Hindawi Publishing Corporation
Journal of Function Spaces and Applications
Volume 2013, Article ID 715789, 5 pages
http://dx.doi.org/10.1155/2013/715789

Research Article

The Use of an Isometric Isomorphism on the Completion of the Space of Henstock-Kurzweil Integrable Functions

Luis Ángel Gutiérrez Méndez, Juan Alberto Escamilla Reyna, Francisco Javier Mendoza Torres, and María Guadalupe Morales Macías

Facultad de Ciencias Físico Matemáticas, Benemérita Universidad, Autónoma de Puebla, Avenida San Claudio y 18 Sur, Colonia San Manuel, 72570, Puebla, Mexico

Correspondence should be addressed to Luis Ángel Gutiérrez Méndez; gutierrezmendezluisangel@yahoo.com.mx

Received 19 April 2013; Accepted 5 June 2013

Academic Editor: Nelson Merentes

Employing an isometrically isomorphic space, we determine new properties for the completion of the space of the Henstock-Kurzweil integrable functions with the Alexiewicz norm.

1. Introduction

Let $[a,b]$ be a compact interval in $\mathbb{R}$. In the vector space of Henstock-Kurzweil integrable functions on $[a,b]$ with values in $\mathbb{R}$, the Alexiewicz seminorm is defined as

$$\|f\|_A = \sup_{a\le r\le b}\left|\int_a^r f\right|. \tag{1}$$

The corresponding normed space is built using the quotient space determined by the relation $f \sim g$ if and only if $f = g$, *except in a set of Lebesgue measure zero or, equivalently, if they have the same indefinite integral.* This normed space will be denoted by $(\mathcal{HK}[a,b], \|\cdot\|_A)$.

It is known that $(\mathcal{HK}[a,b], \|\cdot\|_A)$ is neither complete nor of the second category [1]. However, $(\mathcal{HK}[a,b], \|\cdot\|_A)$ is a separable space [1] and, consequently, its completion also has the same property. In addition, $(\mathcal{HK}[a,b], \|\cdot\|_A)$ has "nice" properties, from the point of view of functional analysis, since it is an ultrabornological space [2]. As $(\mathcal{HK}[a,b], \|\cdot\|_A)$ is not complete, it is natural to study its completion, which will be denoted by $(\widehat{\mathcal{HK}[a,b]}, \|\cdot\|_A)$.

Talvila in [3] makes an analysis to determine some properties of the Henstock-Kurzweil integral on $(\widehat{\mathcal{HK}[a,b]}, \|\cdot\|_A)$, such as integration by parts, Hölder inequality, change of variables, convergence theorems, the Banach lattice structure, the Hake theorem, the Taylor theorem, and second mean value theorem. Talvila makes this analysis by means of the space of the distributions that are derivatives of the continuous functions on $[a,b]$, which is an isometrically isomorphic space to $(\widehat{\mathcal{HK}[a,b]}, \|\cdot\|_A)$. Making use of this same isometrically isomorphic space, Bongiorno and Panchapagesan in [4] establish characterizations for the relatively weakly compact subsets of $(\mathcal{HK}[a,b], \|\cdot\|_A)$ and $(\widehat{\mathcal{HK}[a,b]}, \|\cdot\|_A)$.

In this paper, we make an analysis on $(\widehat{\mathcal{HK}[a,b]}, \|\cdot\|_A)$ by means of another isometrically isomorphic space to prove that $(\widehat{\mathcal{HK}[a,b]}, \|\cdot\|_A)$*has the Dunford-Pettis property, it has a complemented subspace isomorphic to* c_0*, it does not have the Radon-Riesz property, it is not weakly sequentially complete, and it is not isometrically isomorphic to the dual of any normed space;* hence, we will also prove that $(\widehat{\mathcal{HK}[a,b]}, \|\cdot\|_A)$*is neither reflexive nor has the Schur property*. Then, as an application of the above results, we prove that $(\widehat{\mathcal{HK}[a,b]}, \|\cdot\|_A)$ is not isomorphic to the dual of any normed space and that the space of all bounded, linear, weakly compact operators from $(\widehat{\mathcal{HK}[a,b]}, \|\cdot\|_A)$ into itself is not a complemented subspace in the space of all bounded, linear operators from $(\widehat{\mathcal{HK}[a,b]}, \|\cdot\|_A)$ into itself.

2. Preliminaries

In this section, we restate the conventions, notations, and concepts that will be used throughout this paper.

All the vector spaces are considered over the field of the real numbers or complex numbers.

Let X be a normed space. By X^*, we denote the dual space of X. A topological property that holds with respect to the weak topology of X is said to be a *weak* property or to hold *weakly*. On the other hand, if a topological property holds without specifying the topology, the norm topology is implied.

Let X, Y be two Banach spaces. We denote by $L(X,Y)$ ($W(X,Y)$, resp.) we denote the Banach space of all bounded, linear (bounded, linear, weakly compact resp.), operators from X into Y. If $X = Y$, then we write $L(X)$ (resp. $W(X)$) instead of $L(X,X)$ (resp. $W(X,X)$).

The symbols c_0, l_1, and l_∞ represent, as usual, the vector spaces of all sequences of scalars convergent to 0, all sequences of scalars absolutely convergent, and all bounded sequences of scalars, respectively, neither one with nor usual norm.

Let K be a compact metric space. We denote by $\mathscr{C}(K)$ the vector space of all continuous functions of scalar-values on K together with the norm defined by $\|F\|_\infty = \sup\{|F(x)| : x \in K\}$.

By $\mathscr{B}_c[a,b]$ we denote the following collection:

$$\{F : [a,b] \longrightarrow \mathbb{R} \mid F \text{ is continuous on } [a,b] \text{ and } F(a) = 0\} \tag{2}$$

which is a closed subspace of $\mathscr{C}[a,b]$ and $(\mathscr{B}_c[a,b], \|\cdot\|_\infty)$ is therefore a Banach space.

Definition 1. Let X *and* Y be normed spaces and let $T : X \to Y$ be a lineal operator. We have the following.

(i) T is an *isomorphism* if it is one-to-one and continuous and its inverse mapping T^{-1} is continuous on the range of T. Moreover, if $\|T(x)\| = \|x\|$, for all $x \in X$, it is said that T is an *isometric isomorphism*.

(ii) X and Y are *isomorphic*, which is denoted by $X \cong Y$, if there exists an isomorphism from X onto Y.

(iii) X and Y are *isometrically isomorphic* if there exists an isometric isomorphism from X onto Y.

The following result is key to our principal results.

Theorem 2 (see [4]). *The space* $(\mathscr{B}_c[a,b], \|\cdot\|_\infty)$ *is isometrically isomorphic to* $(\widehat{\mathscr{H}\mathscr{K}[a,b]}, \|\cdot\|_A)$.

According to Theorem 2, we will prove that $(\widehat{\mathscr{H}\mathscr{K}[a,b]}, \|\cdot\|_A)$ has the Dunford-Pettis property.

Definition 3. Let X be a Banach space. It is said that X has the *Dunford-Pettis property* if for every sequence $\{x_n\}$ in X converging weakly to 0 and every sequence $\{x_n^*\}$ in X^* converging weakly to 0, the sequence $\{x_n^*(x_n)\}$ converges to 0.

If a Banach space X has the Dunford-Pettis property, then not necessarily every closed subspace of X inherits such property, except when the subspace is complemented in X [5].

Definition 4. Let Y be a subspace of a normed space X. It is said that Y is complemented in X if it is closed in X and there exists a closed subspace W in X such that $X = Y \oplus W$.

Theorem 5 (see [5]). *Let X be a Banach space with the Dunford-Pettis property. If Y is a complemented subspace in X, then Y has the Dunford-Pettis property.*

To prove that $(\widehat{\mathscr{H}\mathscr{K}[a,b]}, \|\cdot\|_A)$ has a complemented subspace isomorphic to c_0 the following result is essential.

Theorem 6 (see [6]). *Let K be a compact metric space. If X is an infinite-dimensional complemented subspace of $\mathscr{C}(K)$, then X contains a complemented subspace isomorphic to c_0.*

On the other hand, making use again of Theorem 2 we will prove that $(\widehat{\mathscr{H}\mathscr{K}[a,b]}, \|\cdot\|_A)$ is neither weakly sequentially complete nor has the Radon-Riesz property.

Definition 7. Let X be a normed space, and let $\{x_n\}$ be a sequence in X and $x \in X$.

(i) If $\{x_n\}$ weakly converges whenever $\{x_n\}$ is weakly Cauchy, then it is said that X is *weakly sequentially complete*.

(ii) If $\{x_n\}$ converges to x whenever $\{x_n\}$ weakly converges to x and $\|x_n\| \to \|x\|$, then it is said that X has the *Radon-Riesz property* or the *Kadets-Klee property*.

(iii) If $\{x_n\}$ converges to x whenever $\{x_n\}$ weakly converges to x, then it is said that X has the *Schur property*.

The following result establishes a characterization of weakly Cauchy sequences and weakly convergent sequences of the space $\mathscr{C}[a,b]$.

Theorem 8 (see [7]). *Let $\{F_n\}$ and F be a sequence and an element, respectively, in the space $\mathscr{C}[a,b]$. Then we have the following.*

(1) *The sequence $\{F_n\}$ is weakly convergent to F if and only if*

(i) $\lim_{n\to\infty} F_n(x) = F(x)$, *for all* $x \in [a,b]$,

(ii) *there exists* $M > 0$ *such that* $\|F_n\|_\infty \le M$, *for all* $n \in \mathbb{N}$.

(2) *The sequence $\{F_n\}$ is weakly Cauchy if and only if*

(i) $\lim_{n\to\infty} F_n(x)$ *exists, for all* $x \in [a,b]$,

(ii) *there exists* $M > 0$ *such that* $\|F_n\|_\infty \le M$, *for all* $n \in \mathbb{N}$.

We will also prove that $(\widehat{\mathscr{H}\mathscr{K}[a,b]}, \|\cdot\|_A)$ is not isometrically isomorphic to the dual of any normed space and, as a consequence, we will prove that $(\widehat{\mathscr{H}\mathscr{K}[a,b]}, \|\cdot\|_A)$ is not reflexive, for which we will use again Theorem 2 and the concept of extremal point.

Definition 9. Let X be a vector space, $K \subseteq X$, and $z \in K$. It is said that z is an *extremal point* of K if for all $x, y \in K$ such that $z = (1/2)(x+y)$ it holds that $z = x = y$.

If X is a normed space, then its closed unit ball will be denoted by B_X and the collection of all extremal points of B_X as $\operatorname{ext}(B_X)$.

The following theorem establishes that the extremal points are preserved under isometric isomorphisms.

Theorem 10 (see [8]). *Let X and Y be Banach spaces, $K \subseteq X$ and let $T : X \to Y$ be an isometric isomorphism. Then x is an extremal point of K if and only if $T(x)$ is an extremal point of $T(K)$.*

Corollary 11 (see [9]). *An infinite-dimensional normed space whose closed unit ball has only finitely many extreme points is not isometrically isomorphic to the dual of any normed space.*

3. Principal Results

Lemma 12. *The space $(\mathscr{B}_c[a,b], \|\cdot\|_\infty)$ has the Dunford-Pettis property.*

Proof. Let $H : \mathscr{C}[a,b] \to \mathbb{R}$ be the functional defined by $H(F) = F(a)$. It is clear that H is bounded and therefore $\ker(H)$ is a hyperplane of the space $\mathscr{C}[a,b]$, that is,

$$\mathscr{C}[a,b] = \ker(H) \oplus W, \tag{3}$$

where $\ker(H) = \{F \in \mathscr{C}[a,b] : F(a) = 0\} = \mathscr{B}_c[a,b]$ and W is a one-dimension, subspace in $\mathscr{C}[a,b]$. Then, as $\mathscr{C}[a,b]$ has the Dunford-Pettis property [10] and according to Theorem 5, we obtain the desired conclusion. □

Lemma 13. *The space $(\mathscr{B}_c[a,b], \|\cdot\|_\infty)$ has a complemented subspace isomorphic to c_0.*

Proof. Using the proof of Lemma 12, we can see that $\mathscr{B}_c[a,b]$ is a complemented subspace in $\mathscr{C}[a,b]$. Then, according to Theorem 6, we obtain the desired conclusion. □

Lemma 14. *The space $(\mathscr{B}_c[a,b], \|\cdot\|_\infty)$ is not weakly sequentially complete.*

Proof. Without loss of generality, suppose that $|b-a| \geq 1$. Let F_n be the function defined by

$$F_n(x) = \begin{cases} 0, & \text{if } x \in \left[a, b - \dfrac{1}{n}\right], \\ n(x-b)+1, & \text{if } x \in \left(b - \dfrac{1}{n}, b\right], \end{cases} \tag{4}$$

for all $n \in \mathbb{N}$. Thus,

(i) $\lim_{n\to\infty} F_n(x)$ exists, for all $x \in [a,b]$,

(ii) $\|F_n\|_\infty = 1$, for all $n \in \mathbb{N}$.

Therefore, according to Theorem 8 item (2), it follows that the sequence $\{F_n\}$ is weakly Cauchy in $(\mathscr{B}_c[a,b], \|\cdot\|_\infty)$.

Now, suppose that there exists a function $G \in (\mathscr{B}_c[a,b], \|\cdot\|_\infty)$ such that the sequence $\{F_n\}$ converges weakly to G. Then according to Theorem 8 item (1), it holds that $G(x) = \lim_{n\to\infty} F_n(x)$, for all $x \in [a,b]$, that is,

$$G(x) = \begin{cases} 0, & \text{if } x \in [a,b), \\ 1, & \text{if } x = b. \end{cases} \tag{5}$$

However, since G is not continuous, it follows that the sequence $\{F_n\}$ does not converge weakly in $(\mathscr{B}_c[a,b], \|\cdot\|_\infty)$. □

Lemma 15. *The space $(\mathscr{B}_c[a,b], \|\cdot\|_\infty)$ does not have the Radon-Riesz property.*

Proof. Without loss of generality, suppose that $|b-a| \geq 1$. Let F_n be the function defined by

$$F_n(x) = \begin{cases} \left(\dfrac{n}{n(b-a)-1}\right)(x-a), & \text{if } x \in \left[a, b - \dfrac{1}{n}\right], \\ 2n(b-x)-1, & \text{if } x \in \left(b - \dfrac{1}{n}, b - \dfrac{1}{2n}\right], \\ 2n(x-b)+1, & \text{if } x \in \left(b - \dfrac{1}{2n}, b\right], \end{cases} \tag{6}$$

for all $n \geq 2$. Thus,

(i) the sequence $\{F_n\}$ converges pointwise to F, where $F(x) = (x-a)/(b-a)$ for all $x \in [a,b]$,

(ii) $\|F_n\|_\infty = 1$, for all $n \geq 2$.

Therefore, according to Theorem 8 item (1), it follows that the sequence $\{F_n\}$ converges weakly to F in $(\mathscr{B}_c[a,b], \|\cdot\|_\infty)$; in addition, as $\|F\|_\infty = 1$, it holds that $\|F_n\|_\infty \to \|F\|_\infty$. However, the sequence $\{F_n\}$ does not converge to F in $(\mathscr{B}_c[a,b], \|\cdot\|_\infty)$. □

It is not difficult to prove that if a Banach space has the Dunford-Pettis property, or if it has a complemented subspace isomorphic to c_0, or if it is not weakly sequentially complete, or if it has the Radon-Riesz property, then these properties are preserved under isometric isomorphisms. Therefore, according to Theorem 2 and Lemmas 12, 13, 14 and 15, we obtain the following result.

Proposition 16. *The space $(\widehat{\mathscr{H}\mathscr{K}[a,b]}, \|\cdot\|_A)$*

(1) *has the Dunford-Pettis property,*

(2) *has a complemented subspace isomorphic to c_0,*

(3) *is not weakly sequentially complete,*

(4) *does not have the Radon-Riesz property.*

Remark 17. According to Definition 7 and Proposition 16 item (4), it follows that $(\widehat{\mathscr{H}\mathscr{K}[a,b]}, \|\cdot\|_A)$ does not have the *Schur property*.

Lemma 18. *The collection of all extremal points of the closed unit ball of the space $(\mathscr{B}_c[a,b], \|\cdot\|_\infty)$ is empty.*

Proof. Let $F \in B_{\mathscr{B}_c[a,b]}$. Since F is continuous on $[a,b]$, it holds that for $\varepsilon = 1/2$ there exits $\delta > 0$ such that

$$|F(x)| < \frac{1}{2}, \quad \forall x \in [a, a+\delta). \tag{7}$$

Now, define the following functions:

$$G(x) = \begin{cases} F(x) + r(x), & \text{if } x \in [a, a+\delta], \\ F(x), & \text{if } x \in [a+\delta, b], \end{cases}$$
$$H(x) = \begin{cases} F(x) - r(x), & \text{if } x \in [a, a+\delta], \\ F(x), & \text{if } x \in [a+\delta, b], \end{cases} \tag{8}$$

where the function r can be any continuous function defined over the interval $[a, a+\delta]$ such that $r(a) = 0 = r(a+\delta)$ and $\|r\|_\infty < 1/2$.

Since the functions G and H are continuous, $\|G\|_\infty = 1 = \|H\|_\infty$, and $F = (1/2)(G+H)$, it holds that F cannot be an extremal point; therefore, $\operatorname{ext}(B_{\mathscr{B}_c[a,b]}) = \emptyset$. □

It is a known fact that the collection $\operatorname{ext}(B_{\mathscr{C}[a,b]})$ is formed only by the constant functions ± 1. However, since in general there is not a relationship between the extremal points of the closed unit ball of a subspace with the extremal points of the closed unit ball of all space, we need Lemma 18 for the following result.

Proposition 19. *The space $(\widehat{\mathscr{H}\mathscr{K}[a,b]}, \|\cdot\|_A)$ is not isometrically isomorphic to the dual of any normed space.*

Proof. By Lemma 18 and Theorems 2 and 10, it holds that $\operatorname{ext}(B_{\widehat{\mathscr{H}\mathscr{K}[a,b]}}) = \emptyset$. Then, as the space $(\widehat{\mathscr{H}\mathscr{K}[a,b]}, \|\cdot\|_A)$ is dimensionality infinite and according to Corollary 11, we obtain the desired conclusion. □

Corollary 20. *The space $(\widehat{\mathscr{H}\mathscr{K}[a,b]}, \|\cdot\|_A)$ is not reflexive.*

Proof. Suppose that the space $(\widehat{\mathscr{H}\mathscr{K}[a,b]}, \|\cdot\|_A)$ is reflexive. Then $(\widehat{\mathscr{H}\mathscr{K}[a,b]}, \|\cdot\|_A)$ coincides under the canonical imbedding with its second dual. Therefore, the space $(\widehat{\mathscr{H}\mathscr{K}[a,b]}, \|\cdot\|_A)$ is isometrically isomorphic to the dual of the space $(\widehat{\mathscr{H}\mathscr{K}[a,b]}, \|\cdot\|_A)^*$, which is a contradiction by Proposition 19. □

In general, it is important to know when a Banach space enjoys certain functional analysis properties. However, in certain contexts, also it is useful to know when a Banach space does not have certain properties; Propositions 22 and 24 are examples of both facts.

Lemma 21 (see [11]). *Let X, Y be two Banach spaces. Assume that X and Y contain a complemented copy of c_0. Then $W(X,Y)$ is uncomplemented in $L(X,Y)$.*

Proposition 22. *The space $W(\widehat{\mathscr{H}\mathscr{K}[a,b]})$ is uncomplemented in the space $L(\widehat{\mathscr{H}\mathscr{K}[a,b]})$.*

Proof. According to Proposition 16 item (2), it holds that $(\widehat{\mathscr{H}\mathscr{K}[a,b]}, \|\cdot\|_A)$ contains a complemented copy of c_0. Therefore, on the basis of Lemma 21, we obtain the desired conclusion. □

By Proposition 19, we can see that $(\widehat{\mathscr{H}\mathscr{K}[a,b]}, \|\cdot\|_A)$ is not isometrically isomorphic to the dual of any normed space. However, we can ask ourselves the following. Is there a normed space X such that $(\widehat{\mathscr{H}\mathscr{K}[a,b]}, \|\cdot\|_A)$ is isomorphic to the dual of X? To answer this question, we need of the following result.

Lemma 23 (see [7]). *Let X be a normed space and let Y be a Banach space with the Dunford-Pettis property that does not have the Schur property. If X^* contains a copy of Y, then X contains a copy of l_1.*

Proposition 24. *The space $(\widehat{\mathscr{H}\mathscr{K}[a,b]}, \|\cdot\|_A)$ is not isomorphic to the dual of any normed space.*

Proof. Suppose that there exists a normed space X such that

$$\left(\widehat{\mathscr{H}\mathscr{K}[a,b]}, \|\cdot\|_A\right) \cong X^*. \tag{9}$$

Since $(\mathscr{H}\mathscr{K}[a,b], \|\cdot\|_A)$ is separable [1] and consequently $(\widehat{\mathscr{H}\mathscr{K}[a,b]}, \|\cdot\|_A)$ also is separable, it follows from (9) that X^* is separable.

On the other hand, by Proposition 16 item (2) and according to the isomorphism from (9), it holds that X^* has a complemented subspace isomorphic to c_0. Since c_0 has the Dunford-Pettis property [9] and does not have the Schur property [9], it holds that X has a copy of l_1, by Lemma 23.

As X has a copy of l_1, it holds that l_1^* is isometrically isomorphic to $X^*/l_1^\perp$, where $l_1^\perp$ denotes the annihilator of l_1. Since l_1^* is isometrically isomorphic to l_∞ it holds that, in particular,

$$l_\infty \cong \frac{X^*}{l_1^\perp}. \tag{10}$$

Therefore, since X^* is separable and according to the isomorphism from (10), we obtain that l_∞ is separable, which is a contradiction. □

On this way, we can see that Proposition 19 and Corollary 20 are consequences of the above result. We did not do it this way because one of the principal objectives of this paper is to show the importance of knowing explicitly a closed subspace of $\mathscr{C}[a,b]$ which is isometrically isomorphic to $(\widehat{\mathscr{H}\mathscr{K}[a,b]}, \|\cdot\|_A)$ and it is a known fact that every separable Banach space of infinite dimension is isometrically isomorphic to a closed subspace of $\mathscr{C}[a,b]$; however, this information is not sufficient to prove, in particular, the results that we have shown in this paper.

References

[1] C. Swartz, *Introduction to Gauge Integrals*, World Scientific, Singapore, 2001.

[2] J. L. Gámez, *Integraciones de denjoy de funciones con valores en espacios de banach [Ph.D. thesis]*, Universidad Complutense de Madrid, 1997.

[3] E. Talvila, "The distributional Denjoy integral," *Real Analysis Exchange*, vol. 33, no. 1, pp. 51–82, 2008.

[4] B. Bongiorno and T. V. Panchapagesan, "On the Alexiewicz topology of the Denjoy space," *Real Analysis Exchange*, vol. 21, no. 2, pp. 604–614, 1995-1996.

[5] J. Diestel, "A survey of results related to the Dunford-Pettis property," *AMS Contemporary Mathematics*, vol. 2, pp. 15–60, 1980.

[6] F. Albiac and N. J. Kalton, *Topics in Banach Space Theory*, Springer, New York, NY, USA, 2006.

[7] J. Diestel, *Sequences and Series in Banach Spaces*, Springer, New York, NY, USA, 1984.

[8] A. Curnock, "An introduction to extreme points and applications in isometric Banach space theory," in *Proceedings of the the Analysis Group*, Goldsmiths College, University of London, May 1998, part of early doctoral work.

[9] R. E. Megginson, *An Introduction to Banach Space Theory*, Springer, New York, NY, USA, 1998.

[10] R. G. Bartle, N. Dunford, and J. Schwartz, "Weak compactness and vector measures," *Canadian Journal of Mathematics*, vol. 7, pp. 289–305, 1955.

[11] G. Emmanuele, "Remarks on the uncomplemented subspace $W(E, F)$," *Journal of Functional Analysis*, vol. 99, no. 1, pp. 125–130, 1991.