta, \end{aligned} \quad (30)$$

which ensures that

$$\langle -y^*, w \rangle \leq 1, \quad \forall w \in \Omega. \quad (31)$$

This gives $\sigma_\Omega(-y^*) \leq 1$, and hence, the proof of (2) is complete.

(3) Let $(h, k) \in E \times F$. Assume first that $(h, k) \in T(\text{gph } M; (\bar{x}, \bar{y})) = \text{gph } \tilde{T}_M$. Then, by Corollary 2, we have $\mathcal{F}_{\tilde{T}_M,\Omega}(h, k) = 0$. Also, we have $\langle (x^*, y^*), (h, k) \rangle \leq 0$, for all $(x^*, y^*) \in N(\text{gph } M, (\bar{x}, \bar{y}))$. Hence, by (2), we have $\langle (x^*, y^*), (h, k) \rangle \leq 0$, for all $(x^*, y^*) \in \partial \mathcal{F}_{M,\Omega}(\bar{x}, \bar{y})$, and so $\sigma_{\partial \mathcal{F}_{M,\Omega}(\bar{x}, \bar{y})}(h, k) \leq 0$. Now we use the assumption $(\bar{x}, \bar{y}) \in \text{core}(\text{dom } \mathcal{F}_{M,\Omega})$ to write $\mathcal{F}'_{M,\Omega}((\bar{x}, \bar{y}); (h, k)) = \sigma_{\partial \mathcal{F}_{M,\Omega}(\bar{x}, \bar{y})}(h, k)$ and so $\mathcal{F}'_{M,\Omega}((\bar{x}, \bar{y}); (h, k)) \leq 0$. Hence by Part (1) we obtain the needed equality

$$\mathcal{F}'_{M,\Omega}((\bar{x}, \bar{y}); (h, k)) = 0 = \mathcal{F}_{\tilde{T}_M,\Omega}(h, k). \quad (32)$$

Assume now that $(h, k) \notin T(\text{gph } M; (\bar{x}, \bar{y}))$. Without loss of generality, we assume that $\mathcal{F}_{\tilde{T}_M,\Omega}(h, k) < \infty$. Put $\alpha := \mathcal{F}_{\tilde{T}_M,\Omega}(h, k)$. Fix any $\delta \geq 0$ with $(k + \delta \Omega) \cap \tilde{T}_M(h) \neq \emptyset$. There exists $w \in \Omega$ such that $k + \delta w \in \tilde{T}_M(h)$; that is, $(h, k + \delta w) \in T(\text{gph } M; (\bar{x}, \bar{y}))$. Therefore, by Part (2), for any $(x^*, y^*) \in \partial \mathcal{F}_{M,\Omega}(\bar{x}, \bar{y})$, we have $\sigma_\Omega(-y^*) \leq 1$ and $(x^*, y^*) \in N(\text{gph } M; (\bar{x}, \bar{y}))$, and hence, we get the following

$$\begin{aligned} & \langle (x^*, y^*), (h, k) \rangle \\ &= \langle (x^*, y^*), (h, k + \delta w) \rangle + \delta \langle -y^*, w \rangle \\ &\leq \delta \sigma_\Omega(-y^*) \leq \delta, \quad \forall (x^*, y^*) \in \partial \mathcal{F}_{M,\Omega}(\bar{x}, \bar{y}). \end{aligned} \quad (33)$$

This ensures that

$$\mathcal{F}'_{M,\Omega}((\bar{x}, \bar{y}), (h, k)) = \sigma_{\partial \mathcal{F}_{M,\Omega}(\bar{x}, \bar{y})}(h, k) \leq \delta, \quad (34)$$

and since δ is taken arbitrarily, we take the infimum to obtain that

$$\begin{aligned} & \mathcal{F}'_{M,\Omega}((\bar{x}, \bar{y}), (h, k)) \\ &\leq \inf \{ \delta > 0 : \text{such that } (k + \delta \Omega) \cap \tilde{T}_M(h) \neq \emptyset \} \\ &= \mathcal{F}_{\tilde{T}_M,\Omega}(h, k). \end{aligned} \quad (35)$$

Let us prove the reverse inequality

$$\mathcal{F}_{\tilde{T}_M,\Omega}(h, k) \leq \mathcal{F}'_{M,\Omega}((\bar{x}, \bar{y}), (h, k)). \quad (36)$$

Obviously, we assume that $\mathcal{F}'_{M,\Omega}((\bar{x}, \bar{y}), (h, k)) < \infty$. Put $\alpha := \mathcal{F}'_{M,\Omega}((\bar{x}, \bar{y}), (h, k))$. Let $(t_j)_j$ be a net of positive real numbers converging to zero and satisfying the limit in the definition of the directional derivative; that is, $\alpha = \lim_j T_{t_j^{-1}[M(\bar{x}+t_j h)-\bar{y}],\Omega}(k)$. Let $\delta_j > 0$ and $w_j \in \Omega$ such that

$$\begin{aligned} & T_{t_j^{-1}[M(\bar{x}+t_j h)-\bar{y}],\Omega}(k) \leq \delta_j < T_{t_j^{-1}[M(\bar{x}+t_j h)-\bar{y}],\Omega}(k) + t_j, \\ & k + \delta_j w_j \in t_j^{-1} [M(\bar{x} + t_j h) - \bar{y}]; \end{aligned} \quad (37)$$

that is, $(h, k + \delta_j w_j) \in t_j^{-1} [\text{gph } M - (\bar{x}, \bar{y})]$.

Since $\text{gph } M$ is closed convex, we have $T(\text{gph } M; (\bar{x}, \bar{y})) = \text{cl}(\mathbb{R}_+[\text{gph } M - (\bar{x}, \bar{y})])$, and so $(h, k + \delta_j w_j) \in T(\text{gph } M; (\bar{x}, \bar{y}))$, for all j ; that is, $k + \delta_j w_j \in \tilde{T}_M(h)$, for all j .

On the other hand we clearly have $\mathcal{F}_{\tilde{T}_M,\Omega}(h, k) \leq \rho_\Omega(s - k)$, for all $s \in \tilde{T}_M(h)$. Consequently, we obtain for all j that

$$\begin{aligned} & \mathcal{F}_{\tilde{T}_M,\Omega}(h, k) \leq \rho_\Omega((k + \delta_j w_j) - k) = \delta_j \rho_\Omega(w_j) \\ &\leq \delta_j < T_{t_j^{-1}[M(\bar{x}+t_j h)-\bar{y}],\Omega}(k) + t_j, \quad \forall j. \end{aligned} \quad (38)$$

Taking the limit on this inequality, we obtain that

$$\mathcal{F}_{\tilde{T}_M,\Omega}(h, k) \leq \mathcal{F}'_{M,\Omega}((\bar{x}, \bar{y}); (h, k)), \quad (39)$$

thus completing the proof of (3).

(4) The inclusion $T(\text{gph } M, (\bar{x}, \bar{y})) \subset \{(h, k) \in E \times F : \mathcal{T}'_{M, \Omega}((\bar{x}, \bar{y}); (h, k)) = 0\}$ follows from the assertion (3) and Corollary 2. Conversely, let $(h, k) \in E \times F$ with $\mathcal{T}'_{M, \Omega}((\bar{x}, \bar{y}); (h, k)) = 0$. Let (t_j) be a net of positive real numbers converging to zero and satisfying the limit $\mathcal{T}'_{M, \Omega}((\bar{x}, \bar{y}); (h, k)) = \lim_{t_j \rightarrow 0} \mathcal{T}_{t_j^{-1}[M(\bar{x}+t_j h) - \bar{y}], \Omega}(k)$. For any $j \in J$, we can find that $\delta_j > 0$ and $w_j \in \Omega$ such that

$$\mathcal{T}_{t_j^{-1}[M(\bar{x}+t_j h) - \bar{y}], \Omega}(k) \leq \delta_j < \mathcal{T}_{t_j^{-1}[M(\bar{x}+t_j h) - \bar{y}], \Omega}(k) + t_j,$$

$$k + \delta_j w_j \in t_j^{-1} [M(\bar{x} + t_j h) - \bar{y}];$$

$$\text{that is, } (h, k + \delta_j w_j) \in t_j^{-1} [\text{gph } M - (\bar{x}, \bar{y})]. \quad (40)$$

Since the graph of M is closed convex, we obtain that

$$(h, k + \delta_j w_j) \in t_j^{-1} [\text{gph } M - (\bar{x}, \bar{y})] \subset T(\text{gph } M; (\bar{x}, \bar{y})). \quad (41)$$

On the other hand, we have $\lim_j \delta_j = \lim_{t_j \rightarrow 0} \mathcal{T}_{t_j^{-1}[M(\bar{x}+t_j h) - \bar{y}], \Omega}(k) = 0$, and hence, by the boundedness of the set Ω , we get the convergence of $\delta_j w_j$ to zero. This ensures the convergence of $k + \delta_j w_j \rightarrow k$ which ensures by the closedness of the tangent cone that $(h, k) \in T(\text{gph } M, (\bar{x}, \bar{y}))$, and hence, the proof of (4) is finished.

(5) The inclusion $\text{cl}_{w^*}(\mathbb{R}_+ \partial \mathcal{T}_{M, \Omega}((\bar{x}, \bar{y}))) \subset N(\text{gph } M; (\bar{x}, \bar{y}))$ follows directly from part (2). So we prove the reverse one. To do that, we start by proving the inclusion

$$(\partial \mathcal{T}_{M, \Omega}((\bar{x}, \bar{y})))^- \subset T(\text{gph } M; (\bar{x}, \bar{y})). \quad (42)$$

Fix any $(h, k) \in (\partial \mathcal{T}_{M, \Omega}((\bar{x}, \bar{y})))^-$. Then

$$\langle (x^*, y^*); (h, k) \rangle \leq 0, \quad \forall (x^*, y^*) \in \partial \mathcal{T}_{M, \Omega}((\bar{x}, \bar{y})). \quad (43)$$

Since $\partial \mathcal{T}_{M, \Omega}((\bar{x}, \bar{y})) \neq \emptyset$ (because $0 \in \partial \mathcal{T}_{M, \Omega}((\bar{x}, \bar{y}))$) and $(\bar{x}, \bar{y}) \in \text{core}(\text{dom } \mathcal{T}_{M, \Omega})$, then $\mathcal{T}'_{M, \Omega}((\bar{x}, \bar{y}); (h, k)) = \sigma_{\partial \mathcal{T}_{M, \Omega}((\bar{x}, \bar{y}))}(h, k)$. Thus

$$\mathcal{T}'_{M, \Omega}((\bar{x}, \bar{y}); (h, k)) = \sigma_{\partial \mathcal{T}_{M, \Omega}((\bar{x}, \bar{y}))}(h, k) \leq 0. \quad (44)$$

Since the reverse inequality $\mathcal{T}'_{M, \Omega}((\bar{x}, \bar{y}); (h, k)) \geq 0$ follows directly from part (1), then we get the equality form $\mathcal{T}'_{M, \Omega}((\bar{x}, \bar{y}); (h, k)) = 0$, and hence, part (4) ensures that $(h, k) \in T(\text{gph } M; (\bar{x}, \bar{y}))$. Therefore, $(\partial \mathcal{T}_{M, \Omega}((\bar{x}, \bar{y})))^- \subset T(\text{gph } M; (\bar{x}, \bar{y}))$. Since clearly we have $(\partial \mathcal{T}_{M, \Omega}((\bar{x}, \bar{y})))^- = (\text{cl}_{w^*}(\mathbb{R}_+ \partial \mathcal{T}_{M, \Omega}((\bar{x}, \bar{y}))))^-$, then

$$(\text{cl}_{w^*}(\mathbb{R}_+ \partial \mathcal{T}_{M, \Omega}((\bar{x}, \bar{y}))))^- \subset T(\text{gph } M; (\bar{x}, \bar{y})), \quad (45)$$

which ensures that

$$\begin{aligned} & N(\text{gph } M; (\bar{x}, \bar{y})) \\ &= (T(\text{gph } M; (\bar{x}, \bar{y})))^- \\ &\subset ((\text{cl}_{w^*}(\mathbb{R}_+ \partial \mathcal{T}_{M, \Omega}((\bar{x}, \bar{y}))))^-)^- \\ &= \text{cl}_{w^*}(\mathbb{R}_+ \partial \mathcal{T}_{M, \Omega}((\bar{x}, \bar{y}))). \end{aligned} \quad (46)$$

The last equality follows from the fact that $E \times F$ is locally convex. Thus, the proof of the theorem is complete. $\square$

Many corollaries can be deduced from this theorem. We state the following one [18] by taking Ω to be the closed unit ball of a normed vector space F and M to be a constant set-valued mapping; that is, $M(x) = S$.

Corollary 9. Assume that F is a normed vector space, S is a closed convex subset in F , and $\bar{y} \in S$. Then, one has

$$\begin{aligned} (1) \quad d'_S(\bar{y}; k) &= \lim_{t \rightarrow 0^+} t^{-1} d_S(\bar{y} + tk) \\ &= \lim_{t \rightarrow 0^+} d_{t^{-1}[S - \bar{y}]}(k) \quad \forall k \in F; \\ (2) \quad \partial d_S(\bar{y}) &= N(S; \bar{y}) \cap \{y^* \in F^* : \|y^*\| \leq 1\}; \\ (3) \quad d'_S(\bar{y}; k) &= d_{T(S; \bar{y})}(k), \quad \forall k \in F; \\ (4) \quad T(S; \bar{y}) &= \{k \in F : d'_S(\bar{y}; k) = 0\}. \end{aligned} \quad (47)$$

4. Points outside the Graph of the Set-Valued Mapping

Now, we consider those points outside the graph of set-valued mappings M with nonempty closed values. Let $(\bar{x}, \bar{y}) \notin \text{gph } M$. Clearly, by Corollary 2, we have $\mathcal{T}_{M, \Omega}(\bar{x}, \bar{y}) > 0$. Denote $r := \mathcal{T}_{M, \Omega}(\bar{x}, \bar{y}) > 0$, and define the enlargement set-valued mapping

$$M_r(x) := \{y \in F : \mathcal{T}_{M, \Omega}(x, y) \leq r\}. \quad (48)$$

Clearly, the graph of M_r is defined as $\text{gph } M_r := \{(x, y) \in E \times F : \mathcal{T}_{M, \Omega}(x, y) \leq r\}$. The following lemma is needed. It extends many existing results from normed spaces to Hausdorff topological vector spaces and from the case of sets to the case of set-valued mapping (see for instance [15, 19]). Obviously, we have $\bar{y} \in M_r(\bar{x})$ and $M(\bar{x}) \subset M_r(\bar{x})$. We have also to point out that, due to Lemma 3, the convexity of the graph $\text{gph } M$ ensures the convexity of the graph of M_r whenever Ω is bounded convex. However, the l.s.c. of $\mathcal{T}_{M, \Omega}$ ensures the closedness of the graph of $\text{gph } M_r$. It follows then, from Lemma 3, the convexity of the minimal time function $\mathcal{T}_{M_r, \Omega}$ whenever $\mathcal{T}_{M, \Omega}$ is lower semicontinuous and Ω is bounded convex.

Lemma 10. Let E and F be two Hausdorff topological vector spaces. Assume that M has nonempty closed values, $(\bar{x}, \bar{y}) \notin \text{gph } M$, and Ω is a bounded convex set. Then, for any $(x, y) \in E \times F$ we have

$$\mathcal{T}_{M, \Omega}(x, y) \leq \mathcal{T}_{M_r, \Omega}(x, y) + r \quad (49)$$

and for any $(x, y) \notin \text{gph } M_r$ we have

$$\mathcal{T}_{M, \Omega}(x, y) = \mathcal{T}_{M_r, \Omega}(x, y) + r. \quad (50)$$

Proof. First we prove the inequality

$$\mathcal{T}_{M, \Omega}(\bar{x}, y) \leq \mathcal{T}_{M_r, \Omega}(\bar{x}, y) + r, \quad \text{for any } y \in F. \quad (51)$$

Assume that $(x, y) \in E \times F$ with $\mathcal{T}_{M_r, \Omega}(x, y) < \infty$. The case $\mathcal{T}_{M_r, \Omega}(x, y) = \infty$ is obvious. Fix any $\delta \geq 0$ satisfying $[y + \delta\Omega] \cap M_r(x) \neq \emptyset$. Then, there exists $w \in \Omega$ such that $y + \delta w \in M_r(x)$; that is, $\mathcal{T}_{M_r, \Omega}(x, y + \delta w) \leq r$. Then, for any $\epsilon > 0$, we can choose $\lambda > 0$ and $z \in \Omega$ such that

$$\begin{aligned} & \mathcal{T}_{M, \Omega}(x, y + \delta w) \\ & \leq \lambda < \mathcal{T}_{M, \Omega}(x, y + \delta w) + \epsilon, \quad y + \delta w + \lambda z \in M(x). \end{aligned} \quad (52)$$

By convexity of Ω , we have $\delta w + \lambda z \in (\delta + \lambda)\Omega$, and hence,

$$y + \delta w + \lambda z \in M(x) \cap [y + (\delta + \lambda)\Omega] \neq \emptyset. \quad (53)$$

Therefore,

$$\begin{aligned} \mathcal{T}_{M, \Omega}(x, y) & \leq \delta + \lambda \\ & < \delta + \mathcal{T}_{M, \Omega}(x, y + \delta w) + \epsilon \\ & < \delta + r + \epsilon. \end{aligned} \quad (54)$$

Since δ is taken arbitrarily satisfying $(y + \delta\Omega) \cap M_r(x) \neq \emptyset$, we obtain by taking the infimum over all δ the inequality

$$\mathcal{T}_{M, \Omega}(x, y) \leq \mathcal{T}_{M_r, \Omega}(x, y) + r + \epsilon, \quad \forall \epsilon > 0, \quad (55)$$

and by taking $\epsilon \rightarrow 0$ we get the first desired inequality

$$\mathcal{T}_{M, \Omega}(x, y) \leq \mathcal{T}_{M_r, \Omega}(x, y) + r. \quad (56)$$

Assume now that $y \notin M_r(x)$, and let us prove the reverse inequality; that is, $\mathcal{T}_{M_r, \Omega}(x, y) + r \leq \mathcal{T}_{M, \Omega}(x, y)$. The case $\mathcal{T}_{M, \Omega}(x, y) = \infty$ is obvious so we suppose that $\mathcal{T}_{M, \Omega}(x, y) < \infty$. Take arbitrarily a nonnegative number $\delta \geq 0$ for which $[y + \delta\Omega] \cap M(x) \neq \emptyset$. Then, there exists $w \in \Omega$ such that $y + \delta w \in M(x)$. Clearly, $\mathcal{T}_{M, \Omega}(x, y) \leq \delta$, and since $y \notin M_r(x)$, we have $\delta > r$. Therefore,

$$y + (\delta - r)w + rw \in M(x) \cap [y + (\delta - r)\Omega] \neq \emptyset, \quad (57)$$

and so

$$\begin{aligned} & \mathcal{T}_{M, \Omega}(x, y + (\delta - r)w) \\ & \leq r, \quad \text{that is, } y + (\delta - r)w \in M_r(x). \end{aligned} \quad (58)$$

Thus,

$$y + (\delta - r)w \in M_r(x) \cap [y + (\delta - r)\Omega] \neq \emptyset. \quad (59)$$

Then,

$$\mathcal{T}_{M_r, \Omega}(x, y) \leq \delta - r. \quad (60)$$

Since δ is taken arbitrarily with $[y + \delta\Omega] \cap M(x) \neq \emptyset$, we can take the infimum over all those δ , and hence, we obtain that

$$\mathcal{T}_{M_r, \Omega}(x, y) + r \leq \mathcal{T}_{M, \Omega}(x, y). \quad (61)$$

The proof is complete. $\square$

The previous lemma extends Lemma 3.4 in [2] from the case of sets to the case of set-valued mappings in Hausdorff topological vector spaces. Also, it extends the inequality (4.41) on page 97 in [15] from the case where F is a normed vector space to the case where F is a Hausdorff topological vector space and from the case of distance function to images Δ_M to the case of minimal time function $\mathcal{T}_{M, \Omega}$.

The first consequence of Lemma 10 is the following proposition in which we establish a relationship between the directional derivatives of $\mathcal{T}_{M, \Omega}$ and $\mathcal{T}_{M_r, \Omega}$.

Proposition 11. *Let E and F be two Hausdorff topological vector spaces. Assume that M has closed convex graph, $(\bar{x}, \bar{y}) \notin \text{gph } M$, and Ω is a bounded convex set. Let $r := \mathcal{T}_{M, \Omega}(\bar{x}, \bar{y}) > 0$. Then, for any $(h, k) \in E \times F$, one has*

$$\mathcal{T}'_{M, \Omega}((\bar{x}, \bar{y}); (h, k)) \leq \mathcal{T}'_{M_r, \Omega}((\bar{x}, \bar{y}); (h, k)). \quad (62)$$

Proof. It follows, directly from the first part of the previous lemma, the fact that $\mathcal{T}_{M_r, \Omega}(\bar{x}, \bar{y}) = 0$ and the definition of the directional derivative. $\square$

In the following theorem, we characterize the convex tangent cone of the graph of the enlargement set-valued mapping M_r as the set of all directions (h, k) in $E \times F$ for which the directional derivative of $T_{M, \Omega}$ is nonnegative.

Theorem 12. *Let E and F be two Hausdorff topological vector spaces. Assume that M has a closed convex graph, $(\bar{x}, \bar{y}) \notin \text{gph } M$, and Ω is a bounded convex set. Assume that $\mathcal{T}_{M, \Omega}$ is l.s.c. at $(\bar{x}, \bar{y})$. Then, one has*

$$\begin{aligned} & T(\text{gph } M_r; (\bar{x}, \bar{y})) \\ & = \{(h, k) \in E \times F : \mathcal{T}'_{M_r, \Omega}((\bar{x}, \bar{y}); (h, k)) = 0\} \\ & = \{(h, k) \in E \times F : \mathcal{T}'_{M, \Omega}((\bar{x}, \bar{y}); (h, k)) \leq 0\}. \end{aligned} \quad (63)$$

Proof. The first equality follows directly from part (4) in Theorem 8. The direct inclusion of the second equality follows directly from the previous proposition. So it remains to prove the inclusion

$$\begin{aligned} & \{(h, k) \in E \times F : \mathcal{T}'_{M, \Omega}((\bar{x}, \bar{y}); (h, k)) \leq 0\} \\ & \subset T(\text{gph } M_r; (\bar{x}, \bar{y})). \end{aligned} \quad (64)$$

Let $(h, k) \in E \times F$ with $\mathcal{T}'_{M, \Omega}((\bar{x}, \bar{y}); (h, k)) \leq 0$. Let $(t_j)_j$ be a net in $(0, \infty)$ converging to 0 and satisfying the limit

$$\begin{aligned} & \mathcal{T}'_{M, \Omega}((\bar{x}, \bar{y}); (h, k)) \\ & = \lim_j t_j^{-1} [\mathcal{T}_{M, \Omega}(\bar{x} + t_j h, \bar{y} + t_j k) - \mathcal{T}_{M, \Omega}(\bar{x}, \bar{y})]. \end{aligned} \quad (65)$$

Use the lower semicontinuity of $\mathcal{T}_{M, \Omega}$ at $(\bar{x}, \bar{y})$ and the assumption $\bar{y} \notin M(\bar{x})$ to find some $j_0 \in J$ such that

$$\mathcal{T}_{M, \Omega}(\bar{x} + t_j h, \bar{y} + t_j k) > 2t_j^2 > 0, \quad \text{for any } j \geq j_0. \quad (66)$$

Using now the assumption $\mathcal{T}'_{M,\Omega}((\bar{x}, \bar{y}); (h, k)) \leq 0$ ensures the existence of some $j_1 \geq j_0$ such that

$$\mathcal{T}_{M,\Omega}(\bar{x} + t_j h, \bar{y} + t_j k) \leq r + t_j^2, \quad \text{for any } j \geq j_1. \quad (67)$$

Thus, combining (66) and (67), we obtain

$$0 < 2t_j^2 < \mathcal{T}_{M,\Omega}(\bar{x} + t_j h, \bar{y} + t_j k) \leq r + t_j^2, \quad \text{for any } j \geq j_1. \quad (68)$$

Use now the definition of the minimal time function to choose for any $j \geq j_1$ some $w_j \in \Omega$ and $\delta_j \geq 0$ such that

$$\begin{aligned} \mathcal{T}_{M,\Omega}(\bar{x} + t_j h, \bar{y} + t_j k) &\leq \delta_j < \mathcal{T}_{M,\Omega}(\bar{x} + t_j h, \bar{y} + t_j k) + t_j^2, \\ \bar{y} + t_j k + \delta_j w_j &\in M(\bar{x} + t_j h). \end{aligned} \quad (69)$$

Hence,

$$0 \leq \delta_j - 2t_j^2 < r, \quad \forall j \geq j_1. \quad (70)$$

Let $z_j := \bar{y} + t_j k + 2t_j^2 w_j$ and $k_j := k + 2t_j w_j$. Then,

$$\begin{aligned} z_j + (\delta_j - 2t_j^2) w_j &= \bar{y} + t_j k + \delta_j w_j \in M(\bar{x} + t_j h) \\ \cap (z_j + (\delta_j - 2t_j^2) \Omega) &\neq \emptyset. \end{aligned} \quad (71)$$

So $\mathcal{T}_{M,\Omega}(\bar{x} + t_j h, z_j) \leq \delta_j - 2t_j^2 \leq r$; that is, $z_j \in M_r(\bar{x} + t_j h)$.

Therefore,

$$\bar{y} + t_j k_j = \bar{y} + t_j k + 2t_j^2 w_j \in M_r(\bar{x} + t_j h), \quad (72)$$

and so

$$(h, k_j) \in t_j^{-1} [gph M_r - (\bar{x}, \bar{y})] \subset T(gph M_r; (\bar{x}, \bar{y})). \quad (73)$$

Now, the boundedness of Ω ensures the convergence of $2t_j w_j$ to zero, which gives the convergence of k_j to k . Thus, by the closedness of the convex tangent cone, we get $(h, k) \in T(gph M_r, (\bar{x}, \bar{y}))$, and hence, the proof is finished. $\square$

Various corollaries can be deduced from the above theorem by taking special cases of M and Ω and the spaces E and F . We state the following one which has been proved in Corollary 4.4 in [15] (see also [20]) by taking F to be a normed vector space, Ω to be the closed unit ball of F , and M to be a constant set-valued mapping; that is, $M(x) = S$. In this case, $M_r(x)$ coincides with the enlargement set $S(r)$ defined by $S(r) = \{y \in F : d_S(y) \leq r\}$.

Corollary 13. *Let F be a normed vector space, S a closed convex subset in F , and $\bar{y} \in S$. Then, one has*

$$\begin{aligned} d'_S(\bar{y}; \bar{k}) &\leq d'_{S(r)}(\bar{y}; \bar{k}) \leq d_{T(S(r); \bar{y})}(\bar{k}), \quad \forall \bar{k} \in F, \\ T(S(r); \bar{y}) &= \{k \in F : d'_{S(r)}(\bar{y}; \bar{k}) = 0\} \\ &= \{k \in F : d'_S(\bar{y}; \bar{k}) \leq 0\}. \end{aligned} \quad (74)$$

Now, we turn to establish some relationships between the convex normal cone to $gph M_r$ and the convex subdifferential of $\mathcal{T}_{M,\Omega}$ at $(\bar{x}, \bar{y}) \notin gph M$.

Theorem 14. *Let E and F Hausdorff topological vector spaces, M a set-valued mapping with nonempty closed values, Ω a bounded convex set in E , and $(\bar{x}, \bar{y}) \notin gph M$. Assume that $\mathcal{T}_{M,\Omega}$ is l.s.c. at $(\bar{x}, \bar{y})$, E and F are locally convex, and $\partial \mathcal{T}_{M,\Omega}((\bar{x}, \bar{y})) \neq \emptyset$. Then,*

$$\begin{aligned} N(gph M_r; (\bar{x}, \bar{y})) &= cl_{w^*}(\mathbb{R}_+ \partial \mathcal{T}_{M,\Omega}((\bar{x}, \bar{y}))) \\ &= cl_{w^*}(\mathbb{R}_+ \partial \mathcal{T}_{M,\Omega}((\bar{x}, \bar{y}))). \end{aligned} \quad (75)$$

Proof. The first equality follows directly from Theorem 8. We will prove the equality $N(gph M_r; (\bar{x}, \bar{y})) = cl_{w^*}(\mathbb{R}_+ \partial \mathcal{T}_{M,\Omega}((\bar{x}, \bar{y})))$. Assume that $\partial \mathcal{T}_{M,\Omega}(\bar{x}, \bar{y}) \neq \emptyset$. First, we prove that

$$\partial \mathcal{T}_{M,\Omega}(\bar{x}, \bar{y}) \subset N(gph M_r; (\bar{x}, \bar{y})). \quad (76)$$

Since $\partial^C \mathcal{T}_{M,\Omega}(\bar{x}, \bar{y}) \neq \emptyset$, we take some $(x^*, y^*) \in E^* \times F^*$ such that

$$\begin{aligned} \langle (x^*, y^*), (h, k) \rangle \\ \leq \mathcal{T}'_{M,\Omega}((\bar{x}, \bar{y}); (h, k)), \quad \forall (h, k) \in E \times F. \end{aligned} \quad (77)$$

Hence, for any $(h, k) \in T(gph M_r; (\bar{x}, \bar{y}))$, we have $\mathcal{T}'_{M,\Omega}((\bar{x}, \bar{y}); (h, k)) \leq 0$ (by Theorem 12), and so

$$\langle (x^*, y^*), (h, k) \rangle \leq \mathcal{T}'_{M,\Omega}((\bar{x}, \bar{y}); (h, k)) \leq 0; \quad (78)$$

that is, $(x^*, y^*) \in (T(gph M_r; (\bar{x}, \bar{y})))^\circ = N(gph M_r; (\bar{x}, \bar{y}))$. Since $N(gph M_r; (\bar{x}, \bar{y}))$ is a closed convex cone in $E^* \times F^*$, we obtain that

$$cl_{w^*}(\mathbb{R}_+ \partial \mathcal{T}_{M,\Omega}(\bar{x}, \bar{y})) \subset N(gph M_r; (\bar{x}, \bar{y})). \quad (79)$$

Conversely, we have to prove that

$$N(gph M_r; (\bar{x}, \bar{y})) \subset cl_{w^*}(\mathbb{R}_+ \partial \mathcal{T}_{M,\Omega}(\bar{x}, \bar{y})). \quad (80)$$

We will prove that the negative polar of $cl_{w^*}(\mathbb{R}_+ \partial \mathcal{T}_{M,\Omega}(\bar{x}, \bar{y}))$ is included in $T(gph M_r; (\bar{x}, \bar{y})) = (N(gph M_r; (\bar{x}, \bar{y})))^\circ$; that is, (by Theorem 12)

$$\begin{aligned} (cl_{w^*}(\mathbb{R}_+ \partial \mathcal{T}_{M,\Omega}(\bar{x}, \bar{y})))^\circ \\ \subset \{(h, k) \in E \times F : \mathcal{T}'_{M,\Omega}((\bar{x}, \bar{y}); (h, k)) \leq 0\}. \end{aligned} \quad (81)$$

Note first that

$$(cl_{w^*}(\mathbb{R}_+ \partial \mathcal{T}_{M,\Omega}(\bar{x}, \bar{y})))^\circ = (\partial \mathcal{T}_{M,\Omega}(\bar{x}, \bar{y}))^\circ. \quad (82)$$

So consider any $(h, k) \in (\partial \mathcal{T}_{M,\Omega}(\bar{x}, \bar{y}))^\circ$. Then, we have

$$\langle (x^*, y^*), (h, k) \rangle \leq 0, \quad \forall (x^*, y^*) \in \partial \mathcal{T}_{M,\Omega}(\bar{x}, \bar{y}). \quad (83)$$

Hence, (since $\partial \mathcal{T}_{M,\Omega}(\bar{x}, \bar{y}) \neq \emptyset$ and $(\bar{x}, \bar{y}) \in \text{core}(\text{dom } T_{M,\Omega})$)

$$\begin{aligned} & \mathcal{T}'_{M,\Omega}((\bar{x}, \bar{y}); (h, k)) \\ &= \sigma_{\partial \mathcal{T}_{M,\Omega}(\bar{x}, \bar{y})}(h, k) \\ &= \sup_{(x^*, y^*) \in \partial \mathcal{T}_{M,\Omega}(\bar{x}, \bar{y})} \langle (x^*, y^*), (h, k) \rangle \leq 0, \end{aligned} \quad (84)$$

which ensures by Theorem 12 that $(\bar{h}, \bar{k}) \in T(\text{gph } M_r; (\bar{x}, \bar{y}))$. Thus, completing the proof. $\square$

The following corollary extends Corollary 1 in [14] from the case of sets to the case of set-valued mappings and from normed vector spaces to Hausdorff Topological vector spaces. It says that if we put the calmness of $\mathcal{T}_{M,\Omega}$ instead of its l.s.c. and the assumption $0 \notin \partial \mathcal{T}_{M,\Omega}(\bar{x}, \bar{y})$ instead of the assumption $\partial \mathcal{T}_{M,\Omega}(\bar{x}, \bar{y}) \neq \emptyset$ in Theorem 14 we may remove the weak star closedness in the second equation in Theorem 14. Recall that f is said to be calm at $\bar{u}$ if there exists a closed balanced neighborhood V of zero and $U \in \mathcal{N}(\bar{u})$ such that

$$|f(u) - f(\bar{u})| \leq \rho_V(\bar{u} - u), \quad \forall u \in U. \quad (85)$$

Corollary 15. *Let E and F Hausdorff topological vector spaces, $M : E \rightrightarrows F$ a set-valued mapping with a closed convex graph, Ω be a bounded convex set in F , and $(\bar{x}, \bar{y}) \notin \text{gph } M$. If $\mathcal{T}_{M,\Omega}$ is calm at $(\bar{x}, \bar{y})$ with $0 \notin \partial \mathcal{T}_{M,\Omega}(\bar{x}, \bar{y})$, then one has*

$$N(\text{gph } M_r; (\bar{x}, \bar{y})) = \mathbb{R}_+ \partial \mathcal{T}_{M,\Omega}(\bar{x}, \bar{y}). \quad (86)$$

Proof. Using the calmness of $\mathcal{T}_{M,\Omega}$ at $(\bar{x}, \bar{y})$ and Banach-Alaoglu theorem (see Theorem 3.5 in [21]), we get the weak star compactness of $\partial^C \mathcal{T}_{M,\Omega}(\bar{x}, \bar{y})$ in $E^* \times F^*$. Therefore, the assumption $0 \notin \partial^C \mathcal{T}_{M,\Omega}(\bar{x}, \bar{y})$ with the weak star compactness of $\partial^C \mathcal{T}_{M,\Omega}(\bar{x}, \bar{y})$ ensures the weak star closedness of the cone generated by $\partial^C \mathcal{T}_{M,\Omega}(\bar{x}, \bar{y})$; that is, $\text{cl}_{w^*}(\mathbb{R}_+ \partial^C \mathcal{T}_{M,\Omega}(\bar{x}, \bar{y})) = \mathbb{R}_+ \partial^C \mathcal{T}_{M,\Omega}(\bar{x}, \bar{y})$. Thus, the conclusion follows from Theorem 14. $\square$

The next results depend on the nonemptiness of the minimum set for the minimal time function $\mathcal{T}_{M,\Omega}$ defined as follows

$$\pi_{M(x),\Omega}(u) := \{y \in M(x) : \mathcal{T}_{M,\Omega}(x, y) = \rho_\Omega(y - u)\}. \quad (87)$$

We begin with the following lemma.

Lemma 16. *Let E and F be two Hausdorff topological vector spaces, $M : E \rightrightarrows F$ a set-valued mapping with closed convex graph, Ω a bounded convex set in F , and $(\bar{x}, \bar{y}) \notin \text{gph } M$ with $\mathcal{T}_{M,\Omega}(\bar{x}, \bar{y}) < \infty$ and $\pi_{M(\bar{x}),\Omega}(\bar{y}) \neq \emptyset$. For every $y \in \pi_{M(\bar{x}),\Omega}(\bar{y})$ and for every $t \in (0, 1]$, one has*

$$\mathcal{T}_{M,\Omega}(\bar{x}, ty + (1-t)\bar{y}) = (1-t)\mathcal{T}_{M,\Omega}(\bar{x}, \bar{y}). \quad (88)$$

This ensures that $y \in \pi_{M(\bar{x}),\Omega}(ty + (1-t)\bar{y})$, for any $t \in (0, 1]$.

Proof. Let $y \in \pi_{M(\bar{x}),\Omega}(\bar{y})$. Then $y \in M(\bar{x})$ with $\mathcal{T}_{M,\Omega}(\bar{x}, \bar{y}) = \rho_\Omega(y - \bar{y}) < \infty$. First, we prove the inequality

$$\begin{aligned} & \mathcal{T}_{M,\Omega}(\bar{x}, ty + (1-t)\bar{y}) \\ & \leq (1-t)\mathcal{T}_{M,\Omega}(\bar{x}, \bar{y}), \quad \forall t \in (0, 1]. \end{aligned} \quad (89)$$

Put $\alpha := \rho_\Omega(y - \bar{y}) > 0$, and fix any $t \in (0, 1]$. Clearly, by definition of the Minkowski function, we can find for any $\epsilon > 0$ some $\alpha_0 \geq 0$ such that $y - \bar{y} \in \alpha_0 \Omega$ and $\alpha_0 \in [\alpha, \alpha + \epsilon]$, and so

$$\begin{aligned} y &= ty + (1-t)\bar{y} + \alpha_0(1-t) \frac{(y - \bar{y})}{\alpha_0} \in M(\bar{x}) \\ & \cap [ty + (1-t)\bar{y} + \alpha_0(1-t)\Omega] \neq \emptyset, \end{aligned} \quad (90)$$

which ensures that

$$\begin{aligned} & \mathcal{T}_{M,\Omega}(\bar{x}, ty + (1-t)\bar{y}) \\ & \leq \alpha_0(1-t) < (1-t)[\rho_\Omega(y - \bar{y}) + \epsilon] \\ & \leq (1-t)\mathcal{T}_{M,\Omega}(\bar{x}, \bar{y}) + \epsilon, \quad \forall \epsilon > 0. \end{aligned} \quad (91)$$

Taking $\epsilon \rightarrow 0^+$ finishes the proof of the first inequality.

By the first part, we have $\mathcal{T}_{M,\Omega}(\bar{x}, ty + (1-t)\bar{y}) < \infty$ whenever $\mathcal{T}_{M,\Omega}(\bar{x}, \bar{y}) < \infty$ and so for any $\epsilon > 0$, we can find some $w \in \Omega$, some $\delta > 0$, and some $\delta_0 > 0$ such that

$$\begin{aligned} & \mathcal{T}_{M,\Omega}(\bar{x}, ty + (1-t)\bar{y}) \\ & \leq \delta < \mathcal{T}_{M,\Omega}(\bar{x}, ty + (1-t)\bar{y}) + \epsilon, \\ & ty + (1-t)\bar{y} + \delta w \in M(\bar{x}), \\ & y - \bar{y} \in \delta_0 \Omega \quad \text{with} \quad 0 < \rho_\Omega(y - \bar{y}) \leq \delta_0 < \rho_\Omega(y - \bar{y}) + \epsilon. \end{aligned} \quad (92)$$

Then, by convexity of Ω , we have $\delta_0 t((y - \bar{y})/\delta_0) + \delta w \in (\delta_0 t + \delta)\Omega$, and so

$$\begin{aligned} \bar{y} + t(y - \bar{y}) + \delta w &= \bar{y} + \delta_0 t \frac{(y - \bar{y})}{\delta_0} + \delta w \in M(\bar{x}) \\ & \cap [\bar{y} + (\delta_0 t + \delta)\Omega] \neq \emptyset. \end{aligned} \quad (93)$$

Hence,

$$\begin{aligned} & \mathcal{T}_{M,\Omega}(\bar{x}, \bar{y}) \\ & \leq \delta_0 t + \delta \\ & < t(\rho_\Omega(y - \bar{y}) + \epsilon) + \mathcal{T}_{M,\Omega}(\bar{x}, ty + (1-t)\bar{y}) + \epsilon \\ & < t\mathcal{T}_{M,\Omega}(\bar{x}, \bar{y}) + \mathcal{T}_{M,\Omega}(\bar{x}, ty + (1-t)\bar{y}) + 2\epsilon, \end{aligned} \quad (94)$$

and so by taking $\epsilon \rightarrow 0^+$, we obtain that

$$(1-t)\mathcal{T}_{M,\Omega}(\bar{x}, \bar{y}) \leq \mathcal{T}_{M,\Omega}(\bar{x}, ty + (1-t)\bar{y}). \quad (95)$$

This completes the proof of the lemma. $\square$

We use this lemma to prove the following proposition on directional derivatives and convex subdifferentials of $\mathcal{T}_{M,\Omega}$ at points outside the graph.

Proposition 17. *Assume that the assumptions of Lemma 16 are fulfilled. Then, for every $(\bar{x}, \bar{y}) \notin \text{gph } M$ and for every $y \in \pi_{M(\bar{x},\Omega)}(\bar{y})$, one has*

$$\begin{aligned} \mathcal{T}'_{M,\Omega}((\bar{x}, \bar{y}); (0, y - \bar{y})) &= -\mathcal{T}_{M,\Omega}(\bar{x}, \bar{y}), \\ \partial\mathcal{T}_{M,\Omega}(\bar{x}, \bar{y}) &\subset \{(x^*, y^*) \in E^* \times F^* : \sigma_\Omega(-y^*) = 1\}. \end{aligned} \quad (96)$$

Proof. By Lemma 16, we have

$$\begin{aligned} T_{M,\Omega}((\bar{x}, \bar{y}); (0, y - \bar{y})) \\ = \lim_{t \rightarrow 0^+} t^{-1} [\mathcal{T}_{M,\Omega}(\bar{x}, \bar{y} + t(y - \bar{y})) - \mathcal{T}_{M,\Omega}(\bar{x}, \bar{y})] \\ = -\mathcal{T}_{M,\Omega}(\bar{x}, \bar{y}). \end{aligned} \quad (97)$$

We turn now to show the inclusion

$$\partial\mathcal{T}_{M,\Omega}(\bar{x}, \bar{y}) \subset \{(x^*, y^*) \in E^* \times F^* : \sigma_\Omega(-y^*) = 1\}. \quad (98)$$

Fix any $(x^*, y^*) \in \partial\mathcal{T}_{M,\Omega}(\bar{x}, \bar{y})$. Then

$$\langle y^*, k \rangle \leq \mathcal{T}'_{M,\Omega}((\bar{x}, \bar{y}); (0, k)), \quad \forall k \in F. \quad (99)$$

By assumption, we have $\pi_{M(\bar{x},\Omega)}(\bar{y}) \neq \emptyset$. Then, there exists $y_0 \in \pi_{M(\bar{x},\Omega)}(\bar{y})$; that is, $y_0 \in M(\bar{x})$ with $\mathcal{T}_{M,\Omega}(\bar{x}, \bar{y}) = \rho_\Omega(y_0 - \bar{y}) \neq 0$. By the first part, we obtain that

$$\begin{aligned} \langle y^*, y_0 - \bar{y} \rangle &\leq \mathcal{T}'_{M,\Omega}((\bar{x}, \bar{y}); (0, y_0 - \bar{y})) \\ &= -\mathcal{T}_{M,\Omega}(\bar{x}, \bar{y}) = -\rho_\Omega(y_0 - \bar{y}), \end{aligned} \quad (100)$$

and so

$$\left\langle -y^*, \frac{y_0 - \bar{y}}{\rho_\Omega(y_0 - \bar{y})} \right\rangle \geq 1, \quad (101)$$

since $\rho_\Omega(y_0 - \bar{y}) \neq 0$, because $\bar{y} \notin M(\bar{x})$. Thus, for any $\epsilon > 0$, there exist $w \in \Omega$ and $\alpha_0 > 0$ such that

$$0 < \rho_\Omega(y_0 - \bar{y}) \leq \alpha_0 < \rho_\Omega(y_0 - \bar{y}) + \epsilon \quad (102)$$

and $y_0 - \bar{y} \in \alpha_0 \Omega$. Thus,

$$\begin{aligned} \sigma_\Omega(-y^*) &\geq \left\langle -y^*, \frac{y_0 - \bar{y}}{\alpha_0} \right\rangle \\ &= \frac{\rho_\Omega(y_0 - \bar{y})}{\alpha_0} \left\langle -y^*, \frac{y_0 - \bar{y}}{\rho_\Omega(y_0 - \bar{y})} \right\rangle \\ &\geq \frac{\rho_\Omega(y_0 - \bar{y})}{\alpha_0} \geq \frac{\rho_\Omega(y_0 - \bar{y})}{\rho_\Omega(y_0 - \bar{y}) + \epsilon}, \end{aligned} \quad (103)$$

and hence,

$$\sigma_\Omega(-y^*) \geq 1 - \frac{\epsilon}{\rho_\Omega(y_0 - \bar{y}) + \epsilon}, \quad \forall \epsilon > 0. \quad (104)$$

Taking $\epsilon \rightarrow 0^+$ yields

$$\sigma_\Omega(-y^*) \geq 1. \quad (105)$$

Let us prove the reverse inequality

$$\sigma_\Omega(-y^*) \leq 1. \quad (106)$$

Let $(t_j)_j$ be a net in $(0, \infty)$ converging to 0 and satisfying

$$\begin{aligned} \mathcal{T}'_{M,\Omega}((\bar{x}, \bar{y}); (0, k)) \\ = \lim_j t_j^{-1} [\mathcal{T}_{M,\Omega}(\bar{x}, \bar{y} + t_j k) - \mathcal{T}_{M,\Omega}(\bar{x}, \bar{y})]. \end{aligned} \quad (107)$$

By Lemma 10, we have, for $r := \mathcal{T}_{M,\Omega}(\bar{x}, \bar{y})$, the following inequality

$$\begin{aligned} \mathcal{T}_{M,\Omega}(x, y) - \mathcal{T}_{M,\Omega}(\bar{x}, \bar{y}) \\ \leq \mathcal{T}_{M_r,\Omega}(x, y) - \mathcal{T}_{M_r,\Omega}(\bar{x}, \bar{y}), \quad \forall (x, y) \in E \times F, \end{aligned} \quad (108)$$

which ensures that

$$\begin{aligned} t_j^{-1} [\mathcal{T}_{M,\Omega}(\bar{x}, \bar{y} + t_j k) - \mathcal{T}_{M,\Omega}(\bar{x}, \bar{y})] \\ \leq t_j^{-1} \mathcal{T}_{M_r,\Omega}(\bar{x}, \bar{y} + t_j k) \\ = t_j^{-1} \inf_{u \in M_r(\bar{x})} \rho_\Omega(u - (\bar{y} + t_j k)) \\ \leq t_j^{-1} \rho_\Omega(\bar{y} - (\bar{y} + t_j k)) = \rho_\Omega(-k), \end{aligned} \quad (109)$$

and so

$$t_j^{-1} [\mathcal{T}_{M,\Omega}(\bar{x}, \bar{y} + t_j k) - \mathcal{T}_{M,\Omega}(\bar{x}, \bar{y})] \leq \rho_\Omega(-k). \quad (110)$$

Taking the limit on j , we get that

$$\begin{aligned} \mathcal{T}'_{M,\Omega}((\bar{x}, \bar{y}); (0, k)) \\ = \lim_j t_j^{-1} [\mathcal{T}_{M,\Omega}(\bar{x}, \bar{y} + t_j k) - \mathcal{T}_{M,\Omega}(\bar{x}, \bar{y})] \leq \rho_\Omega(-k). \end{aligned} \quad (111)$$

Using the fact that $(x^*, y^*) \in \partial\mathcal{T}_{M,\Omega}(\bar{x}, \bar{y})$, we write

$$\begin{aligned} \langle -y^*, -k \rangle &= \langle (x^*, y^*); (0, k) \rangle \leq \mathcal{T}_{M,\Omega}((\bar{x}, \bar{y}); (0, k)) \\ &\leq \rho_\Omega(-k), \quad \forall k \in F. \end{aligned} \quad (112)$$

Fix any $v \in \Omega$, and take $k := -v$ in the previous inequality we get

$$\langle -y^*, v \rangle = \langle -y^*, -k \rangle \leq \rho_\Omega(v) \leq 1. \quad (113)$$

This ensures that $\sigma_\Omega(-y^*) \leq 1$, and hence, the proof is finished. $\square$

The following theorem establishes another relationship between the convex subdifferential of $\mathcal{T}_{M,\Omega}$ and the convex normal cone of M_r at points outside the graph $\text{gph } M$.

Theorem 18. Let E and F be two Hausdorff topological vector spaces, M a set-valued mapping with nonempty closed values of F , Ω be a bounded convex set in F , and $(\bar{x}, \bar{y}) \notin \text{gph } M$. Assume that $\pi_{M(\bar{x}), \Omega}(\bar{y}) \neq \emptyset$. Then, one has

$$\begin{aligned} \partial \mathcal{T}_{M, \Omega}(\bar{x}, \bar{y}) &\subset N(\text{gph } M_r; (\bar{x}, \bar{y})) \\ &\cap \{(x^*, y^*) \in E^* \times F^* : \sigma_{\Omega}(-y^*) = 1\}. \end{aligned} \quad (114)$$

If, in addition, $\mathcal{T}_{M, \Omega}$ is calm at $(\bar{x}, \bar{y})$, then one has

$$\begin{aligned} \partial \mathcal{T}_{M, \Omega}(\bar{x}, \bar{y}) &= N(\text{gph } M_r; (\bar{x}, \bar{y})) \\ &\cap \{(x^*, y^*) \in E^* \times F^* : \sigma_{\Omega}(-y^*) = 1\}. \end{aligned} \quad (115)$$

Proof. The first inclusion follows directly from the proof of Theorem 14 and Proposition 17. Let us prove the reverse inclusion. Fix any $(x^*, y^*) \in N(\text{gph } M_r; (\bar{x}, \bar{y}))$ with $\sigma_{\Omega}(-y^*) = 1$. First, let us check that $0 \notin \partial \mathcal{T}_{M, \Omega}(\bar{x}, \bar{y})$. Indeed, by the assumption $\pi_{M(\bar{x}), \Omega}(\bar{y}) \neq \emptyset$ and Proposition 17, there exists $z_0 \in \pi_{M(\bar{x}), \Omega}(\bar{y})$ such that

$$\mathcal{T}'_{M, \Omega}((\bar{x}, \bar{y}); (0, z_0 - \bar{y})) = -\mathcal{T}_{M, \Omega}(\bar{x}, \bar{y}) < 0, \quad (116)$$

which ensures that $0 \notin \partial \mathcal{T}_{M, \Omega}(\bar{x}, \bar{y})$. Therefore, by Corollary 15, there exists some $\lambda > 0$ and $(u^*, v^*) \in \partial \mathcal{T}_{M, \Omega}(\bar{x}, \bar{y})$, such that $(x^*, y^*) = \lambda(u^*, v^*)$. Using the first part of the proof, we obtain that $\sigma_{\Omega}(-v^*) = 1$, and so by combining this equality with the assumption $\sigma_{\Omega}(-y^*) = 1$, we get $\lambda = 1$. Thus completing the proof. $\square$

5. Application of $\mathcal{T}_{M, \Omega}$: Calmness and Exact Penalization

The primary goal in the present section is to make clear that the scalar function $\mathcal{T}_{M, \Omega}$ can also be a powerful tool in the study of the calmness property of optimization problems in Hausdorff topological vector spaces. Here, we are interested in the concept of calmness of perturbed optimization problems with a constraint defined by a set used and studied by Burke [22, 23] in normed vector spaces. We will adapt his definition for a general perturbed problem with a constraint defined by a set-valued mapping in Hausdorff topological vector spaces, and we will prove that it is equivalent to the existence of an exact penalization in terms of the minimal time function associated with the set-valued mapping defining the constraint of the problem.

Consider the problem $(\mathcal{P})$, which consists in minimizing the function f over all $x \in E$ satisfying $0 \in M(x)$,

$$(\mathcal{P}) \begin{cases} \text{minimize} & f(x) \\ \text{subject to} & 0 \in M(x), \end{cases} \quad (117)$$

where $M : E \rightrightarrows F$ is a closed set-valued mapping between two normed vector spaces E and F and $f : E \rightarrow \mathbb{R} \cup \{+\infty\}$ is an extended real-valued function. We begin with the definition of calmness.

Definition 19. Let f , M , E , and F be as in the statement of $(\mathcal{P})$, and consider the following perturbed problem

$$(\mathcal{P}_y) \begin{cases} \text{minimize} & f(x) \\ \text{subject to} & y \in M(x). \end{cases} \quad (118)$$

Let $S : F \rightrightarrows E$ be the feasible set-valued mapping associated with $(\mathcal{P}_y)$; that is,

$$S(y) := \{x \in \text{dom } f : y \in M(x)\}. \quad (119)$$

Let $(\bar{x}, \bar{y}) \in \text{gph } S$ and let Ω be a bounded set in F . One will say that the problem $(\mathcal{P}_{\bar{y}})$ is calm at $\bar{x}$ with respect to Ω if there exist a constant $\bar{\alpha} \geq 0$ and $X \in \mathcal{N}_E(\bar{x})$ such that for every $x \in X$ and any $y \in M(x)$ one has

$$f(\bar{x}) \leq f(x) + \bar{\alpha} \rho_{\Omega}(y - \bar{y}). \quad (120)$$

The constant $\bar{\alpha}$ and X are called the *modulus of calmness* and *neighborhood of calmness* for $(\mathcal{P}_{\bar{y}})$ at $\bar{x}$, respectively.

Remark 20. When F is assumed to be normed, the above definition coincides with the definition used in [22, 23] by taking Ω to be the closed unit ball in F . Observe that if $(\mathcal{P}_{\bar{y}})$ is calm at $\bar{x}$ with respect to a bounded set Ω , then $\bar{x}$ is necessarily a local solution to $(\mathcal{P}_{\bar{y}})$.

For any problem $(\mathcal{P}_y)$, any real number $\alpha \geq 0$, and any bounded set Ω , we will associate the function $P_{\alpha, y, \Omega}$ defined by

$$P_{\alpha, y, \Omega}(x) := f(x) + \alpha \mathcal{T}_{M, \Omega}(x, y). \quad (121)$$

In the following theorem we state our main result in this section. It establishes a relationship between the calmness property and the existence of an exact penalization of the general perturbed problem $(\mathcal{P}_{\bar{y}})$ in terms of the minimal time function to images associated with the set-valued mapping M defining the constraint of the problem.

Theorem 21. Let $(\bar{x}, \bar{y}) \in \text{gph } S$. If $(\mathcal{P}_{\bar{y}})$ is calm at $\bar{x}$ with respect to Ω with modulus $\bar{\alpha}$ and neighborhood $X \in \mathcal{N}_E(\bar{x})$, then $\bar{x}$ is a minimum over X of the function $P_{\alpha, \bar{y}, \Omega}$, for all $\alpha \geq \bar{\alpha}$. If, in addition, Ω is convex and $0 \in \Omega$, the converse holds.

Proof. Let $\epsilon > 0$. Given any $x \in X \cap \text{dom } M \neq \emptyset$. Then there exist $t_0 \geq 0$ and $w \in \Omega$ such that $\bar{y} + t_0 w \in M(x)$ and $\mathcal{T}_{M, \Omega}(\bar{x}, \bar{y}) \leq t_0 < \mathcal{T}_{M, \Omega}(\bar{x}, \bar{y}) + \epsilon$. Thus, if $\alpha \geq \bar{\alpha}$, we obtain from the calmness hypothesis that

$$f(\bar{x}) \leq f(x) + \alpha \rho_{\Omega}(y - \bar{y}), \quad \forall x \in X, \forall y \in M(x). \quad (122)$$

Fix any $x \in X$. Then by taking $y := \bar{y} + t_0 w$ in the previous inequality and observing that $\rho_{\Omega}(w) \leq 1$ we obtain that

$$\begin{aligned} f(\bar{x}) &\leq f(x) + \alpha \rho_{\Omega}(t_0 w) \leq f(x) + t_0 \alpha \rho_{\Omega}(w) \\ &\leq f(x) + t_0 \alpha \leq f(x) + \alpha \mathcal{T}_{M, \Omega}(\bar{x}, \bar{y}) + \alpha \epsilon. \end{aligned} \quad (123)$$

Hence

$$P_{\alpha, \bar{y}, \Omega}(\bar{x}) = f(\bar{x}) \leq P_{\alpha, \bar{y}, \Omega}(x) + \alpha \epsilon, \quad \forall x \in X. \quad (124)$$

Since $\epsilon > 0$ was taken arbitrarily, the direct implication is established by taking $\epsilon \rightarrow 0$.

Conversely, we assume that Ω is convex and $0 \in \Omega$. Fix any $x \in X$ and any $y \in M(x)$. We claim first the inequality $\mathcal{T}_{M,\Omega}(x, \bar{y}) \leq \rho_\Omega(y - \bar{y})$. Clearly the case $\rho_\Omega(y - \bar{y}) = \infty$ is obvious. So we assume that $\rho_\Omega(y - \bar{y}) < \infty$. Since $y \in M(x)$, we have $\mathcal{T}_{M,\Omega}(x, y) = 0$. Then by the finiteness of $\mathcal{T}_{M,\Omega}(x, y)$ we can find for any $\epsilon > 0$ some $\lambda_0 \geq 0$ such that

$$\begin{aligned} 0 &= \mathcal{T}_{M,\Omega}(x, y) \\ &\leq \lambda_0 < \mathcal{T}_{M,\Omega}(x, y) + \epsilon = \epsilon, \quad (y + \lambda_0 \Omega) \cap M(x) \neq \emptyset. \end{aligned} \quad (125)$$

Hence, there exists $w \in \Omega$ such that $y + \lambda_0 w \in M(x)$. Therefore, for any $v \in \Omega$ and any $\beta > 0$, we have

$$\begin{aligned} y + \lambda_0 w &= y - \beta v + (\lambda_0 w + \beta v) \\ &= y - \beta v + (\lambda_0 + \beta) \left(\frac{\lambda_0}{\lambda_0 + \beta} w + \frac{\beta}{\lambda_0 + \beta} v \right) \in M(x). \end{aligned} \quad (126)$$

Since Ω is convex, we have $(\lambda_0/(\lambda_0 + \beta))w + (\beta/(\lambda_0 + \beta))v \in \Omega$, and so $(y - \beta v + (\lambda_0 + \beta)\Omega) \cap M(x) \neq \emptyset$, which ensures by the definition of the minimal time function that

$$\mathcal{T}_{M,\Omega}(x, y - \beta v) \leq \lambda_0 + \beta < \epsilon + \beta, \quad \forall \epsilon > 0. \quad (127)$$

Since $\rho_\Omega(y - \bar{y}) < \infty$, we let $\beta := \rho_\Omega(y - \bar{y}) + \epsilon < \infty$ and $v := (y - \bar{y})/\beta$. Since Ω is convex and $0 \in \Omega$, we have $\{h \in E : \rho_\Omega(h) < 1\} \subset \Omega$, and hence $v \in \Omega$. Thus, (127) ensures that

$$\begin{aligned} \mathcal{T}_{M,\Omega}(x, \bar{y}) &= \mathcal{T}_{M,\Omega}(x, y - \beta v) < \epsilon + \beta \\ &= \rho_\Omega(y - \bar{y}) + 2\epsilon, \quad \forall \epsilon > 0. \end{aligned} \quad (128)$$

After taking $\epsilon \rightarrow 0$, this inequality ensures the desired inequality; that is, $\mathcal{T}_{M,\Omega}(x, \bar{y}) \leq \rho_\Omega(y - \bar{y})$. Therefore, for any $x \in X$ and any $y \in M(x)$, we have

$$\begin{aligned} f(\bar{x}) &= P_{\bar{\alpha}, \bar{y}, \Omega}(\bar{x}) \leq P_{\bar{\alpha}, \bar{y}, \Omega}(x) = f(x) + \bar{\alpha} \mathcal{T}_{M,\Omega}(x, \bar{y}) \\ &\leq f(x) + \bar{\alpha} \rho_\Omega(y - \bar{y}). \end{aligned} \quad (129)$$

Hence, $(\mathcal{P}_{\bar{y}})$ is calm at $\bar{x}$ with respect to Ω , and the proof of the theorem is complete. $\square$

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